Paley-Wiener type theorems for a differential operator connected with symmetric spaces

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1. Introduction

This paper deals with spectral decomposition of the following differential operators

$$\omega_{p,q} : d^2/dt^2 + g \cdot d/dt$$

where $g(t) = p \cdot \coth t + 2q \cdot \coth 2t$, with p and q positive real numbers and t contained in the open interval $]0, \infty[$.

The radial part of the Laplace-Beltrami operator on a symmetric space of rank one is of the form $\omega_{p,q}$. Here (p, q) are certain pairs of non-negative integers. See Harish-Chandra [11], p. 302, Araki [1].

The main result in this paper is Theorem 4, which generalizes the classical Paley-Wiener theorem, characterizing the Fourier transform of C^∞-functions with compact support, and the theorem that the set of Schwartz-functions is mapped onto itself by the Fourier transform. The results in Theorem 4 (i) and (ii) are well known for symmetric space, see Gangolli [8], Harish-Chandra [11], [12], Helgason [14], [15], Trombi and Varadarajan [23]. However our proof does not use the theory of Lie groups and symmetric spaces at all. Theorem 4 (iii) is probably only known in a special case, see Ehrenpreis and Mautner [6].

The main difficulty in the proof of Theorem 4 is getting the best possible estimates for the eigenfunctions of $\omega_{p,q}$. This is done in Theorem 2. In our proof we use heavily the fact that the eigenfunctions of $\omega_{p,q}$ are essentially hypergeometric functions. It seems, however, that similar results should be obtainable for other differential operators having the same type of singularities as $\omega_{p,q}$. In fact if $p + q < 1$ some results of Dym [5] can be applied to $\omega_{p,q}$ and give a Paley-Wiener theorem which however is weaker than ours. Note also that the classical Hankel-transform is a spectral decomposition of the differential operator $\omega^k = d^2/dt^2 + k \cdot t^{-1} d/dt$, which has the same type of singularity at zero as $\omega_{p,q}$ (for $k = p + q$). For the Hankel transform it is quite easy (using well known facts about Bessel functions) to prove a Paley-Wiener theorem. A prelimenary version of the proof of Theorem 2 used perturbation theory relating eigenfunctions of $\omega_{p,q}$ to eigenfunctions of ω^k.

Results of Ehrenpreis and Mautner [6], Gasper [9], Muckenhoupt and Stein [20], and Schwartz [21] suggest that one should develop further harmonic analysis with respect to $\omega_{p,q}$.

I would like to express my thanks to prof. S. Helgason for taking interest in this work, and to prof. L. Carleson for his hospitality at Institut Mittag-Leffler.

2. Statement of results

1. *Eigenfunctions of* $\omega_{p,q}$

The singular points for $\omega = \omega_{p,q}$ are 0 and ∞. The function $g(t)$ has the form $g(t) = (p + q)t^{-1} + G(t)$, where $G(t)$ is analytic in the closed interval $[0, \infty[$ and $\lim_{t\to\infty} g(t) = p + 2q$. Define $\varrho = 2^{-1}(p + 2q)$. Well known results about solutions of singular second order differential equations [3], Chap IV, § 8, [18], § 7, give the following facts about the equation:

$$\omega_{p,q}\varphi + (\lambda^2 + \varrho^2)\varphi = 0 \quad \text{on} \quad]0, \infty[\tag{2.1}$$

There exists a unique solution $\varphi_\lambda(t)$ satisfying $\varphi_\lambda(0) = 1$ and $\varphi_\lambda'(0) = 0$, and it follows that $\varphi_\lambda(t)$ is analytic for $t \in [0, \infty[$ and that $\varphi_\lambda(t) = \varphi_{-\lambda}(t)$ and $\overline{\varphi_\lambda(t)} = \varphi_{\bar\lambda}(t)$. There exists a linearly independent solution behaving at zero, for $p + q \neq 1$ like $t^{1-(p+q)}$, and for $p + q = 1$ like $\log t$.

For $\operatorname{Im}\lambda \geq 0$ there exists a unique solution $\Phi_\lambda(t)$ satisfying

$$\Phi_\lambda(t) = e^{(i\lambda-\varrho)t}(1 + o(1)) \quad \text{as} \quad t \to \infty$$

and it follows that $\Phi_\lambda(t)$ is analytic for $t \in]0, \infty[$ and that $\overline{\Phi_\lambda(t)} = \Phi_{-\bar\lambda}(t)$.

There exists a linearly independent solution behaving at ∞, for $\lambda \neq 0$ like $e^{(-i\lambda-\varrho)t}$, and for $\lambda = 0$ like $te^{-\varrho t}$.

PROPOSITION 1. *For each fixed* $t \in]0, \infty[$, *as function of* λ, $\varphi_\lambda(t)$ *is an entire function; and* $\Phi_\lambda(t)$ *is holomorphic in the upper half plane, and extends to a holomorphic function in* $\Omega = \mathbf{C}\setminus\{-i\mathbf{N}\}$. *For all* $\lambda \in \Omega$ $\Phi_\lambda(t)$ *is a solution of* (2.1) *and satisfies*

$$\Phi_\lambda(t) = e^{(i\lambda-\varrho)t}(1 + o(1)) \quad \textit{as} \quad t \to \infty\,.$$

For $\lambda \neq 0$, such that λ, $-\lambda \in \Omega$, Φ_λ and $\Phi_{-\lambda}$ are linearly independent because of their behaviour at ∞. Hence there exist $c^+(\lambda)$ and $c^-(\lambda)$ such that

$$\varphi_\lambda(t) = c^+(\lambda)\Phi_\lambda(t) + c^-(\lambda)\Phi_{-\lambda}(t)\,.$$

Obviously $c^+(-\lambda) = c^-(\lambda)$, so we define $c(\lambda) = c^+(\lambda)$ and have $\overline{c(\lambda)} = c(-\bar{\lambda})$.

THEOREM 2.

(i) *For all* $n \in \mathbf{Z}^+$ *there exists* $K_n > 0$ *such that for all* $\lambda = \xi + i\eta \in \mathbf{C}$, $t \in [0, \infty[$*:*

$$\text{(ia)} \quad \left|\frac{d^n}{dt^n}\varphi_\lambda(t)\right| \le K_n(1 + |\lambda|)^n(1 + t)e^{(|\eta|-\varrho)t}$$

$$\text{(ib)} \quad \left|\frac{d^n}{d\lambda^n}\varphi_\lambda(t)\right| \le K_n(1 + t)^{n+1}e^{(|\eta|-\varrho)t}$$

(ii) *For all* $c > 0$, $\varepsilon > 0$ *and* $n \in \mathbf{Z}^+$ *there exists* $K_n > 0$ *such that for all* $\lambda = \xi + i\eta \in \mathbf{C}$ *with* $\eta \ge -\varepsilon|\xi|$ *and all* $t \in [c, \infty[$*:*

$$\Phi_\lambda(t) = e^{(i\lambda-\varrho)t}(1 + e^{-2t}\Theta(\lambda, t)) \quad \textit{and} \quad \left|\frac{d^n}{dt^n}\Theta(\lambda, t)\right| \le K_n$$

(iii) *For all* $\varepsilon > 0$ *there exists* $K > 0$ *such that for all* $\lambda = \xi + i\eta \in \mathbf{C}$ *with* $\eta \ge -\varepsilon|\xi|$*:*

$$|\lambda c(-\lambda)| \le K(1 + |\lambda|)^{1-2^{-1}(p+q)}$$
$$|c(-\lambda)|^{-1} \le K(1 + |\lambda|)^{2^{-1}(p+q)}$$

2. *Generalized Fourier transform*

Notice that for $p = q = 0$, we have $\varphi_\lambda(t) = \cos\lambda t$, $\Phi_\lambda(t) = e^{i\lambda t}$, $c(\lambda) = 1/2$, $\Theta(\lambda, t) = 0$. In this case the estimates are trivial, but not essentially better. Restricting our attention to even functions f on $\mathbf{R}$, the classical Fourier cosine transform $\tilde{f}(\lambda) = (2/\pi)^{1/2}\int_0^\infty f(t)\cos\lambda t dt$ and the inversion formula $f(t) = (2/\pi)^{1/2}\int_0^\infty f(\lambda)\cos\lambda t d\lambda$ is a spectral decomposition of $\omega_{0,0} = d^2/dt^2$. We shall now discuss the similar »Fourier transform» related to $\omega_{p,q}$.

First note that

$$\omega_{p,q} = \Delta(t)^{-1}\frac{d}{dt}\left(\Delta(t)\frac{d}{dt}\right) \quad \text{where} \quad \Delta(t) = (e^t - e^{-t})^p(e^{2t} - e^{-2t})^q\,,$$

and thus $\omega_{p,q}$ is formally self-adjoint with respect to the measure $\Delta(t)dt$ on $[0, \infty[$. The operator in $L^2(\Delta)$ defined by $\omega_{p,q}$ with domain

$$D^0_{p,q} = \{u \in L^2(\Delta)\,|\,u \text{ and } u' \text{ are absolutely continuous and } \omega_{p,q}u \in L^2(\Delta)\}$$

can be restricted to a domain $D_{p,q}$, such that $\omega_{p,q}$ becomes self-adjoint. $D_{p,q}$ contains at least functions in $D^0_{p,q}$ which are differentiable at zero. For this see e.g. [16], p. 208 or [18], § 9, $\omega_{p,q}$ has limit-point at ∞; and at zero there is limit-point if $p+q \geq 3$, and limit-circle if $p+q<3$. In this last case $D_{p,q} \neq D^0_{p,q}$ and choosing $\lambda_1 \in \mathbf{C}$ with $\operatorname{Im} \lambda_1^2 > 0$ we can define

$$D_{p,q} = \{u \in D^0_{p,q} \mid \lim_{t\to 0} (\Delta(t) \cdot (\varphi_{\lambda_1}(t)\overline{u'(t)} - \varphi'_{\lambda_1}(t)\overline{u(t)})) = 0\}.$$

PROPOSITION 3. *For $f \in L^2(\Delta)$ and $\lambda \in \mathbf{R}^+$ define*

$$\tilde{f}(\lambda) = (2\pi)^{-1/2} \int_0^\infty f(t)\varphi_\lambda(t)\Delta(t)dt$$

the integral converging in $L^2(|c(\lambda)|^{-2})$. $f \to \tilde{f}$ is a linear, normpreserving map of $L^2(\Delta)$ onto $L^2(|c(\lambda)|^{-2})$, the inverse given by

$$f(t) = (2\pi)^{-1/2} \int_0^\infty \tilde{f}(\lambda)\varphi_\lambda(t)|c(\lambda)|^{-2}d\lambda$$

the integral converging in $L^2(\Delta)$. A function $f \in L^2(\Delta)$ belongs to $D_{p,q}$ if and only if $(\lambda^2+\varrho^2)\tilde{f}(\lambda) \in L^2(|c(\lambda)|^{-2})$ and in that case

$$\widetilde{\omega_{p,q}f}(\lambda) = -(\lambda^2+\varrho^2)\tilde{f}(\lambda).$$

3. *Paley-Wiener theorems*

For all $\lambda \in \mathbf{C}$ for which the integral converges we denote also

$$\tilde{f}(\lambda) = (2\pi)^{-1/2} \int_0^\infty f(t)\varphi_\lambda(t)\Delta(t)dt .$$

Let us define the following function spaces:

$\mathcal{D}_R$ = {even, C^∞ on $\mathbf{R}$, support contained in $[-R, R]$} for $0 < R < \infty$
$\mathcal{S}$ = {even, rapidly decreasing on $\mathbf{R}$}
$\mathcal{D}^r$ = $\{(\cosh t)^{-2\varrho/r} \cdot \mathcal{S}\}$ for $0 < r \leq 2$
$\mathcal{H}_R$ = {even, entire, rapidly decreasing of exponential type R that is: for all n $P_n(\Psi) = \sup_{\lambda\in\mathbf{C}} |(1+\lambda^n)e^{-R|\eta|}\Psi(\lambda)| < +\infty$}
For $0 < r \leq 2$ let $D_r = \{\xi + i\eta \in \mathbf{C} \mid |\eta| \leq (2r^{-1}-1)\varrho\}$
$\mathcal{H}^r$ = {even, holomorphic in the interior of D_r, C^∞ in D_r; rapidly decreasing, that is: for all m, n $P_{n,m}(\Psi) = \sup_{\lambda\in D_r} |(1+\lambda^n)d^m/d\lambda^m\Psi(\lambda)| < +\infty$}.

Now notice that $\mathcal{H}^2 = \mathcal{S}$ and that $\mathcal{D}^r \subset L^r(\Delta)$ and if $r \leq s$ then $\mathcal{D}^r \subseteq \mathcal{D}^s \subset L^2(\Delta)$.

Give $\mathcal{D}_R$ the topology of uniform convergence of all derivatives, $\mathcal{H}_R$ the topology defined by the semi-norms P_n, and $\mathcal{H}^r$ the topology defined by the semi-norms $P_{n,m}$. Then $\mathcal{S} = \mathcal{H}^2$ has the usual Schwartz-topology, and

$$\mathcal{D}^r = (\cosh t)^{-\frac{2}{r}\varrho} \cdot \mathcal{S}$$

shall be given the topology from $\mathcal{S}$. Clearly $\mathcal{D}_R$ and $\mathcal{D}^r$ are invariant under $\omega = \omega_{p,q}$, and we notice that the semi-norms

$$Q_n(f) = \sup_{t \in [0,R]} |\omega^n f(t)|$$

are continuous on $\mathcal{D}_R$; the seminorms

$$Q_{n,m}(f) = \sup_{t \in [0,\infty[} |(\cosh t)^{2\varrho/r}(1+t)^n \omega^m f(t)|$$

are continuous on $\mathcal{D}^r$; and that the topology on $\mathcal{H}^r$ can equivalently be defined by the semi-norms

$$P^0_{n,m}(\Psi) = \sup_{\lambda \in D_r} \left| \frac{d^n}{d\lambda^n} ((\lambda^2 + \varrho^2)^m \Psi(\lambda)) \right| .$$

Note that all the spaces are Frechet spaces.

Now let $\mathcal{D} = \bigcup_{R>0} \mathcal{D}_R$ and $\mathcal{H}_\infty = \bigcup_{R>0} \mathcal{H}_R$ both given the inductive limit topology. And define

${}_\infty\mathcal{H} = \{$entire; even; slowly increasing functions of exponential type, that is: there exist $N \in \mathbf{N}$, $R > 0$ such that: $\sup_{\lambda \in C} (1 + |\lambda|)^{-N} e^{-R|\eta|} |\Psi(\xi + i\eta)| < +\infty\}$.

Let $\mathcal{E}$ denote the set of all even C^∞functions on $\mathbf{R}$, given the topology of compact convergence of all derivatives.

The dual spaces $\mathcal{D}'$, $(\mathcal{D}^2)'$ and $\mathcal{E}'$ shall be called respectively the distributions, the tempered distributions and the distributions of compact support (with respect to the density Δ). A function f is identified with the distribution:

$$g \to \int_0^\infty g(x) f(x) \Delta(x) dx \quad g \in \mathcal{D} .$$

Similar definitions hold on the Fourier transform side for distributions with respect to the density $|c(\lambda)|^{-2}$.

THEOREM 4. *The Fourier transform* $f \to \tilde{f}$ *defines a linear, bijective map between:*

(i) $\mathcal{D}$, *the space of even* C^∞*-functions of compact support, and* $\mathcal{H}_\infty$, *the space of even, entire, rapidly decreasing functions of exponential type.*

(ii) $\mathcal{D}^r$ *and* $\mathcal{H}^r$ *for* $0 < r \leq 2$.

(iii) $\mathcal{E}'$ *and* ${}_\infty\mathcal{H}$.

In (i) *and* (ii) *the map is also bicontinuous.*

(i) and (iii) are the generalized Paley-Wiener theorem.

For $r = 2$, (ii) generalizes the theorem about the Fourier transform of Schwartz functions. For the dual spaces $(\mathcal{D}^2)'$ and $\mathcal{S}'$ we define the Fourier transform as the transpose of the inverse Fourier transform. And thus we have that the tempered distributions correspond under the Fourier transform. Note also that $\mathcal{D}^2 \subset (\mathcal{D}^2)'$, and that the definition of Fourier transform agree on the two spaces, due to the Parseval equation.

4. *Convolution structure*

It is natural to consider a convolution structure associated with this type of eigenfunction expansions. Roughly speaking, the convolution of two functions should be defined by means of the pointwise product on the Fourier transform side.

As T. Koornwinder pointed out to the author it is a simple consequence of the results in [10] and [19] that this convolution is defined by means of a positive kernel. In a joint paper with T. Koornwinder this convolution structure will be discussed further. Here we just state without proof the following easy result:

THEOREM 5.

(i) *Let* k, l, r *be positive numbers or* ∞, *such that* $1/k + 1/l - 1 = 1/r$, *then, for* $f \in L^k(\Delta)$ *and* $g \in L^l(\Delta)$, $f * g$ *is well defined as a function in* $L^r(\Delta)$ *and*

$$\|f * g\|_r \leq \|f\|_k \|g\|_l \, .$$

(ii) *When ever well defined the following holds*

$$\widetilde{f * g}(\lambda) = \tilde{f}(\lambda)\tilde{g}(\lambda) \, .$$

(iii) $(L^1(\Delta), *)$ *is a semi-simple Banach algebra.*

Using Theorem 5, we shall prove the following theorem, part of which is needed in the proof of Theorem 4 (iii).

THEOREM 6. *There exists an approximate identity* $\{v_\varepsilon\}_{\varepsilon>0}$ *in the convolution algebra* $\mathcal{D}$, *which also acts as approximate identity in:*

$$L^r(\Delta) \text{ for } 0 < r < \infty \quad \text{and} \quad \mathcal{D}^r \text{ for } 0 < r \leq 2 \, .$$

In the case of a symmetric space, this convolution is just the convolution on the corresponding semi-simple Lie group.

3. The proof of Proposition 1 and Theorem 2

From now on we always assume that $p + q > 0$. Making in (2.1) the change of variable $z = -(\sinh t)^2$ we get $z \in]-\infty, 0[$

$$z(z-1)\varphi''(z) + \{(a+b+1)z - c\}\varphi'(z) + ab\varphi(z) = 0 \tag{3.1}$$

$$a = a(\lambda) = 2^{-1}(\varrho + i\lambda), \quad b = b(\lambda) = 2^{-1}(\varrho - i\lambda), \quad c = 2^{-1}(p + q + 1)$$

this is the hypergeometric differential equation and it follows that

$$\varphi_\lambda(t) = F(a, b, c; z)$$

F being the hypergeometric function. We state a couple of well-known formulas for F, [7]: for $z \in]-\infty, 0]$ if $\operatorname{Re}(b) > 0$ and $\operatorname{Re}(c - b) > 0$:

$$F(a, b, c; z) = \Gamma(c)\Gamma(b)^{-1}\Gamma(c-b)^{-1} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}dt \tag{3.2}$$

$$\frac{d}{dz} F(a, b, c; z) = abc^{-1}F(a+1, b+1, c+1; z) \tag{3.3}$$

Note that if $p_1 = p$, $q_1 = q + 2$ then we get $a_1 = a + 1$, $b_1 = b + 1$, $c_1 = c + 1$ and $\varrho_1 = \varrho + 2$. Setting $G(p, q, \lambda; t) = F(a, b, c; z) = \varphi_\lambda(t)$ we have

$$\frac{d}{dt} G(p, q, \lambda; t) = -2^{-1}(p+q+1)^{-1}(\lambda^2 + (2^{-1}p + q)^2) \sinh 2t \, G(p, q+2, \lambda; t) \tag{3.4}$$

The functions φ_λ can be considered as a continuous orthogonal system. The discrete analogue of this system is given by Jacobi polynomials $P_n^{(\alpha,\beta)}$ with $\alpha = \frac{1}{2}(p + q - 1)$, $\beta = \frac{1}{2}(q - 1)$. We may call the functions Jacobi functions.

T. Koornwinder [19] has derived an integral representation for Jacobi polynomials which has an analogue for Jacobi functions. It takes in our notation the following form:

For all $t \in [0, \infty[$ *and* $\lambda \in \mathbf{C}$ *we have for*

$p > 0$, $q > 0$,

$$\varphi_\lambda(t) = \varphi_\lambda^{(p,q)}(t) = \tau(p, q) \int_0^1 \int_0^\pi \alpha(t, r, \theta)^\gamma d\mu(\theta) d\nu(r) \tag{3.5}$$

with $\quad \tau(p, q) = 2\pi^{-1/2}\Gamma(2^{-1}(p+q+1))(\Gamma(2^{-1}p)\Gamma(2^{-1}q))^{-1}$

$$\alpha(t, r, \theta) = 2^{-1}(\cosh 2t + 1 + (\cosh 2t - 1)r^2 + 2\sinh 2t \cdot r\cos\theta)$$

$$d\mu(\theta) = (\sin\theta)^{q-1}d\theta, \quad d\nu(r) = (1 - r^2)^{p/2-1} \cdot r^q dr$$

$$\gamma = -2^{-1}(2^{-1}p + q + i\lambda)$$

$p = 0,\ \ q \neq 0\,,$

$$\varphi_\lambda^{(0,q)}(t) = \tau(0, q) \int_0^\pi \alpha(t, 1, \theta)^\gamma d\mu(\theta) \tag{3.6}$$

with $$\tau(0, q) = \pi^{-1/2}\Gamma(2^{-1}(q+1))\Gamma(2^{-1}q)^{-1}$$

$p \neq 0,\ \ q = 0\,,$

$$\varphi_\lambda^{(p,0)}(t) = \varphi_{2\lambda}^{(0,p)}(2^{-1}t) = \tau(0, p) \int_0^\pi \alpha(2^{-1}t, 1, \theta)^{2\gamma} d\mu(\theta) \tag{3.7}$$

These formulas are explicit forms of Harish-Chandras formula for the spherical functions: $\varphi_\lambda(t) = \int_K \exp\,(i\lambda - \varrho)(tk)dk$ in the case of symmetric spaces, see Helgasons book [13], p. 432. Formula (3.6) is well known [7], formula 3.15 (22).

For convenience of the reader we give an elementary proof of formula (3.5) by using (3.6) and a fractional integral for hypergeometric functions. The idea of the proof in the similar case of Jacobi polynomials, see [19], is due to prof. R. Askey.

Proof. By analytic continuation with respect to complex y we find from formula (2.11) in Askey and Fitch [2] that for $y < 0,\ \mu > 0,\ c > 0$.

$$(-y)^{c+\mu-1}(1-y)^{a-c}F(a, b+\mu, c+\mu; y) =$$
$$= \Gamma(c+\mu)(\Gamma(c)\Gamma(\mu))^{-1} \int_y^0 (x-y)^{\mu-1}(1-x)^{a-c-\mu}(-x)^{c-1}F(a, b, c, x)dx\,.$$

Now using

$$\varphi_\lambda^{(p,q)}(t) = F(-\gamma, y + q + \tfrac{1}{2}p, \tfrac{1}{2}q + \tfrac{1}{2} + \tfrac{1}{2}p;\ -(\sinh t)^2)$$

we find that

$$(\sinh t)^{p+q-1}(\cosh t)^{-(2\gamma+q+1)}\varphi_\lambda^{(p,q)}(t) = \Gamma((\tfrac{1}{2}p + q + 1))(\Gamma(\tfrac{1}{2}(q+1))\Gamma(\tfrac{1}{2}p))^{-1}\cdot$$
$$\int_0^{(\sinh t)^2} ((\sinh t)^2 - (\sinh s)^2)^{\frac{1}{2}p+1}(\cosh s)^{-(2\gamma+p+q+1)}(\sinh s)^{q-1}\varphi_\lambda^{(0,q)}d(\sinh s)^2$$

using formula (3.6) for $\varphi_\lambda^{(0,q)}(s)$ and making the change of variables

$$r = \sinh s \cosh t\,(\sinh t \cosh s)^{-1}$$

formula (3.5) follows. Q.e.d.

For the study of $\Phi_\lambda(t)$ the easiest is to write formally:

$$\Phi_\lambda(t) = e^{(i\lambda-\varrho)t}\sum_{m=0}^\infty \Gamma_m(\lambda)e^{-mt}\,.$$

Inserting in the differential equation one finds the following recursion formula for $\Gamma_m(\lambda)$:

$$\Gamma_0 = 1,\ \ \Gamma_{2n-1} = 0 \ \text{ and } \ 4n(n - i\lambda)\Gamma_{2n} = \sum_{r=0}^{n-1} (2r - i\lambda + \varrho)(2p + \delta_r^n 4q)\Gamma_{2r}$$

where $\delta_r^n = 0$ for $r \equiv n + 1 \pmod 2$, $\delta_r^n = 1$ for $r \equiv n \pmod 2$.

LEMMA 7. *Let $D \subset \mathbf{C}$ be one of the following sets:*
(i) *D is compact, contained in Ω*
(ii) *$D = \{\xi + i\eta \mid \eta \geq -\varepsilon|\xi|\}$ for some $\varepsilon \geq 0$.*
There exist constants K and $d > 0$ such that

$$|\Gamma_m(\lambda)| \leq K(1 + m)^d \ \textit{for all} \ m \in \mathbf{Z}^+,\ \lambda \in D\,.$$

Proof (see Helgason [14]). For $\lambda \in \Omega$ and $r \in \mathbf{Z}^+$ define

$$c_r(\lambda) = 4r|r - i\lambda|; \qquad \gamma_r(\lambda) = 4\varrho|2r - i\lambda + \varrho|$$

and inductively:

$$b_0(\lambda) = 1, \qquad b_n(\lambda) = c_n(\lambda)^{-1}\sum_{r=0}^{n-1} b_r(\lambda)\gamma_r(\lambda)$$

then clearly $|\Gamma_{2n}(\lambda)| \leq b_n(\lambda)$ for all $n \in \mathbf{Z}^+$, $\lambda \in \Omega$.

$$b_n(\lambda)c_n(\lambda) = \sum_{r=0}^{n-2} b_r(\lambda)\gamma_r(\lambda) + b_{n-1}(\lambda)\gamma_{n-1}(\lambda) = b_{n-1}(\lambda)c_{n-1}(\lambda)(1 + \gamma_{n-1}(\lambda)c_{n-1}(\lambda)^{-1}) =$$
$$= \ldots = b_1(\lambda)c_1(\lambda)\prod_{r=1}^{n-1}(1 + \gamma_r(\lambda)c_r(\lambda)^{-1})\,.$$

We claim that for each set D there exists a $c > 0$ such that for all $r \in \mathbf{N}$, $\lambda \in D$:

$$\gamma_r(\lambda)c_r(\lambda)^{-1} \leq c \cdot r^{-1}\,. \tag{3.8}$$

Take

$$(r\gamma_r(\lambda)c_r(\lambda)^{-1})^2 = \varrho^2((2r + \varrho + \eta)^2 + \xi^2)((r + \eta)^2 + \xi^2)^{-1} = \alpha(r, \lambda)\,.$$

For $\eta \geq 0$ or $(2r + \varrho + \eta) \leq 0$ it is clear that

$$(2r + \varrho + \eta)^2 \leq k(r + \eta)^2 \ \text{ for some } \ k > 0$$

and so $\alpha(r, \lambda)$ is bounded. If $\eta < 0$ and $2r + \varrho + \eta > 0$ then

$$\alpha(r, \lambda) \leq K(((2 + \varrho)r + \eta)^2 + \xi^2)((r + \eta)^2 + \xi^2)^{-1} = \beta(r, \lambda)\,.$$

Now in case (i) $\beta(r, \lambda)$ is continuous on the compact space $\mathbf{N} \cup \{\infty\} \times D$, and thus bounded. In case (ii) $\beta(r, \lambda) = \beta(1, r^{-1}\lambda)$ and is bounded since $\beta(1, \lambda)$ is bounded on D. This proves (3.8). Now since $\log(1 + x) \leq x$ for $x \geq 0$, we conclude that for $n \in \mathbf{N}$:

$$\prod_{r=1}^{n-1} (1 + \gamma_r(\lambda)c_r(\lambda)^{-1}) \leq \exp (c \sum_{r=1}^{n-1} r^{-1}) \leq e^c \cdot n^c$$

also

$$b_1(\lambda)c_1(\lambda)c_n(\lambda)^{-1} = \varrho|\varrho - i\lambda||n(n - i\lambda)|^{-1}$$

is bounded for $n \in \mathbf{N}$ and $\lambda \in D$. This completes the proof of Lemma 7. Q.e.d.

From Lemma 7 it follows that the expansion for $\Phi_\lambda(t)$ converges uniformly on sets of the form $\{(t, \lambda) \in [c, \infty[\times D\}$, where c is a constant greater than zero. This shows, that for all $\lambda \in \Omega$, $\Phi_\lambda(t)$ is a solution of (2.1) satisfying

$$\Phi_\lambda(t) = (1 + o(1)) \exp (i\lambda - \varrho)t \quad \text{as} \quad t \to \infty .$$

Proposition 1, and Theorem 2 (ii) now follow easily.

LEMMA 8. *With notation from* (3.1), $c(\lambda)$ *is given by*

$$c(\lambda) = \frac{2^{2b}\Gamma(c)\Gamma(i\lambda)}{\Gamma(c - b)\Gamma(a)}$$

$\lambda c(-\lambda)$ *is holomorphic in* Ω, *and the zeros of* $\lambda c(-\lambda)$ *is contained in the set* $-i[\varepsilon, \infty[$ *for some* $\varepsilon > 0$.

Proof. The Wronski-determinant of φ_λ and Φ_λ is independent of t so we get:

$$\begin{aligned} W(\varphi_\lambda, \Phi_\lambda) &= \Delta(t)(\varphi_\lambda'(t)\Phi_\lambda(t) - \varphi_\lambda(t)\Phi_\lambda'(t)) = \\ &= \lim_{t \to \infty} \Delta(t)c(-\lambda)(\Phi_{-\lambda}'(t)\Phi_\lambda(t) - \Phi_{-\lambda}(t)\Phi_\lambda'(t)) = -2i\lambda c(-\lambda) \end{aligned} \tag{3.9}$$

this proves that $\lambda c(-\lambda)$ is holomorphic in Ω.

For $\eta > 0$ we get

$$\lim_{t \to \infty} e^{(i\lambda+\varrho)t}\varphi_\lambda(t) = \lim_{t \to \infty} (e^{(i\lambda+\varrho)t}c(\lambda)e^{(i\lambda-\varrho)t} + e^{(i\lambda+\varrho)t}c(-\lambda)e^{(-i\lambda-\varrho)t}) = c(-\lambda)$$

Now assuming $\eta > 0$, $\operatorname{Re}(b) > 0$ and $\operatorname{Re}(c - b) > 0$, that is $0 < \eta < p + \frac{1}{2}$, then by (3.2):

$$c(-\lambda) = \lim_{z \to -\infty} (-4z)^a \frac{\Gamma(c)}{\Gamma(b)\Gamma(c - b)} \int_0^1 t^{b-1}(1 - t)^{c-b-1}(1 - zt)^{-a}dt =$$

$$= \frac{2^{2a}\Gamma(c)}{\Gamma(b)\Gamma(c - b)} \int_0^1 t^{b-1}(1 - t)^{c-b-1}t^{-a}dt = \frac{2^{2a}\Gamma(c)\Gamma(-i\lambda)}{\Gamma(b)\Gamma(c - a)}$$

this formula holds for all $\lambda \in \Omega \setminus \{0\}$ by analytic continuation. The rest is now clear. Q.e.d.

COROLLARY 9. *Given* $\varepsilon \geq 0$ *there exists* $K > 0$ *such that*

$$|\lambda c(-\lambda)| \leq K(1 + |\lambda|)^{1-2^{-1}(p+q)}$$

$$|c(-\lambda)|^{-1} \leq K(1 + |\lambda|)^{2^{-1}(p+q)}$$

for all $\lambda = \xi + i\eta$ *with* $\eta \geq -\varepsilon|\xi|$.

Proof. Since $\lambda c(-\lambda)$ and $c(-\lambda)^{-1}$ are continuous for $\eta \geq -\varepsilon|\xi|$ we just have to consider the behaviour at ∞. Stirling's formula [22], p. 151, gives that for any $\alpha \in \mathbf{C}$ and any $\delta > 0$

$$\log \Gamma(\alpha + z) = (z + \alpha - 2^{-1}) \log z - z + 2^{-1} \log 2\pi + O(|z|^{-1})$$

uniformly in $\{z|\ |\arg(z)| \leq \pi - \delta\}$ as $|z| \to \infty$.

Corollary 9 follows by using this in the formula for $c(\lambda)$. Q.e.d.

This proves Theorem 2 (iii).

In the proof of Theorem 2 (i) we treat the case (3.5), the other cases are easier. The proof consists of a series of lemmas.

Let $\lambda = \xi + i\eta \in \mathbf{C}$, since $\varphi_\lambda(t) = \varphi_{-\lambda}(t)$ we assume for the most $\eta \geq 0$.

LEMMA 10. $\alpha^{(n)}(t, r, \theta) = \dfrac{d^n}{dt^n}\alpha(t, r, \theta)$ *satisfies*

(i) $\alpha^{(n)} = 4\alpha^{(n-2)}$ *for* $n \geq 3$

(ii) $2^n e^{-2t} \leq \alpha^{(n)} \leq 2^n e^{2t}$ *for* $n = 0, 1, 2, \ldots$

Proof. Obvious.

LEMMA 11. *For all* $\lambda \in \mathbf{C}$ *and* $t \in [0, \infty[$ *we have:*

(i) $|\varphi_\lambda(t)| \leq \varphi_{i\eta}(t)$

(ii) $|\eta| \geq \varrho \Rightarrow |\varphi_\lambda(t)| \leq e^{(|\eta| - \varrho)t}$

(iii) $|\eta| \leq \varrho \Rightarrow |\varphi_\lambda(t)| \leq 1$

Proof. (i) and (ii) follows from (1) by taking absolute values inside the integration, using $\operatorname{Re}(\gamma) = 2^{-1}(\eta - \varrho) \geq 0$. (iii) follows from (ii) using the maximum modulus principle on $\{\lambda|\ |\eta| \leq \varrho\}$. Q.e.d.

LEMMA 12. *There exists* $K > 0$ *such that for all* $t \in [0, \infty[$:

$$0 < \varphi_0(t) \leq K(1 + t)e^{-\varrho t}.$$

Proof. This follows from the discussion just before Proposition 1. That $(1 + t)$ cannot be avoided, can be seen from the fact that $\varphi_0(t)$ and $\Phi_0(t)$ are linearly independent, cf. (3.9). Q.e.d.

LEMMA 13. *There exists* $K > 0$ *such that for all* $\lambda \in \mathbf{C}$ *and* $t \in [0, \infty[$:

$$|\varphi_\lambda(t)| \leq K(1 + t)e^{(|\eta|-\varrho)t} .$$

Proof. For $\lambda \in \mathbf{R}$ it follows from Lemma 11 (i) and Lemma 12. Phragmén-Lindelöf's theorem [22], p. 177, applies for fixed t to the function $e^{i\lambda t}\varphi_\lambda(t)$ in the domain $\{\lambda | \eta \geq 0\}$ and gives the result. Q.e.d.

LEMMA 14. *There exists* $K > 0$ *such that for all* $\lambda \in \mathbf{C}$, $t \in [0, \infty[$ *and* $n \in \mathbf{Z}^+$:

$$\left|\frac{d^n}{d\lambda^n}\varphi_\lambda(t)\right| \leq t^n \varphi_{i\eta}(t) \leq K(1 + t)^{n+1}e^{(|\eta|-\varrho)t} .$$

Proof.

$$\frac{d^n}{d\lambda^n}\varphi_\lambda(t) = \tau(p, q)(-2^{-1}i)^n \int_0 \int_0^\pi \alpha^{-2^{-1}(\varrho+i\lambda)}(\log \alpha)^n d\mu(\theta) d\nu(r)$$

From Lemma 10 (ii) follows that $|\log \alpha(t, r, \theta)| \leq 2t$, thus the lemma follows by taking absolute values, and using Lemma 13. Q.e.d.

LEMMA 15. *For each* $n \in \mathbf{Z}^+$ *exists* $K_n > 0$ *such that for all* $\lambda \in \mathbf{C}$ *and* $t \in [0, \infty[$:

$$\left|\frac{d^n}{dt^n}\varphi_\lambda(t)\right| \leq K_n(1 + t)(1 + |\lambda|)^n e^{(|\eta|-\varrho)t} .$$

Proof. Using Lemma 10 (i) we find with certain constants $K_{s,e}$:

$$\frac{d^n}{dt^n}\varphi_\lambda(t) =$$

$$= \int_0^1 \int_0^\pi \sum_{s=0}^n \sum_{e=\max\{0,2s-n\}}^s K_{s,e}\gamma(\gamma - 1) \dots (\gamma - s + 1) \cdot \alpha^{\gamma-s}(\alpha')^e(\alpha'')^{s-e} d\mu(\theta) d\nu(r)$$

Taking absolute values we find using Lemma 10 (ii)

$$\left|\frac{d^n}{dt^n}\varphi_\lambda(t)\right| \leq K(1 + |\lambda|)^n e^{(|\eta|+\varrho)t} . \tag{3.10}$$

Thus we can apply Phragmén-Lindelöf's theorem for fixed t to the function $e^{i\lambda t}(i + \lambda)^{-n} d^n/dt^n \varphi_\lambda(t)$ in the domain $\{\lambda | \eta \geq 0\}$ and the lemma follows when we have proved the estimates for $\lambda \in \mathbf{R}$:

1. $|\lambda| \leq K$, $t \in [0, \infty[$ the proof is reduced to the case $n = 0$. Using (3.4): For certain constants $K_{\alpha,r}$ and $K^0_{\alpha,r}$ we get

$$\left|\frac{d^n}{dt^n}\varphi_\lambda(t)\right| \le \sum_{\alpha=0}^{n} \sum_{r=\max\{0,2\alpha-n\}}^{\alpha} K_{\alpha,r}(\cosh 2t)^{\alpha-r}(\sinh 2t)^r |G(p, q+2\alpha, \lambda, t)| \le$$
$$\le \sum_{\alpha,r} K^0_{\alpha,r} e^{2\alpha t}(1+t)e^{(|\eta|-(2^{-1}p+q+2\alpha))t} \le K(1+t)e^{(|\eta|-\varrho)t} \qquad \text{Q.e.d.}$$

2. $\lambda \in \mathbf{R}$, $t \in [0, K]$ it follows from (3.10).
3. $|\lambda| \ge K$ and $t \in [K, \infty[$ it follows from

$$\varphi_\lambda(t) = c(\lambda)\Phi_\lambda(t) + c(-\lambda)\Phi_{-\lambda}(t)$$

and theorem 2 (ii) and (iii). 1., 2. and 3. prove the lemma for $\lambda \in \mathbf{R}$. Q.e.d.

This finishes the proof of Theorem 2 (i), and thus Theorem 2 is proved.

4. The proof of Proposition 3, Theorem 4 and 6

Proposition 3 is contained in the general spectral theory of differential operators. Referring to Dunford and Schwartz [4], also for notation, we give a short outline of the calculations leading to proposition 3. Where [4] uses λ we shall use μ, and we take $\mu = (\lambda^2 + \varrho^2)$, and assume that $\operatorname{Im}\lambda \ge 0$.

Transforming the differential equation (2.1) by $\psi(t) = \Delta^{1/2}(t)\varphi(t)$ we get

$$\frac{d^2}{dt^2}\psi + \{(\lambda^2+\varrho^2) - h\}\psi = 0 .$$

Here h is the function

$$h(t) = 2^{-1}\frac{d}{dt}g(t) + 4^{-1}g(t)^2 .$$

Let τ be the differential operator

$$\tau = \tau_{p,q} = -\frac{d^2}{dt^2} + h ,$$

then the solutions of $\tau\psi = \mu\psi$ is given by $\psi = \Delta^{1/2}\varphi$, where $\omega_{p,q}\varphi = -(\lambda^2+\varrho^2)\varphi$.

From [4], XIII, 3, theorem 16, and (3.9) we find the Green's function related to τ, for $s < t$:

$$K(t, s, \mu) = -(2i\lambda c(-\lambda))^{-1}\Delta^{1/2}(t)\Phi_\lambda(t)\Delta^{1/2}(s)\varphi_\lambda(s) .$$

First we take [4], XIII, 5, theorem 18, corollary 20 with

$$\Lambda =]-\infty, \varrho^2[, \quad U = \{\operatorname{Re}\mu < \varrho^2\} \quad \text{and}$$
$$\sigma_1(t,\lambda) = \Delta^{1/2}(t)\varphi_\lambda(t), \quad \sigma_2(t,\lambda) = \Delta^{1/2}(t)\Phi_\lambda(t)$$

This gives $\Theta^+_{21} = (2i\lambda c(-\lambda))^{-1}$ and $\Theta^+_{ij} = 0$ for $(i,j) \ne (2,1)$. Now since $\{\Theta^+_{ij}\}$ is analytic in U and real in Λ then $\{\varrho_{ij}(\Lambda)\} = 0$.

Secondly we take $\Lambda =]\varrho^2 - \varepsilon, \infty[$, $U = \{\operatorname{Re}\mu > \varrho^2 - \varepsilon\}$ and

$$\sigma_1(t, \lambda) = \Delta^{1/2}(t)\varphi_\lambda(t)$$
$$\sigma_2(t, \lambda) = \Delta^{1/2}(t) i\lambda (c(\lambda)\Phi_\lambda(t) - c(-\lambda)\Phi_{-\lambda}(t)) ,$$

this gives

$$K(t, s, \mu) = -(2i\lambda c(-\lambda))^{-1}((2c(\lambda))^{-1}\sigma_1(t, \lambda) + (2i\lambda c(\lambda))^{-1}\sigma_2(t, \lambda))\sigma_1(t, \lambda)$$

and thus

$$\Theta_{11}^+ = -(4i\lambda c(\lambda)c(-\lambda))^{-1},\ \Theta_{21}^+ = (4\lambda^2 c(\lambda)c(-\lambda))^{-1} \text{ and } \Theta_{12}^+ = \Theta_{22}^+ = 0 .$$

Now since $\{\Theta_{ij}^+(\mu)\}$ is continuous in the intersection between U and the closed lower half plane, Θ_{21}^+ is real on Λ, and Θ_{11}^+ is real on $]\varrho^2 - \varepsilon, \varrho^2]$, we find

$$\varrho_{ij}(\Lambda) = 0 \text{ for } (i, j) \neq (1, 1) \text{ and } \varrho_{11}(]\varrho^2 - \varepsilon, \varrho^2]) = 0 ,$$

and for $\mu \geq \varrho^2$ or equivalently $\lambda \geq 0$

$$d\varrho_{11} = (4\pi\lambda c(\lambda)c(-\lambda))^{-1}d\mu = (4\pi|c(\lambda)|^2)^{-1}d\lambda .$$

Now Proposition 3 follows from [4], XIII, 5, theorems 13, 14 and 16.

We now turn to the proof of Theorem 4. This proof follows rather closely the ideas of Chap. I in Hörmander's book [17]. In the following we shall often use Theorem 2, and Proposition 3 without reference. Let K denote a suitable constant, which every time it occurs may have a new value.

1. We take $f \in \mathcal{D}_R$. $\tilde{f} = \int_0^R f(t)\varphi_\lambda(t)\Delta(t)dt$ is clearly entire and even, and $\widetilde{\omega^n f}(\lambda) = (-(\lambda^2 + \varrho^2))^n\tilde{f}(\lambda)$. We find

$$|(1 + \lambda)^{2n}\tilde{f}(\lambda)| \leq K \int_0^R |\omega^n f(t)\varphi_\lambda(t)\Delta(t)|dt \leq KQ_n(f)e^{R|\eta|} .$$

This shows that $\tilde{f} \in \mathcal{H}_R$ and that $f \to \tilde{f}$ is continuous.

2. We take $\Psi \in \mathcal{H}_R$. The inverse Fourier-transform is for $t > 0$ given by:

$$\check{\Psi}(t) = \int_0^\infty \Psi(\lambda)\varphi_\lambda(t)(c(\lambda)c(-\lambda))^{-1}d\lambda = \int_{-\infty}^\infty \Psi(\lambda)c(-\lambda)^{-1}\Phi_\lambda(t)d\lambda$$

by Cauchy's theorem, and the fact that the integrand is well behaved at ∞, we find

$$\check{\Psi}(t) = \int_{-\infty}^\infty \Psi(\xi + i\eta)c(-\xi - i\eta)^{-1}\Phi_{\xi+i\eta}(t)d\xi \quad \text{for } \eta \geq 0$$

thus

$$|\check{\Psi}(t)| \leq K e^{(-\eta-\varrho)t} \int_{-\infty}^{\infty} |\Psi(\xi+i\eta)|(1+|\xi+i\eta|)^{\frac{p+q}{2}} d\xi \leq K e^{\eta(R-t)}.$$

Since this holds for all $\eta \geq 0$, we find that $\check{\Psi}(t) = 0$ for $t > R$. In order to see that $\check{\Psi}(t)$ is C^∞, we consider for $t \in [0, \infty[$

$$\int_0^\infty \left| \Psi(\lambda) \frac{d^n}{dt^n} \varphi_\lambda(t) \right| |c(\lambda)|^{-2} d\lambda \leq K \int_0^\infty |\Psi(\lambda)(1+\lambda)^{n+p+q}| d\lambda < +\infty$$

this shows that $\check{\Psi}(t)$ is C^∞ and that

$$\frac{d^n}{dt^n} \check{\Psi}(t) = \int_0^\infty \Psi(\lambda) \frac{d^n}{dt^n} \varphi_\lambda(t) |c(\lambda)|^{-2} d\lambda.$$

Thus $\check{\Psi} \in \mathcal{D}_R$. Since $f \to \tilde{f}$ is continuous from $\mathcal{D}_R$ onto $\mathcal{H}_R$, and $\mathcal{D}_R$ and $\mathcal{H}_R$ are Frechet spaces the map is bicontinuous.

This finishes the proof of Theorem 4 (i).

3. We take $f \in \mathcal{D}^r$, $f(t) = (\cosh t)^{-\frac{2}{r}\varrho} g(t)$ with $g \in \mathcal{S}$. For $\lambda = i\eta \in D_r$, that is $|\eta| \leq (2r^{-1} - 1)\varrho$, we find

$$\int_0^\infty \left| g(t)(\cosh t)^{-\frac{2}{r}\varrho} \frac{d^n}{d\lambda^n} \varphi_\lambda(t) \Delta(t) \right| dt \leq K \int_0^\infty |g(t)(1+t)^{n+1}| dt < \infty.$$

This shows that $\tilde{f}(\lambda)$ is C^∞ in D_r and holomorphic in the interior, and that

$$\frac{d^n}{d\lambda^n} \tilde{f}(\lambda) = \int_0^\infty f(t) \frac{d^n}{d\lambda^n} \varphi_\lambda(t) \Delta(t) dt \quad \text{for all } \lambda \in D_r$$

and that

$$\left| \frac{d^n}{d\lambda^n} ((\lambda^2 + \varrho^2)^m \tilde{f}(\lambda)) \right| \leq K \int_0^\infty \left| \omega^m f(t) \frac{d^n}{d\lambda^n} \varphi_\lambda(t) \Delta(t) \right| dt \leq$$

$$\leq K \sup_{t \in [0,\infty[} |(\cosh t)^{\frac{2}{r}\varrho} \omega^m f(t)(1+t)^{n+3}| = K Q_{n+3,m}(f)$$

for all $\lambda \in D_r$. This shows that $f \in \mathcal{H}^r$ and that $f \to \tilde{f}$ is continuous.

4. Finally we take $\Psi \in \mathcal{H}^r$. For $t \in [0, \infty[$ consider:

$$\int_0^\infty \left| \Psi(\lambda) \frac{d^n}{dt^n} \varphi_\lambda(t) \right| |c(\lambda)|^{-2} d\lambda \leq K \int_0^\infty |\Psi(\lambda)|(1 + |\lambda|)^{n+p+q} d\lambda < + \infty$$

therefore $\check{\Psi}$ is C^∞. It is just left to find the behaviour of $\check{\Psi}(t)$ at ∞, so assume $t \geq c > 0$ and we get for $\eta = (2r^{-1} - 1)\varrho$

$$\begin{aligned} \check{\Psi}(t) &= \int_{-\infty}^{\infty} \Psi(\xi + i\eta) c(-\xi - i\eta)^{-1} e^{(i\xi - \eta - \varrho)t}(1 + e^{-2t}\Theta(\xi + i\eta, t)) d\xi = \\ &= e^{\frac{2}{r}\varrho t} \left(\int_{-\infty}^{\infty} \Psi(\xi + i\eta) c(-\xi - i\eta)^{-1} e^{i\xi t} d\xi + \right. \qquad (4.1) \\ &\left. + e^{-2t} \int_{-\infty}^{\infty} \Psi(\xi + i\eta) c(-\xi - i\eta)^{-1} \Theta(\xi + i\eta, t) e^{i\xi t} d\xi \right) = e^{\frac{2}{r}\varrho t} (g_1(t) + e^{-2t} g_2(t)) \end{aligned}$$

Now g_1 is rapidly decreasing since it is the usual Fourier-transform of a rapidly decreasing function; and since all derivatives of $\Theta(\xi + i\eta, t)$ with respect to t is bounded uniformly in ξ, it follows easily that $e^{-2t} g_2(t)$ is rapidly decreasing. It follows that $\check{\Psi} \in \mathcal{D}^r$. Since $f \to \tilde{f}$ is continuous from $\mathcal{D}^r$ onto $\mathcal{H}^r$, and $\mathcal{D}^r$ and $\mathcal{H}^r$ are Frechet spaces the map is bicontinuous.

This finishes the proof of Theorem 4 (ii).

Before the proof of Theorem 4 (iii), we shall prove Theorem 6.

LEMMA 16. *Let v be an even C^∞-function, positive and with support contained in $[-1, 1]$, such that*

$$\int_0^\infty v(t) \Delta(t) dt = 1 .$$

Define

$$v_\varepsilon(t) = \varepsilon^{-1} \Delta(t)^{-1} \Delta(\varepsilon^{-1} t) v(\varepsilon^{-1} t) ,$$

then, for $\varepsilon > 0$, v_ε satisfies the same conditions as v, except that supp $v_\varepsilon \subseteq [-\varepsilon, \varepsilon]$.

There exists constants K and K_n such that for all $\lambda = \xi + i\eta \in \mathbf{C}$

(i) $|\tilde{v}_\varepsilon(\lambda) - 1| \leq \varepsilon \cdot K(1 + |\lambda|) e^{|\eta|}$

(ii) $|d^n/d\lambda^n \tilde{v}_\varepsilon(\lambda)| \leq \varepsilon^n e^{\varepsilon|\eta|} K_n$.

Proof.

(i) $$|\tilde{v}_\varepsilon(\lambda) - 1| = \left|\int_0^\infty v(t)(\varphi_\lambda(\varepsilon t) - 1)\Delta(t)dt\right| \leq \int_0^1 v(t)\varepsilon t|\varphi'_\lambda(\Theta)|\Delta(t)dt \leq$$

$$\leq \varepsilon \cdot K \int_0^1 v(t)t(1 + \Theta)(1 + |\lambda|)e^{(|\eta|-\varrho)t} \leq \varepsilon \cdot K(1 + |\lambda|)e^{|\eta|},$$

where $\Theta = \Theta(\varepsilon, t) \in [0, \varepsilon t]$,

(ii) $$\left|\frac{d^n}{d\lambda^n}\tilde{v}_\varepsilon(\lambda)\right| = \left|\int_0^\infty v_\varepsilon(t)\frac{d^n}{d\lambda^n}\varphi_\lambda(t)\Delta(t)dt\right| \leq \int_0^\infty v_\varepsilon(t)t^n\varphi_{i\eta}(t)\Delta(t)dt \leq$$

$$\leq \int_0^1 v(t)(\varepsilon t)^n\varphi_{i\eta}(\varepsilon t)\Delta(t)dt \leq \varepsilon^n e^{\varepsilon|\eta|}K. \qquad \text{Q.e.d.}$$

Proof of Theorem 6. First consider $f \in \mathscr{D}$. For all $t \in [0, \infty[$ we get

$$|v_\varepsilon * f(t) - f(t)| = \left|\int_0^\infty (\tilde{v}_\varepsilon(\lambda) - 1)\tilde{f}(\lambda)\varphi_\lambda(t)|c(\lambda)|^{-2}d\lambda\right| \leq \varepsilon \cdot K.$$

Since the support of $v_\varepsilon * f - f$ for $\varepsilon \leq 1$ is contained in a fixed compact set, we get for $0 < r \leq \infty$:

$$\|v_\varepsilon * f - f\|_r \to 0 \quad \text{for} \quad \varepsilon \to 0.$$

Using Theorem 5 (i) and the fact that $\mathscr{D}$ is dense in $L^r(\Delta)$ for all $0 < r < \infty$, it follows easily that v_ε is an approximate identity in $L^r(\Delta)$. For $f \in \mathscr{D}$ we find, as above, for $n \in \mathbf{N}$:

$$\left|\frac{d^n}{dt^n}(v_\varepsilon * f - f)(t)\right| \leq \varepsilon \cdot K,$$

this shows that v_ε is an approximate identity in $\mathscr{D}$.

Now take $f \in \mathscr{S}^r$ for $0 < r \leq 2$. Let $\eta = (2r^{-1} - 1)\varrho$, then as in (4.1) we get

$$v_\varepsilon * f(t) - f(t) = e^{-\frac{2}{r}\varrho t}(g_1(t, \varepsilon) + g_2(t, \varepsilon))$$

for $t \geq c > 0$ where

$$g_1(t, \varepsilon) = \int_{-\infty}^\infty \tilde{f}(\xi + i\eta)(\tilde{v}_\varepsilon(\xi + i\eta) - 1)c(-\xi - i\eta)^{-1}e^{i\xi t}d\xi$$

and

$$g_2(t, \varepsilon) = e^{-2t} \int_{-\infty}^{\infty} \tilde{f}(\xi + i\eta)(\tilde{v}_\varepsilon(\xi + i\eta) - 1)c(-\xi - i\eta)^{-1}\Theta(\xi + i\eta, t)e^{i\xi t}d\xi$$

In order to prove that v_ε is an approximate identity in $\mathcal{I}^r$, it is enough to show that for any given integers n and m, we can find a constant $K > 0$ such that $|t^n(d^m/dt^m g_i(t, \varepsilon))| \leq \varepsilon \cdot K$ for $i = 1, 2$. g_2 is easily taken case of using the factor e^{-2t} and Theorem 2 (ii).

$$t^n \frac{d^m}{dt^m} g_1(t, \varepsilon) = \int_{-\infty}^{\infty} h(\xi)(\tilde{v}_\varepsilon(\xi + i\eta) - 1) \frac{d^n}{d\xi^n} (e^{i\xi t})d\xi ,$$

where $h(\xi) = (i\xi)^m(-i)^n\tilde{f}(\xi + i\eta)c(-\xi - i\eta)^{-1}$. Now doing partial integration n times, and then using Lemma 16 (i) and (ii) the result follows. Q.e.d.

LEMMA 17.
(i) *The following inclusions hold*

$$\mathcal{E}' \subseteq (\mathcal{I}^2)' \subseteq \mathcal{D}' .$$

(ii) *With the natural definition of support, we have for all* $u \in \mathcal{D}'$: $u \in \mathcal{E}'$ *if and only if the support of* u *is compact.*

The proof is straight forward, similar to [17], theorems 1.5.1 and 1.5.2.

For $u \in \mathcal{E}'$ the Fourier transform is defined since $\mu \in (\mathcal{I}^2)'$. It is easily seen to be the function $\tilde{u}(\lambda) = u(\varphi_\lambda)$.

PROPOSITION 18. *Given* $R > 0$, *the Fourier transform is a bijective map between the space of distributions* $\{u \in \mathcal{E}' \mid \operatorname{supp} u \subseteq [0, R]\}$ *and the space of even, entire functions* Ψ, *which for some* $N \in \mathbf{N}$ *satisfy an inequality*

$$|\Psi(\lambda)| \leq K(1 + |\lambda|)^N e^{|\eta|R} \quad \text{for all} \quad \lambda = \xi + i\eta \in \mathbf{C} .$$

Proof. Take $u \in \mathcal{E}'$, $\operatorname{supp} u \subset [0, R]$. Let $\Psi \in C(\mathbf{R})$ be such that $|\Psi| \leq 1$, $\Psi(x) = 1$ for $x \leq \frac{1}{2}$ and $\Psi(x) = 0$ for $x \geq 1$. Define, for $\lambda \neq 0$,

$$\Psi_\lambda(x) = \varphi_\lambda(x)\Psi(|\lambda|(|x| - R)) .$$

Then obviously Ψ_λ is even and C^∞, and

$$\begin{aligned} \Psi_\lambda(x) &= \varphi_\lambda(x) \quad \text{for} \quad x \in [0, R + (2|\lambda|)^{-1}] \\ \Psi_\lambda(x) &= 0 \qquad \text{for} \quad |x| \geq R + |\lambda|^{-1} \end{aligned}$$

It follows that for $x \in \operatorname{supp} \Psi_\lambda$ we have inequalities

$$\left|\frac{d^n}{dt^n}\varphi_\lambda(t)\right| \leq K_n(1+|\lambda|)^n e^{|\eta|R}$$

and thus inequalities

$$\left|\frac{d^n}{dt^n}\Psi_\lambda(t)\right| \leq K_n(1+|\lambda|)^n e^{|\eta|R}.$$

Since u is continuous on $\mathscr{E}$ there is an inequality

$$|u(f)| \leq K \sum_{n\leq N} \sup\left|\frac{d^n}{dt^n} f(t)\right| \quad \text{for } f\in\mathscr{D}.$$

Now it follows that

$$|\tilde{u}(\lambda)| = |u(\varphi_\lambda)| = |u(\Psi_\lambda)| \leq K(1+|\lambda|)^N e^{|\eta|R}.$$

Let U be even, entire and satisfy

$$|U(\lambda)| \leq K(1+|\lambda|)^N e^{|\eta|R} \quad \text{for all } \lambda\in\mathbf{C}.$$

Obviously $U\in\mathcal{S}' = (\mathcal{H}^2)'$ and is thus the Fourier transform of a tempered distribution $u\in(\mathcal{G}^2)'$.

Now take v_ε as in Theorem 6 and define u_ε by $\tilde{u}_\varepsilon = u * v_\varepsilon = \tilde{u}\cdot\tilde{v}_\varepsilon = U\cdot\tilde{v}_\varepsilon$. This shows that $\operatorname{supp} u_\varepsilon \subseteq [0, R+\varepsilon]$.

Now pick arbitrary α and β such that $R<\alpha<\beta$, and $f\in\mathscr{D}$ with $\operatorname{supp} f\subseteq[\alpha,\beta]$. Then, for sufficiently small ε, $u_\varepsilon(f)=0$.

This gives

$$0 = u_\varepsilon(f) = \tilde{u}_\varepsilon(\tilde{f}) = \int_0^\infty \tilde{u}(\lambda)\tilde{v}_\varepsilon(\lambda)\tilde{f}(\lambda)|c(\lambda)|^{-2}d\lambda = \tilde{u}(\tilde{v}_\varepsilon\tilde{f}) = u(v_\varepsilon * f).$$

But $v_\varepsilon * f \to f$ in $\mathcal{G}^2$ as $\varepsilon\to 0$ and thus $u(f)=0$. Therefore $\operatorname{supp} u\subseteq[0,R]$.
Q.e.d.

This finishes the proof of Theorem 4.

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Received July 23, 1971

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