Oscillatory integrals and multipliers on FL^p

LARS HÖRMANDER
University of Lund, Sweden

0. Introduction

It has been known since a long time that the function in $\mathbf{R}^d$

$$m_\alpha(\xi) = (1 - |\xi|^2)^\alpha,\ |\xi| < 1;\ m_\alpha(\xi) = 0,\ |\xi| \geqq 1,$$

where α is a real number, is not a multiplier on $FL^p(\mathbf{R}^d)$ unless

$$\alpha \geqq 0, \alpha > d|1/p - 1/2| - 1/2. \tag{0.1}$$

In fact, this follows from the asymptotic expansion of the Fourier transform $\widehat{m_\alpha}$. When $d = 1$ the sufficiency of (0.1) follows from the estimates of M. Riesz for conjugate functions so we assume $d > 1$ in what follows. Stein [7] showed that

$$\alpha > (d - 1)|1/p - 1/2| \tag{0.2}$$

is always a sufficient condition but it is evidently stronger than (0.1) except when $p = 1$ or $p = \infty$. More recently Fefferman has proved in [2] that (0.1) is a sufficient condition when $|1/p - 1/2| > (d + 1)/4d$, and in [3] he proved that

$$\alpha > \max\,(0, d|1/p - 1/2| - 1/2) \tag{0.3}$$

is a necessary condition if $p \neq 2$. When $d = 2$ Carleson and Sjölin [1] have proved completely that (0.3) is a sufficient condition for m_α to be a multiplier on $FL^p(\mathbf{R}^d)$. The main point in their proof is an L^p estimate for oscillatory integrals which is very interesting in its own right. In a special case they developed an idea of Stein and Fefferman (see [2]) to show that it follows from the Hausdorff-Young inequality but in the general case they used a much more complicated argument. In this note we shall simplify their proof by applying an extension of the Hausdorff-Young inequality also in the general case. This gives somewhat more precise estimates also.

The function m_α in $\mathbf{R}^2$ can be replaced by any function of compact support which is smooth except near a curve with non-zero curvature where it is the distance to the curve raised to the power α. In a recent manuscript Sjölin [6] has extended the result to curves with tangents of higher but always finite order. We simplify his proof here.

In a final section we indicate some of the open problems on L^p estimates for oscillatory integrals in any number of variables. These seem to be of interest not only in the study of multiplier problems but in other contexts as well.

1. L^p estimates for oscillatory integrals

We begin with an extension of the Hausdorff-Young inequality.

THEOREM 1.1. *Let* $a \in C_0^\infty(\mathbf{R}^{2d})$, *let* $\varphi \in C^\infty(\mathbf{R}^{2d})$ *be real valued and set with* $N > 1$

$$T_N f(x) = \int e^{iN\varphi(x,y)} a(x, y) f(y) dy, \quad f \in C_0^\infty(\mathbf{R}^d). \tag{1.1}$$

If $\det \partial^2\varphi/\partial x \partial y \neq 0$ *in* $\operatorname{supp} a$ *and* $1 \leqq p \leqq 2$, $1/p + 1/p' = 1$ *then*

$$\|T_N f\|_{p'} \leqq C N^{-d/p'} \|f\|_p, \quad f \in C_0^\infty(\mathbf{R}^d). \tag{1.2}$$

That this is an extension of the Hausdorff-Young inequality is seen by taking $\varphi(x, y) = \langle x, y\rangle$ and a with $a(0, 0) = 1$. If $f(y)$ is replaced by $f(y\sqrt{N})$ the Hausdorff-Young inequality is the limit of (1.2) when $N \to \infty$.

Proof of Theorem 1.1. The statement is obvious when $p = 1$ so in view of M. Riesz' convexity theorem it suffices to prove it when $p = 2$. In the proof we may assume that f has small support. We have to estimate

$$\|T_N f\|^2 = \iint a_N(y, z) f(y) \overline{f(z)} dy dz$$

where

$$a_N(y, z) = \int e^{iN(\varphi(x,y) - \varphi(x,z))} a(x, y) \overline{a(x, z)} dx.$$

When y and z are close to a given point and $(x, y) \in \operatorname{supp} a$ we have

$$|\partial/\partial x(\varphi(x, y) - \varphi(x, z))| = |\varphi''_{xy}(y - z)| + O(|y - z|^2) \geqq c|y - z|$$

so k partial integrations give if k is any positive integer

$$|a_N(y, z)| \leqq C_k(1 + N|y - z|)^{-k}.$$

If $k = d + 1$ it follows that $\int |a_N(y, z)|dy < CN^{-d}$, $\int |a_N(y, z)|dz < CN^{-d}$. Hence $\|T_N f\|_2^2 \leqq CN^{-d}\|f\|_2^2$ and the theorem is proved.

Remark. Using arguments close to those in Hörmander [4, sections 2.2 and 4.3] it is easy to show that

$$N^{d/2}\|T_N\|_2 \to \sup |a(x, y)|\ |\det (\partial^2\varphi/\partial x\partial y/2\pi)|^{-1/2}$$

provided that $y \to \partial\varphi(x, y)/\partial x$ is injective for fixed x and $(x, y) \in \operatorname{supp} a$.

If the matrix $\partial^2\varphi/\partial x\partial y$ in Theorem 1.1 is allowed to be singular it is much harder to analyse the possible L^p estimates for T_N. The simplest situation occurs when φ is independent of y_d, thus a function of $2d - 1$ variables only. When $d = 2$ we shall prove a slightly improved version of the key estimate of Carleson and Sjölin [1]:

THEOREM 1.2. *Let* $a \in C_0^\infty(\mathbf{R}^3)$, *let* $\varphi \in C^\infty(\mathbf{R}^3)$ *be real valued, and assume that the Jacobian* $D(\partial\varphi/\partial y, \partial^2\varphi/\partial y^2)/Dx$ *has no zero in* $\operatorname{supp} a$. *(Here the variables in* $\mathbf{R}^3$ *have been denoted by* (x, y); $x = (x_1, x_2)$.) *Set*

$$T_N f(x) = \int e^{iN\varphi(x,y)}a(x, y)f(y)dy,\quad f \in C_0^\infty(\mathbf{R}),\quad x \in \mathbf{R}^2. \tag{1.3}$$

Then it follows that

$$\|T_N f\|_q \leqq CN^{-2/q}(q/(q-4))^{1/4}\|f\|_r \quad \text{if } q > 4 \text{ and } 3/q + 1/r = 1. \tag{1.4}$$

Proof. To be able to apply Theorem 1.1 we introduce

$$F_N(x) = (T_N f(x))^2 = \int\int e^{iN(\varphi(x,t)+\varphi(x,s))}a(x, t)a(x, s)f(t)f(s)dsdt.$$

However the hypotheses of Theorem 1.1 are not fulfilled since

$$\det \partial^2(\varphi(x, t) + \varphi(x, s))/\partial x\partial(t, s)) = \begin{vmatrix} \varphi''_{tx_1}(x, t)\varphi''_{tx_2}(x, t) \\ \varphi''_{sx_1}(x, s)\varphi''_{sx_2}(x, s) \end{vmatrix},$$

which vanishes when $t = s$. For t close to s the determinant is equal to $(s - t)D(\varphi'_t, \varphi''_{tt})/D(x_1, x_2) + O((t - s)^2)$ so it is bounded from below by $c|t - s|$ in the support of $a(x, t)a(x, s)$ if a has sufficiently small support. Since $\varphi(x, t) + \varphi(x, s)$ is a symmetric function of t, s it is a C^∞ function Φ of x and $y = (t + s, ts)$. Similarly $2a(x, t)a(x, s)$ is the restriction to $\Omega = \{y;\ 4y_2 \leqq y_1^2\}$ of a C_0^∞ function $b(x, y)$. Since $D(y)/D(t, s) = t - s$, it follows that Φ satisfies the hypotheses of Theorem 1.1, and

$$F_N(x) = \int_\Omega e^{iN\Phi(x,y)}b(x, y)f(t)f(s)|t - s|^{-1}dy.$$

If $1 \leqq p \leqq 2$ it follows from Theorem 1.1 that

$$\|T_N f\|_{2p'}^2 = \|F_N\|_{p'} \leqq CN^{-2/p'} \left(\int\limits_{\Omega} |f(t)f(s)|^p |t-s|^{-p} dy \right)^{1/p}$$

$$= CN^{-2/p'} \left(\frac{1}{2} \int\int |f(t)|^p |f(s)|^p |t-s|^{1-p} ds dt \right)^{1/p}.$$

To estimate the right hand side we use the classical inequality for fractionary integrals

$$\int\int |g(t)g(s)| |s-t|^{\delta-1} ds dt \leqq C\delta^{-1} \|g\|_{\varrho}^2, \quad 1 < 2/\varrho = \delta + 1 \leqq 2.$$

Taking $\delta - 1 = 1 - p$, that is, $\delta = 2 - p$ we obtain

$$\left(\int\int |f(t)f(s)|^p |s-t|^{1-p} ds dt \right)^{1/p} \leqq C(2-p)^{-1/p} \|f\|_{p\varrho}^2, \quad 1 < 2/\varrho = 3 - p.$$

Hence

$$\|T_N f\|_{2p'} \leqq CN^{-1/p'} (2-p)^{-1/2p} \|f\|_{2p/(3-p)}, \quad 1 \leqq p < 2.$$

Here we write $2p' = q$ and $2p/(3-p) = r$. Since $1/r = 3/2p - 1/2$ and $3/q = 3/2p'$ we obtain $3/q + 1/r = 1$ and $q > 4$ which are the only restrictions on q and on r. The estimate (1.4) now follows immediately.

COROLLARY 1.3. *Let I be an open interval on $\mathbf{R}$, let $I \ni y \to \Phi(y)$ be a C^∞ immersion of I as a curve with curvature $\neq 0$, and set with $a \in C_0^\infty(I)$*

$$Sf(x) = \int e^{i\langle x, \Phi(y)\rangle} a(y) f(y) dy, \quad f \in C_0^\infty(\mathbf{R}), \quad x \in \mathbf{R}^2. \tag{1.5}$$

Then it follows that

$$\|Sf\|_q \leqq C(q/(q-4))^{1/4} \|f\|_r \quad \text{if} \quad f \in C_0^\infty(\mathbf{R}), \quad q > 4, \quad 3/q + 1/r = 1. \tag{1.6}$$

Moreover, if $\hat{g}$ is the Fourier transform of g

$$\|a(\hat{g} \circ \Phi)\|_r \leqq C(4-3q)^{-1/4} \|g\|_q \quad \text{if} \quad g \in C_0^\infty(\mathbf{R}^2), \quad 1 \leqq q < \frac{4}{3}, \ \frac{3}{q} + \frac{1}{r} = 3. \tag{1.7}$$

Proof. Since Φ' and Φ'' are assumed to be linearly independent the function $\varphi(x, y) = \langle x, \Phi(y)\rangle$ satisfies the hypothesis of Theorem 1.2 in $\mathbf{R}^2 \times I$. Choose $b \in C_0^\infty(\mathbf{R}^2)$ with $b(0) = 1$ and apply Theorem 1.2 with a replaced by $b(x)a(y)$. This gives the desired bound for

$$N^{2/q}\left(\int |b(x)Sf(Nx)|^q dx\right)^{1/q} = \left(\int |b(x/N)Sf(x)|^q dx\right)^{1/q}.$$

When $N \to \infty$ the estimate (1.6) follows. By duality we obtain (1.7).

The following is a simple combination of Theorems 1.1 and 1.2:

THEOREM 1.4. *Let* $a \in C_0^\infty(\mathbf{R}^4)$, $\varphi \in C^\infty(\mathbf{R}^4)$ *and assume that in* supp a *we have* $\partial^2\varphi/\partial y\partial x \neq 0$ *and*

$$t \in \mathbf{R}^2,\ \partial/\partial y\langle t, \partial\varphi/\partial x\rangle = \partial^2/\partial y^2\langle t, \partial\varphi/\partial x\rangle = 0 \Rightarrow t = 0. \tag{1.8}$$

(Here the variables in $\mathbf{R}^4$ *have been denoted by* (x, y); $x, y \in \mathbf{R}^2$.*) If*

$$T_N f(x) = \int e^{iN\varphi(x,y)}a(x, y)f(y)dy, f \in C_0^\infty(\mathbf{R}^2), x \in \mathbf{R}^2,$$

it follows that (1.4) *is valid.*

Proof. The statement is weaker than Theorem 1.1 if the support of a is close to a point where $\det \partial^2\varphi/\partial y\partial x \neq 0$. Let us therefore assume that the support of a is close to a point say $x = y = 0$ where $\det \partial^2\varphi/\partial y\partial x = 0$. After a linear change of the variables x and y we may assume that at $(0, 0)$

$$\partial^2\varphi/\partial y_j\partial x_1 = 0,\ \ j = 1, 2;\ \ \partial^2\varphi/\partial y_1\partial x_2 \neq 0,\ \ \partial^3\varphi/\partial y_1^2\partial x_1 \neq 0.$$

It follows that the function $(x, y_1) \to \varphi(x, y_1, y_2)$ satisfies the hypothesis of Theorem 1.2 in a neighbourhood of 0. Writing

$$S_N f(x, y_2) = \int e^{iN\varphi(x,y)}a(x, y)f(y)dy_1$$

we have $T_N f(x) = \int S_N f(x, y_2)dy_2$ and Theorem 1.2 gives

$$\|T_N f\|_q \leqq CN^{-2/q}(q/(q-4))^{1/4}\int dy_2\left(\int |f(y_1, y_2)|^r dy_1\right)^{1/r}.$$

We can assume that the support of f is in a fixed compact set and the double integral can then be estimated by $\|f\|_r$ in view of Hölder's inequality. The proof is complete.

Example 1.5. Let $\varphi(x, y) = \Phi(x - y)$ where $\Phi \in C^\infty(\mathbf{R}^2\setminus\{0\})$ is positively homogeneous of degree 1, and let $a \in C_0^\infty(\mathbf{R}^2 \times \mathbf{R}^2)$ vanish near the diagonal. Then the hypotheses of Theorem 1.4 are fulfilled if $\Phi''(z) \neq 0$ when $z \neq 0$. In fact, the equation $\Phi''(z)t = 0$ is fulfilled by $t = z$ since Φ' is homogeneous of degree 0, and $\Phi'''(z)z = -\Phi''(z) \neq 0$.

2. Some convolution operators and multipliers

The preceding example leads to the following theorem of Carleson and Sjölin [1].

THEOREM 2.1. *Let* $\Phi \in C^\infty(\mathbf{R}^2 \diagdown \{0\})$ *be real valued, positively homogeneous of degree* 1, *and assume that* $\Phi''(x) \neq 0$ *for every* $x \neq 0$. *If*

$$Kf(x) = \int e^{i\Phi(x-y)} a(x-y) f(y) dy, \quad f \in C_0^\infty(\mathbf{R}^2), \tag{2.1}$$

where $a \in C^\infty(\mathbf{R}^2)$ *and* $a(tz) = t^{-\lambda} a(z)$ *when* $|z| > 1$ *and* $t > 1$, *it follows that* K *is continuous from* L^p *to* L^p *if*

$$\lambda > \max\,(3/2,\, 2|1/p - 1/2| + 1). \tag{2.2}$$

Proof. By passage to the adjoint we can reduce the proof to the case $p \geqq 2$. Choose $\chi \in C_0^\infty(\mathbf{R}^4)$ so that $x \neq y$ if $(x, y) \in \operatorname{supp} \chi$, and set for $t \geqq 1$

$$S_t f(x) = \int e^{it\Phi(x-y)} \chi(x, y) f(y) dy, \quad f \in C_0^\infty(\mathbf{R}^2).$$

Then we have

$$\|S_t f\|_p \leqq C_p(t) \|f\|_p \tag{2.3}$$

where

$$C_p(t) = Ct^{-2/p}(p/(p-4))^{1/4}, p > 4; \quad C_p(t) = Ct^{-1/2}(\log t)^{1/2-1/p}, 2 \leqq p \leqq 4. \tag{2.4}$$

In fact, we may assume that $\operatorname{supp} f$ belongs to a fixed compact set, and for $p > 4$ the assertion is then a consequence of Theorem 1.4 as seen in Example 1.5. When $p = 2$ it follows from Theorem 1.1 applied to a suitable variable as in the proof of Theorem 1.4. Interpolation by the M. Riesz convexity theorem between $p = 2$ and $p = 4 + 1/\log t$ gives the estimate for $p = 4$ and another application of Riesz' theorem proves it for p between 2 and 4.

If ψ is a function such that $\chi(x, y) \neq 0$ implies $\psi(y) = 1$, then

$$\|S_{t,z} f\|_p \leqq C_p(t) \|\psi(. - z) f\|_p, f \in C_0^\infty(\mathbf{R}^2), z \in \mathbf{R}^2, \tag{2.3$'$}$$

where

$$S_{t,z} f(x) = \int e^{it\Phi(x-y)} \chi(x - z, y - z) f(y) dy.$$

It is obvious that

$$\int \chi(x - z, y - z) dz = F(x - y)$$

where $F \in C_0^\infty$ vanishes near 0 and is $\geqq 0$ if χ is. By suitable choice of χ we can obtain any such F, for multiplication of χ by a function $g(x - y)$ leads to the function gF instead of F. Since $S_{t,z}f(x)$ can only be different from 0 for $|x - z| < C$, we have by Hölder's inequality

$$\left| \int S_{t,z}f(x)dz \right|^p \leqq C_1 \int |S_{t,z}f(x)|^p dz.$$

Integration with respect to x gives in view of (2.3)'

$$\|R_t f\|_p \leqq CC_p(t)\|f\|_p, f \in C_0^\infty(\mathbf{R}^2), \tag{2.4}$$

where we have written

$$R_t f(x) = \int S_{t,z}f(x)dz = \int e^{it\Phi(x-y)}F(x-y)f(y)dy.$$

After a change of variables (2.4) takes the form

$$\left\| \int e^{i\Phi(.-y)}F((.-y)/t)f(y)dy \right\|_p \leqq Ct^2C_p(t)\|f\|_p, f \in C_0^\infty.$$

We multiply by $t^{-1-\lambda}$ and integrate from 1 to ∞ with respect to t noting that $\int_1^\infty t^{1-\lambda}C_p(t)dt < \infty$ because $1 - \lambda - 2/p < -1$ and $1 - \lambda - 1/2 < -1$ by (2.2). (Recall that $p \geqq 2$.) If

$$a(x) = \int_1^\infty F(x/t)t^{-1-\lambda}dt \tag{2.5}$$

it follows that K is continuous in L^p. Now a is homogeneous of degree $-\lambda$ for $|x| > R$ if $F(x) = 0$ for $|x| > R$, and every such function can be written as the sum of one of the form (2.5) and one of compact support. This completes the proof.

We shall now consider some multipliers on FL^p. For the relevant facts on multipliers we refer to Hörmander [5, Chapter 1]. We shall denote by M_p the space of multipliers on FL^p.

THEOREM 2.2. *Let I be an interval on* $\mathbf{R}$, *let $\psi \in C^\infty(I)$ be real valued and assume that $\psi'' \neq 0$ on I. If $a \in C_0^\infty(I \times \mathbf{R})$ it follows that*

$$m_\alpha(\xi) = a(\xi)(\xi_2 - \psi(\xi_1))_+^\alpha$$

is in M_p if

$$\alpha > \max(0, 2|1/p - 1/2| - 1/2). \tag{2.6}$$

Here we have used the notation $r_+ = \max(r, 0)$; $r \in \mathbf{R}$.

Proof. Since M_p is a C_0^∞ module we may assume that

$$a(\xi) = a_1(\xi_1)a_2(\xi_2 - \psi(\xi_1)), a_j \in C_0^\infty.$$

m_α is then the Fourier transform of $A(x_2)I(x_1, x_2)$ where $\hat{A}(\xi_2) = a_2(\xi_2)\xi_{2+}^\alpha$ and

$$I(x_1, x_2) = \int e^{2\pi i(\xi_1 x_1 + \psi(\xi_1)x_2)} a_1(\xi_1) d\xi_1.$$

It is a well known consequence of the stationary phase method that the function $I(x)$ is rapidly decreasing except in directions such that $x_1 + \psi'(\xi_1)x_2 = 0$ for some $\xi_1 \in \operatorname{supp} a_1$ which defines ξ_1 as a homogeneous function of x of degree 0. If $\Phi(x_1, x_2) = 2\pi(\xi_1 x_1 + \psi(\xi_1)x_2)$ for this value of ξ_1, we can extend Φ to a homogeneous function of degree 1 satisfying the hypotheses of Theorem 2.1, and $A(x_2)I(x_1, x_2)e^{-i\Phi(x)}$ has an asymptotic expansion in C^∞ homogeneous terms of degree $-\alpha - 3/2$, $-\alpha - 5/2, \ldots$ Hence the theorem follows from Theorem 2.1 and the fact that convolution by any integrable function is bounded in L^p.

The following improvement is due to Sjölin [6] who gave a different proof:

THEOREM 2.2′. *Theorem* 2.2 *remains valid if* ψ'' *has zeros in* I *provided that they are of finite order.*

Proof. Since ψ'' can only have a finite number of zeros in $\operatorname{supp} a_1$ we may assume that there is only one, say at $\xi_1 = 0$. Since composition of any multiplier with a linear transformation in $\mathbf{R}^2$ is another multiplier with the same norm we may assume that

$$\psi(\xi_1) = c\xi_1^m + O(\xi_1^{m+1}), \quad c \neq 0.$$

To examine what happens at $\xi_1 = 0$ we introduce $\psi_\varepsilon(\xi_1) = \varepsilon^{-m}\psi(\varepsilon\xi_1)$. When $\varepsilon \to 0$ we have $\psi_\varepsilon(\xi_1) \to c\xi_1^m$ in C^∞. If $\chi \in C_0^\infty(I \times \mathbf{R})$ vanishes in a neighborhood of 0 it follows from Theorem 2.2 that

$$\chi(\xi)(\xi_2 - \psi_\varepsilon(\xi_1))_+^\alpha$$

is in M_p for $0 < \varepsilon \leqq 1$, and the proof shows that the norm in M_p is independent of ε. In view of the invariance of multipliers under composition with linear maps (Hörmander [5, Theorem 1.13]) it follows that

$$\chi(\xi_1/\varepsilon, \xi_2/\varepsilon^m)\varepsilon^{-m\alpha}(\xi_2 - \psi(\xi_1))_+^\alpha$$

is a multiplier with uniformly bounded norm when $0 < \varepsilon \leqq 1$. If we choose $\chi(\xi) = \varphi(\xi) - \varphi(2\xi_1, 2^m\xi_2)$ where $\varphi \in C_0^\infty(I \times \mathbf{R})$ is equal to 1 near 0 and note that

$$\varphi(\xi)(\xi_2 - \psi(\xi_1))_+^\alpha = \sum_0^\infty \chi(2^k\xi_1, 2^{km}\xi_2)(\xi_2 - \psi(\xi_1))_+^\alpha, \quad \xi \neq 0,$$

it follows that the left hand side is in M_p since $\sum 2^{-mk\alpha} < \infty$. The theorem is proved.

It was convenient in the proof of Theorem 2.2′ to have the singularity of m_α on a curve of the form $\xi_2 = \psi(\xi_1)$ but a partition of unity immediately extends the conclusion to arbitrary curves with no tangent of infinite order.

3. Open problems

We shall now discuss the analogue of Theorem 1.2 for several variables. At the same time we will show that Theorem 1.2 is optimal.

With $a \in C_0^\infty(\mathbf{R}^{2d-1})$ and a real valued $\varphi \in C^\infty(\mathbf{R}^{2d-1})$ we write

$$T_N f(x) = \int e^{iN\varphi(x,y)} a(x, y) f(y) dy, f \in C_0^\infty(\mathbf{R}^{d-1}), x \in \mathbf{R}^d.$$

We shall assume

$$\operatorname{rank} \partial^2\varphi/\partial x \partial y = d - 1 \quad \text{when} \quad (x, y) \in \operatorname{supp} a; \tag{3.1}$$

$$\partial/\partial y\langle \partial\varphi/\partial x, t\rangle = 0, \quad 0 \neq t \in \mathbf{R}^d \Rightarrow \det \partial^2/\partial y^2 \langle \partial\varphi/\partial x, t\rangle \neq 0, (x, y) \in \operatorname{supp} a; \tag{3.2}$$

which reduces to the hypotheses of Theorem 1.2 when $d = 2$. Assume for example that $a = 1$ near 0. The norm of T_N as an operator between L^p spaces is not changed if we replace $\varphi(x, y)$ by $\varphi(x, y) - \varphi(x, 0) - \varphi(0, y) + \varphi(0, 0)$ so we may assume that $\varphi(x, 0) = 0$ and $\varphi(0, y) = 0$ identically. After a linear change of variables x and y we have by (3.1)

$$\varphi(x, y) = \sum_1^{d-1} x_j y_j + \sum_1^{d-1} a_j(x) y_j + \sum_1^{d} x_j b_j(y) + O(|x||y|(|x|^2 + |y|^2)).$$

Here a_j and b_j are quadratic forms. If the support of a is sufficiently small we can take $x_j + a_j$ and $y_j + b_j$ as new variables, $j < d$, and reduce φ to the form

$$\varphi(x, y) = \langle x', y\rangle + x_d\langle Ay, y\rangle/2 + O(|x||y|(|x|^2 + |y|^2|) \tag{3.3}$$

where A is a symmetric matrix and $x = (x', x_d)$. Writing $x' = x_d z$ we have

$$\varphi(x, y) = x_d(\langle z, y\rangle + \langle Ay, y\rangle/2 + \psi(z_d, x, y)),$$

where $\psi(z, x_d, y) = O(|y|(x_d^2 + |y|^2))$. For sufficiently small x_d and z it follows that φ has a unique critical point near 0 as a function of y. If f is 1 in a neighborhood of 0 and has sufficiently small support it follows from the stationary phase method that

$$|T_N f(x_d z, x_d)| \sim (2\pi/Nx_d)^{(d-1)/2} |\det A|^{-1/2}$$

when $Nx_d \to \infty$ and $z, x_d \to 0$. It follows that there are positive constants $c_1, \ldots, c_4$ such that for $|z| < c_3, c_1/N < x_d < c_2$ we have

$$|T_N f(x_d z, x_d)| \geqq c_4(Nx_d)^{(1-d)/2}.$$

Hence

$$\int |T_N f|^q dx \geqq c_5 N^{q(1-d)/2} \int_{c_1/N}^{c_2} x_d^{(d-1)(1-q/2)} dx_d.$$

Depending on the convergence or divergence of the integral at 0 we obtain the following conclusions

$$\liminf_{N\to\infty} \|T_N f\|_q N^{(d-1)/2} \geqq C(1/q - 1/2 + 1/2d)^{-1/q},\ 1/2 - 1/2d < 1/q \leqq 1/2; \quad (3.4)$$

$$\liminf_{N\to\infty} \|T_N f\|_q N^{(d-1)/2} (\log N)^{-1/q} \geqq C,\ 1/q = 1/2 - 1/2d; \quad (3.5)$$

$$\liminf_{N\to\infty} \|T_N f\|_q N^{d/q} \geqq C|1/q - 1/2 + 1/2d|^{-1/q},\ 1/q < 1/2 - 1/2d. \quad (3.6)$$

Here C is a positive constant depending on f. Clearly (3.6) is equivalent to

$$\liminf_{N\to\infty} \|T_N f\|_q N^{d/q} \geqq C|1/q - 1/2 + 1/2d|^{1/2d-1/2},\ 1/q < 1/2 - 1/2d, \quad (3.6)'$$

so we conclude that the constant in (1.4) cannot be improved even for a fixed f.

Comparison of (3.6)′ and Theorem 1.2 suggests that for $1/q < 1/2 - 1/2d$ we should have an estimate of the form

$$\|T_N f\|_q \leqq CN^{-d/q}(1/2 - 1/2d - 1/q)^{1/2d-1/2} \|f\|_r. \quad (3.7)$$

If φ satisfies (3.3) we have

$$T_N f(x/N) \to \int e^{i\langle x, \Phi(y)\rangle} a(0, y) f(y) dy,\ N \to \infty,$$

where $\Phi(y) = \partial\varphi(x, y)/\partial x$, $x = 0$. Note that our assumptions on φ mean that $y \to \Phi(y)$ is an immersion of $\mathbf{R}^{d-1}$ as a surface of total curvature $\neq 0$; conversely for every such Φ the function $\langle x, \Phi(y)\rangle$ satisfies (3.1), (3.2). If we set

$$Tf(x) = \int e^{i\langle x, \Phi(y)\rangle} a_0(y) f(y) dy, \quad (3.8)$$

where $a_0(y) = a(0, y)$ is in $C_0^\infty(\mathbf{R}^{d-1})$, it follows from (3.7) that

$$\|Tf\|_q \leqq C(1/2 - 1/2d - 1/q)^{1/2d-1/2} \|f\|_r,\ f \in C_0^\infty(\mathbf{R}^{d-1}). \quad (3.9)$$

(Compare this with Corollary 1.3.)

By (3.3) we have $\Phi(y) = (y, \langle Ay, y\rangle/2) + O(|y|^3)$. Now set $f_\varepsilon(y) = f(y/\varepsilon)$ where $f \in C_0^\infty$ and $\varepsilon > 0$. Then we have $\|f_\varepsilon\|_r = \varepsilon^{(d-1)/r}\|f\|_r$ and

$$(Tf_\varepsilon)(x'/\varepsilon, x_n/\varepsilon^2)\varepsilon^{1-d} \to Sf(x)$$

where we have used the notation

$$Sf(x) = \int e^{i\langle x', y\rangle + ix_d\langle Ay, y\rangle/2} f(y) dy. \quad (3.10)$$

Hence, if $1/r + 1/r' = 1$,

$$\liminf_{\varepsilon \to 0} \varepsilon^{(d+1)/q - (d-1)/r'} \|Tf_\varepsilon\|_q / \|f_\varepsilon\|_r \geqq \|Sf\|_q / \|f\|_r.$$

We conclude that neither (3.7) nor (3.9) can be valid unless

$$1/q < 1/2 - 1/2d, \; (d+1)/(d-1)q + 1/r \leqq 1. \tag{3.11}$$

If there is equality in the second inequality it follows that

$$\|Sf\|_q \leqq C(1/q - 1/2d - 1/q)^{1/2d - 1/2} \|f\|_r, \; f \in C_0^\infty(\mathbf{R}^{d-1}). \tag{3.12}$$

When $d = 2$ the second condition in (3.11) becomes $3/q + 1/r \leqq 1$ which shows that Theorem 1.2 is optimal also with respect to the L^p classes involved.

Question 3.1. Does (3.7) follow from (3.1), (3.2) and (3.11)?

Question 3.2. Does (3.9) follow from (3.11) when $y \to \Phi(y)$ is an immersion defining a surface with total curvature $\neq 0$?

Question 3.3. Is (3.12) valid for any real symmetric non-singular matrix A when (3.11) is valid with equality in the second inequality?

Note that we have proved that a positive answer to one of these implies a positive answer to the following ones. The arguments given in sections 1 and 2 still apply to show that a positive answer to Question 3.1 implies that m_α is a multiplier on $FL^p(\mathbf{R}^d)$ when (0.3) is fulfilled.

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