Invariant Subspaces and Weighted Polynomial Approximation

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1. Introduction

Let μ be a finite positive Borel measure defined and having compact support in the complex plane $\mathbf{C}$. Assume that μ is not a point mass. Let z denote the complex identity function and let $\mathcal{P}$ stand for the polynomials in z. For each $p, 1 \leq p < \infty$, set $H^p(d\mu)$ equal to the closure of $\mathcal{P}$ in $L^p(d\mu)$. In this paper we ask: Does $H^p(d\mu)$ have at least one closed subspace, other than itself and $\{0\}$, which is invariant under multiplication by z? The answer is known to be yes when $p > 2$ and yes in certain cases when $p \leq 2$. When $p = 2$ the question is especially intriguing, since then it is equivalent to the invariant subspace problem for subnormal operators on Hilbert space. Our main objective here is to answer it for certain measures ψdA which are absolutely continuous with respect to planar Lebesgue measure A.

We begin with a few simple observations. If $H^p(\psi dA) = L^p(\psi dA)$ and W is any measureable set with $0 < \int_W \psi dA < \int \psi dA$ then $S = \{f \in H^p(\psi dA): f = 0$ a.e. $- \psi dA$ on $W\}$ is a nontrivial closed subspace invariant under multiplication by z. If $H^p(\psi dA) \neq L^p(\psi dA)$ (and only then) it may happen that there is a point $\zeta \in \mathbf{C}$ such that the map $f \to f(\zeta)$ can be extended from $\mathcal{P}$ to a bounded linear functional on $H^p(\psi dA)$. A linear functional on $H^p(\psi dA)$ associated to a point ζ in this way is called a bounded evaluation for $H^p(\psi dA)$. By taking $S(\zeta)$ to be the closure in $H^p(\psi dA)$ of the polynomials vanishing at ζ, we obtain a closed subspace which is invariant under multiplication by z and, since $(z - \zeta) \in S(\zeta)$ and $1 \notin S(\zeta)$, it is nontrivial. In some cases it can be shown that either $H^p(\psi dA)$ has a bounded evaluation or else $H^p(\psi dA) = L^p(\psi dA)$, thereby assuring the existence of a z-invariant subspace in $H^p(\psi dA)$.

* This work was partially supported by NSF Grant No. G.P. 19533.

Invariant subspaces can be shown to exist in this way whenever ψ is the characteristic function of a compact set. This was done for $p \neq 2$ in [2] and for $p = 2$ in [3]. The theorem we now state was announced in [4] and includes both of these as special cases.

THEOREM 1. *Let* ψ *be a non-negative function having compact support* E *and satisfying*

(1) $\psi \in L^{1+\varepsilon}(E, dA)$ *for some* $\varepsilon > 0$;

(2) $\int_E \log \psi dA > -\infty$.

Then, for each p, *either* $H^p(\psi dA)$ *has a bounded evaluation or else* $H^p(\psi dA) = L^p(\psi dA)$. *In particular,* $H^p(\psi dA)$ *has a nontrivial closed* z*-invariant subspace.*

In the course of this investigation it will become apparent that the hypothesis $\int_E \log \psi dA > -\infty$ assures that every »function» in $H^p(E, \psi dA)$ has a representative which is analytic in the interior of E. It further implies that the collection of all functions in $L^p(E, \psi dA)$ which admit such a representative constitutes a closed subspace of $L^p(E, \psi dA)$. We denote that subspace by $L^p_a(E, \psi dA)$. Whenever $H^p(E, \psi dA)$ and $L^p_a(E, \psi dA)$ coincide the polynomials are said to be complete in $L^p_a(E, \psi dA)$. A major portion of this paper, Section 3, is devoted to the problem of finding conditions under which completeness occurs. In this connection we obtain several results concerning bounded point evaluations and weighted polynomial approximation but no additional information on the existence of invariant subspaces. The weight ψ is assumed to satisfy (1) and (2) of Theorem 1 throughout.

Section 3 is comprised of three theorems. Of these, Theorem 4 is perhaps the most significant. It originates from the following question: Given a compact set X with connected complement and a simply connected Jordan domain Ω lying in X, when is $H^p(X \setminus \Omega, dA) = L^p_a(X \setminus \Omega, dA)$? Subject to certain regularity restrictions on $\partial\Omega$, we obtain a metric condition which is both necessary and sufficient for completeness in this setting. We put no restrictions on ∂X. This problem has been studied extensively by a number of mathematicians, notably from the Soviet school. Progress was due initially to Keldysh and later to Dzrbasyan and Saginjan (see [20]). More recently, Havin and Mazja [14] have contributed in this area. The relationship between bounded point evaluations and completeness is best portrayed in Theorem 2. Although Theorem 1 is included as a special case, the proof essentially makes use of the earlier result. In Theorem 3 we consider the problem of completeness for measures ψdA, where ψ is the modulus of a nonvanishing analytic function on the interior of E. Under suitable conditions we are able to verify that the polynomials are complete in $L^p_a(E, \psi dA)$ if and only if they are complete in $L^p_a(E, dA)$. Theorems 2 and 3 both include prior results of Hedberg [15] and serve to put that work in its proper setting.

2. Invariant subspaces and approximation on nowhere dense sets

To prove Theorem 1 we shall assume that $H^p(\psi dA)$ has no bounded evaluations and deduce that $H^p(\psi dA) = L^p(\psi dA)$. The argument will run as follows: Let g be any function in $L^q(\psi dA)$, $1/p + 1/q = 1$, with the property that $\int fg\psi dA = 0$ for every $f \in H^p(\psi dA)$. We shall use the nonpresence of bounded evaluations on $H^p(\psi dA)$ to prove that the Cauchy transform $\int (g\psi)(\zeta)(\zeta - z)^{-1}dA(\zeta)$, denoted $\widehat{g\psi}(z)$, is zero almost everywhere dA. This will force $g\psi$ to vanish almost everywhere, since $g\psi = -\pi^{-1}\partial\widehat{g\psi}/\partial\bar{z}$ in the sense of distribution theory. That in turn will imply $H^p(\psi dA) = L^p(\psi dA)$ as asserted.

We are thus left with the task of proving $\widehat{g\psi} = 0$ almost everywhere. For this we will need two lemmas concerning the behavior of the Cauchy transform. The first can be found in Deny [8] (see also [7], pp. 75—76) and is restated here for the convenience of the reader. The second is a variation of a lemma of Carleson [6, Lemma 1].

LEMMA 1. *Let μ be any positive measure in the plane of total mass 1 (i.e. $\int d\mu = 1$). If $1 < q < 2$ and $p = q(q-1)^{-1}$ then*

$$\int_{\mathbf{C}} \left\{ \int \frac{d\mu(\zeta)}{|\zeta - z|} \right\}^p dA \leq K \left\{ \sup_{z\in \mathbf{C}} \int \frac{d\mu(\zeta)}{|\zeta - z|^{2-q}} \right\}^{p-1},$$

where K is some constant depending only on q.

LEMMA 2. *Let E be a compact subset of the plane and let $k \in L^q(E, dA)$ for some $q > 1$. Assume that for each $z_0 \in E$ and each $r_0 > 0$ the set*

$$\{r\colon 0 \leq r \leq r_0,\ (|z - z_0| = r) \cap (\mathbf{C}\setminus E) \neq \phi\}$$

has linear measure equal to r_0. If $\hat{k} \equiv 0$ in $\mathbf{C}\setminus E$ then $\hat{k} = 0$ almost everywhere dA.

Proof. For $q > 2$ the Cauchy transform $\hat{k}$ is everywhere continuous and the lemma is a simple consequence of the fact that E has no interior. For smaller values of q it is less obvious. We shall prove that if $1 < q < 2$ then $\hat{k}(z_0) = 0$ at every point $z_0 \in E$ where $\int |k(z)|^q|z - z_0|^{-1}dA < \infty$. Since this integral is finite almost everywhere, regardless of the value of q, that will establish the lemma.

Fix q with $1 < q < 2$ and let z_0 be as above. To simplify notation assume that $z_0 = 0$. Thus, $\int |k(z)|^q|z|^{-1}dA$ and $\int |k(z)||z|^{-1}dA$ are both finite. We shall construct, for each $\delta > 0$, a probability measure ν_δ in such a way that

(a) ν_δ is carried by $(|z| \leq \delta) \cap (\mathbf{C} \setminus E)$;

(b) $\lim_{\delta \to 0} \int \hat{k}(\zeta) d\nu_\delta(\zeta) = \hat{k}(0)$.

Since $\hat{k} = 0$ in $\mathbf{C} \setminus E$ and ν_δ has no mass outside $\mathbf{C} \setminus E$, it will follow from (b) that $\hat{k}(0) = 0$ as asserted.

In order to obtain the ν_δ's put χ equal to the characteristic function of $\mathbf{C} \setminus E$ and let $l(r)$ be the linear measure of $(|z| = r) \cap (\mathbf{C} \setminus E)$. Note that

$$l(r) = r \int_0^{2\pi} \chi(re^{i\theta}) d\theta, \quad 0 \leq r < \infty.$$

Define a measure σ by setting

$$\sigma(X) = \int_X \chi(z) l(|z|)^{-1} dA$$

for every Lebesgue measureable set X. This measure was brought to my attention by James Wells in connection with Carleson's work [6, Lemma 1]. It is evidently carried by $\mathbf{C} \setminus E$ and, because $l(r) > 0$ for almost every r, one can use polar coordinates and Fubini's theorem to verify that $\sigma(\{z\colon |z| \leq \delta\}) = \delta$. We claim that ν_δ can be taken to be the restriction of $\delta^{-1}\sigma$ to $|z| \leq \delta$.

Assume, therefore, that ν_δ has been chosen in this way. By interchanging the order of integration,

$$\int \hat{k}(\zeta) d\nu_\delta(\zeta) = \int_{|z| \leq 2\delta} + \int_{|z| \geq 2\delta} \left\{ \int \frac{d\nu_\delta(\zeta)}{z - \zeta} \right\} k(z) dA(z), \tag{1}$$

provided this operation is permissable. We shall not mention it again, but the validity of (1) is actually a consequence of the estimates obtained below. Our plan is to show that, as $\delta \to 0$, the first integral on the right side of (1) converges to zero and the second converges to $\hat{k}(0)$.

It is an easy matter to check that

(c) $\int (z - \zeta)^{-1} d\nu_\delta(\zeta) \to 1/z$ for every $z \neq 0$ as $\delta \to 0$;

(d) $\int |z - \zeta|^{-1} d\nu_\delta(\zeta) \leq 2/|z|$ for $|z| \geq 2\delta$.

Then, since $k(z)/z \in L^1(dA)$ by hypothesis, it follows from the Lebesgue dominated convergence theorem that

$$\lim_{\delta \to 0} \int_{|z| \geq 2\delta} \left\{ \int (z - \zeta)^{-1} d\nu_\delta(\zeta) \right\} k(z) dA(z) = \int k(z)/z dA = \hat{k}(0).$$

By applying Hölder's inequality and Lemma 1, in that order, to the remaining integral in (1), we see that

$$\left|\int\limits_{|z|\leq 2\delta}\left\{\int (z-\zeta)^{-1}d\nu_\delta(\zeta)\right\}k(z)dA(z)\right|\leq K\left\{\sup_{z\in\mathbf{C}}\int |z-\zeta|^{q-2}d\nu_\delta(\zeta)\right\}^{\frac{p-1}{p}}$$

$$\left\{\int\limits_{|z|\leq 2\delta]}|k(z)|^q dA\right\}^{1/q},$$

where $1/p + 1/q = 1$ and K is a constant that depends only on q. If z is fixed and $\zeta = re^{i\theta}$,

$$\int |z-\zeta|^{q-2}d\nu_\delta(\zeta)\leq \int \big||z|-|\zeta|\big|^{q-2}d\nu_\delta(\zeta) = \delta^{-1}\int_0^\delta\left\{rl(r)^{-1}\int_0^{2\pi}\chi(re^{i\theta})d\theta\right\}\big||z|-r\big|^{q-2}dr$$

$$= \delta^{-1}\int_0^\delta \big||z|-r\big|^{q-2}dr \leq 2\delta^{-1}\int_0^\delta r^{q-2}dr = 2\delta^{q-2}/(q-1).$$

Thus,

$$\left|\int\limits_{|z|\leq 2\delta}\left\{\int (z-\zeta)^{-1}d\nu_\delta(\zeta)\right\}k(z)dA(z)\right|\leq K'\left\{\delta^{q-2}\int\limits_{|z|\leq 2\delta}|k(z)|^q dA\right\}^{1/q},$$

where K' depends only on q. As soon as $\delta < 1$ the right side is dominated by

$$2^{1/q}K'\left\{\int\limits_{|z|\leq 2\delta}|k(z)|^q/|z|dA\right\}^{1/q}$$

and this tends to zero as $\delta\to 0$, since $|k(z)|^q/|z|$ was assumed to be summable. This establishes the assertion that $\lim_{\delta\to 0}\int \hat{k}(\zeta)d\nu_\delta(\zeta) = \hat{k}(0) = 0$. Q.E.D.

In addition to Lemmas 1 and 2, we shall have recourse to a theorem of Szegö which is basic to the invariant subspace theory for the spaces $H^p(d\mu)$ associated to measures supported on the unit circle. For a more thorough discussion of this and related topics the reader can consult either [12] or [17]. The theorem to which we have alluded is the following:

Szegö's Theorem. *Let $d\theta$ be one-dimensional Lebesgue measure on the unit circle $|z| = 1$ and let $h\in L^1(d\theta/2\pi)$, $h\geq 0$. Denote by $\mathcal{P}_0$ the set of all polynomials that vanish at the origin. Then, the distance from $\mathcal{P}_0$ to the constant function 1 in $L^p(hd\theta/2\pi)$ is given by the formula*

$$\inf_{f\in\mathcal{P}_0}\int |1-f|^p hd\theta/2\pi = \exp\left\{(2\pi)^{-1}\int \log hd\theta\right\}. \tag{II}$$

This allows us to estimate the norm of evaluation at the origin when the latter is regarded as a linear functional on the polynomials in the $L^p(hd\theta/2\pi)$ norm. For example, if $\int \log h d\theta > -\infty$ and we set it equal to $2\pi \log 1/K$, $K > 0$, then (11) says $1 \leq K \int |1 - f|^p h d\theta/2\pi$ for all $f \in \mathcal{P}_0$. This is the same as saying $1 \leq K \int |f|^p h d\theta/2\pi$ for every polynomial f having value 1 at the origin. Therefore,

$$|f(0)|^p \leq K \int |f|^p h d\theta/2\pi$$

for all polynomials f. This observation will be important in the proof of Theorem 1.

Proof of Theorem 1. We wish to prove that $H^p(\psi dA) = L^p(\psi dA)$ provided $H^p(\psi dA)$ has no bounded evaluations, ψ has compact support E, $\psi \in L^{1+\varepsilon}(dA)$ and, $\int_E \log \psi dA > -\infty$. To accomplish this we need only prove that $\widehat{g\psi} = 0$ almost everywhere dA, whenever $g \in L^q(\psi dA)$, $1/p + 1/q = 1$, and $\int fg\psi dA = 0$ for each $f \in H^p(\psi dA)$. Assume, therefore, that g is any such function.

The first step is to show that $\widehat{g\psi} = 0$ everywhere in the complement of E. So fix $\zeta \in \mathbf{C}\setminus E$. If f is a polynomial then $[f(z) - f(\zeta)](z - \zeta)^{-1}$ is also a polynomial and $\int (f(z) - f(\zeta))(z - \zeta)^{-1} g(z)\psi(z) dA(z) = 0$. Consequently, if $\widehat{g\psi}(\zeta) \neq 0$ we can write $f(\zeta) = [\widehat{g\psi}(\zeta)]^{-1} \int f(z) g(z)(z - \zeta)^{-1}\psi(z) dA(z)$ for every polynomial f. But, this is impossible, since $g(z)(z - \zeta)^{-1} \in L^q(\psi dA)$ by choice of ζ and $H^p(\psi dA)$ has no bounded evaluations by hypothesis. Therefore, $\widehat{g\psi}(\zeta) = 0$ and hence $\widehat{g\psi} \equiv 0$ in $\mathbf{C}\setminus E$, since ζ was arbitrary.

Step two consists in using Lemma 2 to conclude that $\widehat{g\psi} = 0$ almost everywhere on E. Before that lemma can be applied, however, we must check that $g\psi \in L^{1+\nu}(dA)$ for some $\nu > 0$. For that purpose fix q' with $1 < q' < q$ and put $p' = q'/(q' - 1)^{-1}$. By applying Hölder's inequality to the measure ψdA,

$$\int |g|^{1+\nu}\psi^{\nu}\psi dA \leq \left\{\int |g|^{(1+\nu)q'}\psi dA\right\}^{1/q'} \left\{\int \psi^{\nu p'}\psi dA\right\}^{1/p'}. \tag{III}$$

Since $\psi \in L^{1+\varepsilon}(dA)$ and $g \in L^q(\psi dA)$, the right side of (III) will be finite and $g\psi \in L^{1+\nu}(dA)$ if ν is chosen small enough to make $(1 + \nu)q' \leq q$ and $\nu p' \leq \varepsilon$.

Fix ν in this way and let $\zeta_0 \in E$. Assume for convenience that $\zeta_0 = 0$. Let E_0 denote the union of all those circles $|z| = r$ that lie entirely in E. We claim that $A(E_0) = 0$, whence $|z| = r$ meets $\mathbf{C}\setminus E$ for almost every r with respect to linear measure. To prove this let us assume that $A(E_0) > 0$. Then, since $\int_{E_0} \log \psi dA > -\infty$, it follows from Fubini's theorem that there is a constant $K > 0$ and a set of non-negative real numbers X, having positive linear measure, such that for every $r \in X$, the circle $|z| = r$ is in E_0, $\psi(re^{i\theta}) \in L^1(d\theta/2\pi)$ and

$$\int_0^{2\pi} \log \psi(re^{i\theta}) d\theta \geq 2\pi \log 1/K.$$

Hence, Szegö's theorem implies that

$$|f(0)|^p \leq K \int_0^{2\pi} |f(re^{i\theta})|^p \psi(re^{i\theta}) d\theta / 2\pi \tag{IV}$$

for every polynomial f and every $r \in X$. Integrating both sides of (IV) with respect to rdr, we get

$$|f(0)|^p \int_X rdr \leq K(2\pi)^{-1} \int_0^{2\pi} \int_X |f(re^{i\theta})|^p \psi(re^{i\theta}) rdrd\theta \leq K(2\pi)^{-1} \int_E |f|^p \psi dA,$$

which is tantamount to saying that $H^p(\psi dA)$ has a bounded evaluation at the origin. Since this possibility has been ruled out by hypothesis, we must conclude that $A(E_0) = 0$ as claimed.

We have thus shown that $\widehat{g\psi} = 0$ off E and that the assumptions of Lemma 2 are satisfied. It follows that $\widehat{g\psi} = 0$ almost everywhere and hence $H^p(\psi dA) = L^p(\psi dA)$. Q.E.D.

Consider now Lebesgue measure restricted to a compact set E. Denote the set of continuous functions on E by $C(E)$. Let $P(E)$ and $R(E)$ consist of those functions in $C(E)$ which on E are the uniform limits of polynomials and rational functions, respectively. If E has no interior and $\mathbf{C} \setminus E$ is connected a theorem of Lavrentieff (see [12, p. 48] or [19, p. 297]) says that $P(E) = C(E)$, and so $H^p(E, dA) = L^p(E, dA)$ for every p. On the other hand, if $\mathbf{C} \setminus E$ is disconnected $P(E) \neq C(E)$. This raises the question: If $\mathbf{C} \setminus E$ is disconnected, must $H^p(E, dA) \neq L^p(E, dA)$, at least for some value of p? Of course, one excludes the trivial case where E contains a closed subset E' of full measure with $\mathbf{C} \setminus E'$ connected. The following example was suggested by John Wermer and answers the question negatively.

A set X will be called thick if every disk centered at a point of X meets X in a set of positive measure.

Example. There exists a compact set E such that

(i) E is thick;
(ii) $\mathbf{C} \setminus E$ is disconnected;
(iii) $H^p(E, dA) = L^p(E, dA)$ for every p.

Proof. Take a sequence of disjoint rectangules

$$R_n = \{x + iy: 3 \cdot 2^{-1-n} \leq x \leq 2^{1-n}, -1 \leq y \leq 1\}, \quad n = 1, 2, \ldots,$$

accumulating to the line segment $I = \{iy: -1 \leq y \leq 1\}$. Join the ends of I by a thick arc E_0 having measure no greater than 1 and lying in the half plane $\operatorname{Re} z < 0$. Such an arc can be constructed by modifying the argument in [21].

Fix a point ζ_0 in the region bounded by $E_0 \cup I$. In R_1 join the top and bottom faces by a thick arc E_1. Choose a polynomial P_1 so that $P_1(\zeta_0) = 1$ and $|P_1| \leq 1$ on $E_0 \cup E_1$. Note, we do not mind if $|P_1|$ is large on I. Assume that thick arcs E_j have been constructed joining the top and bottom faces of R_j, $j = 1, 2, \ldots, n$, and polynomials $P_1, \ldots, P_n$ chosen with

(1) $P_j(\zeta_0) = j$

(2) $\left\{\int_{E_{n+1}} |P_j|^j dA\right\}^{1/j} \leq 1/2^k,\quad k = 0, 1, \ldots, j$ and $j = 1, 2, \ldots, n$.

Construct a thick arc E_{n+1} in R_{n+1} joining the top and bottom faces such that

(3) $\left\{\int_{E_{n+1}} |P_j|^n dA\right\}^{1/n} \leq 1/2^n,\quad j = 1, 2, \ldots, n.$

Choose a polynomial P_{n+1} such that

(4) $P_{n+1}(\zeta_0) = n + 1$

(5) $\left\{\int_{E_k} |P_{n+1}|^{n+1} dA\right\}^{1/n+1} \leq 1/2^k,\quad k = 0, 1, \ldots, n + 1.$

Thus, we obtain, inductively, sequences $\{E_j\}_{j=0}^{\infty}$ and $\{P_j\}_{j=0}^{\infty}$. Define $E = \bigcup_{k=0}^{\infty} E_k \cup I$. Then E is compact, $\mathbf{C}\setminus E$ is disconnected and, since each E_k was thick, E is thick.

Now $P_j(\zeta_0) = j$ for every j and for each r, $1 \leq r < \infty$,

$$\left\{\int_E |P_j|^r dA\right\}^{1/r} \leq \sum_{k=0}^{\infty} \left\{\int_{E_k} |P_j|^r dA\right\}^{1/r} \leq \sum_{k=0}^{\infty} 1/2^k = 2,$$

provided $j > r$. Therefore, ζ_0 is not a bounded evaluation for any $H^p(E, dA)$. If $g \in L^q(E, dA)$, $q > 1$, and $\int fg\, dA = 0$ for all polynomials f it follows as in the proof of Theorem 1 that $\int g(z)(z - \zeta_0)^{-1} dA = 0$. Thus, $(z - \zeta_0)^{-1}$ belongs to every $H^p(E, dA)$. Similarly, $R(E)$ is contained in every $H^p(E, dA)$. But, $\mathbf{C}\setminus E$ has only two components and so $R(E) = C(E)$ by a theorem of Mergeljan [19, p. 317] (see also [12, p. 51]). It follows that $H^p(E, dA) = L^p(E, dA)$ for every p. Q.E.D.

Some time ago, Sinanjan [24, Th. 2.4] constructed a compact set E_0 with the property that $R(E_0)$ is dense in $L^p(E_0, dA)$ for every p but, $R(E_0) \neq C(E_0)$. This example was obtained by removing from the closed unit disk a sequence of open disks $\Delta_j = \{z: |z - a_j| < r_j\}$, $j = 1, 2, \ldots$, such that the Δ_j have mutually disjoint closures and $\sum_{j=1}^{\infty} r_j < \infty$. However, for sets E of this type it is known that $H^p(E, dA)$ is never equal to $L^p(E, dA)$ (see [2, p. 305]). It would be interesting to know if there exists a compact E with $H^p(E, dA) = L^p(E, dA)$ for every p, but $R(E) \neq C(E)$.

3. Approximation on sets with interior points

Let E be an arbitrary compact set in the plane and let ψ be a non-negative function defined on E with $\int_E \log \psi dA > -\infty$. Suppose that $\zeta_0 \in E^0$ and let δ be its distance from ∂E. (Here E^0 and ∂E denote the interior and boundary of E respectively.) We have seen in the proof of Theorem 1 that ζ_0 is a bounded evaluation for the space $H^p(W, \psi dA)$ associated to the measure obtained by restricting ψdA to the annulus $W = \{\zeta: \frac{1}{2}\delta \leq |\zeta - \zeta_0| \leq \delta\}$. We can therefore choose a function $g \in L^q(W, \psi dA)$, $1/p + 1/q = 1$, such that $f(\zeta_0) = \int_W fg\psi dA$ for every polynomial f. Let us assume that g has been defined to be zero outside W and let $k(z) = (z - \zeta_0)g(z)\psi(z)$. Then, $\int_W fkdA = 0$ for all polynomials f and $\hat{k}(\zeta_0) = 1$. Since $\hat{k}$ is continuous in $\mathbf{C}\setminus W$, there is a neighborhood U of ζ_0 on which $|\hat{k}| \geq 1/2$. If $\zeta \in U$, $|\zeta - \zeta_0| \leq \delta/4$, and f is a polynomial then

$$f(\zeta) = \hat{k}(\zeta)^{-1} \int_W f(z)k(z)(z-\zeta)^{-1}dA(z)$$

and we have the estimate

$$|f(\zeta)| \leq 4\delta^{-1} \left\{ \int_W |g|^q \psi dA \right\}^{1/q} \left\{ \int_W |f|^p \psi dA \right\}^{1/p}.$$

Thus, in order for a sequence of polynomials to converge in the $L^p(E, \psi dA)$ norm, convergence must be uniform on every compact subset of E^0. Hence, $H^p(E, \psi dA)$ is contained in $L^p_a(E, \psi dA)$, the subspace of $L^p(E, \psi dA)$ consisting of those functions which are analytic in E^0. We are thus led to ask: Given E and ψ as above, which functions in $L^p_a(E, \psi dA)$ belong to $H^p(E, \psi dA)$?

Questions of this nature have a long history dating back to the late 1800's and the work of Runge on uniform approximation. The first results relating specifically to approximation in the mean were obtained by T. Carleman in 1922. He proved that $H^p(D, dA) = L^p_a(D, dA)$ if D is a simply connected Jordan domain. Later Markusevic and Farrell (see [20], p. 112] proved the corresponding theorem for Caratheodory domains and Sinanjan [24] subsequently extended it to closed Caratheodory sets. (A Caratheodory set is a set whose boundary coincides with the boundary of the unbounded component of its complement.) In the mean time, it was discovered that there exist non-Caratheodory sets E for which $H^p(E, dA) = L^p_a(E, dA)$ (see [20], p. 116). In an attempt to explain this phenomenon, Sinanjan [25] recently conjectured that $H^p(E, dA) = L^p_a(E, dA)$ if and only if $H^p(E, dA)$ has no bounded evaluations arising from points outside E^0. We shall see in Theorem 2 that this is indeed the case. In addition, Theorem 2 applies to a more general setting than that envisioned by Sinanjan and it extends a result of Hedberg [15, Corollary 1] on weighted polynomial approximation.

THEOREM 2. *Let E be compact and let ψ be a non-negative function defined on E. Assume that*

(1) $\psi \in L^{1+\varepsilon}(E, dA)$;

(2) $\int_E \log \psi dA > -\infty$.

If $H^p(E, \psi dA)$ has no bounded evaluations outside E^0 then every function F in $L^p_a(E, \psi dA) \cap L^{p(1+\varepsilon)/\varepsilon}(E, dA)$ belongs to $H^p(E, \psi dA)$.

Remark. Theorem 2 is valid for $\varepsilon = \infty$ if $(1 + \varepsilon)/\varepsilon$ is interpreted as being equal to 1. When $\psi \equiv 1$ on E this proves the sufficiency of Sinanjan's conjecture. The proof of necessity is not difficult and we omit it.

To deal with questions concerning approximation on sets with interior points we need the following estimate which is implied by Hedberg's work [16, Lemma 6]. Although the results in [16] are stated in terms of capacity, that notion can be avoided here by making use of the measure σ introduced in the proof of Lemma 2. We leave it to the reader to supply the details.

LEMMA 3. *Let E be compact and let $k \in L^q(E, dA)$, $1 < q \leq 2$. Assume that for each $z_0 \in \partial E$ and each $r_0 > 0$ the set $\{r: 0 \leq r \leq r_0, (|z - z_0| = r) \cap (\mathbf{C}\setminus E) \neq \phi\}$ has full linear measure. If $\hat{k} \equiv 0$ in $\mathbf{C}\setminus E$ and ζ_0 is a point of E^0 at a distance $\delta < 1/e$ from ∂E then*

$$|\hat{k}(\zeta_0)| \leq C \left\{ k^*(\zeta_0)\delta \log 1/\delta + (\Gamma_q(\delta) \int_{|z-\zeta_0|\leq 4\delta} |k(z)|^q dA)^{1/q} \right\}$$

where $k^*(\zeta) = \sup_r (\pi r^2)^{-1} \int_{|z-\zeta|<r} |k(z)| dA$ is the Hardy-Littlewood maximal function, $\Gamma_q(\delta)$ is equal to $\log 1/\delta$ or δ^{q-2} according to whether $q = 2$ or $q < 2$ and, C is a constant depending only on q and the diameter of E.

Proof of Theorem 2. Fix F in $L^p_a(E, \psi dA) \cap L^{p(1+\varepsilon)/\varepsilon}(E, dA)$. Let g be any function in $L^q(\psi dA)$, $1/p + 1/q = 1$, with the property that $\int fg\psi dA = 0$ for every $f \in H^p(\psi dA)$. To prove that $F \in H^p(\psi dA)$ it is sufficient to show $\int Fg\psi dA = 0$. We shall see that $g\psi = 0$ almost everywhere on ∂E and so the problem reduces to showing $\int_{E^0} Fg\psi dA = 0$.

Since $H^p(\psi dA)$ has no bounded evaluations outside E^0, we can argue as in Theorem 1 to conclude that $\widehat{g\psi} = 0$ everywhere in $\mathbf{C}\setminus E$ and almost everywhere on ∂E. Now, $g\psi \in L^{1+\nu}(dA)$, $\nu > 0$, and so it follows from the theory of singular integrals [5] that $\widehat{g\psi}$ is absolutely continuous on almost every line parallel to each of the coordinate axes and that $\partial(\widehat{g\psi})/\partial x$ and $\partial(\widehat{g\psi})/\partial y$ exist almost everywhere (dA) in the usual sense. By a lemma of Schwartz [22, Theorem V, p. 57], these derivatives coincide with the corresponding distribution derivatives and so

$$g\psi = \frac{-1}{2\pi}\left(\frac{\partial}{\partial x} + i\,\frac{\partial}{\partial y}\right)(\widehat{g\psi})$$

almost everywhere in the usual sense. If $X = \{\zeta \in \partial E : \widehat{g\psi}(\zeta) = 0\}$ then, by Fubini's theorem, almost every point of X is a point of linear density (and hence a point of accumulation) for X in the direction of both coordinate axes and so $\partial(\widehat{g\psi})/\partial x = \partial(\widehat{g\psi})/\partial y = 0$ almost everywhere on X. It follows that $g\psi = 0$ almost everywhere on X and hence almost everywhere on ∂E.

To prove that $\int_{E^0} Fg\psi dA = 0$ we shall regard F as an element of $L^{p(1+\varepsilon)/\varepsilon}(E, dA)$ and we shall require that $g\psi$ belong to the dual space. Recall, therefore, that $g\psi \in L^{1+\nu}(E, dA)$ if $1 < q' < q$, $(1+\nu)q' \leq q$, and, $\nu q'(q'-1)^{-1} \leq \varepsilon$. Equivalently, $\nu \leq \min((q/q') - 1, \ \varepsilon(1 - 1/q'))$. As a function of q', it is easy to see that $(q/q') - 1$ is decreasing and takes values between 0 and $q - 1$, while $\varepsilon(1 - 1/q')$ is increasing and takes values between 0 and ε/p. Thus, to obtain the largest possible ν with $g\psi \in L^{1+\nu}(E, dA)$ we must pick q' so that $(q/q') - 1 = \varepsilon(1 - 1/q')$ and set ν equal to that common value. When this is done we find that $1 + \nu = q(1+\varepsilon)/(q+\varepsilon)$, which is the index conjugate to $p(1+\varepsilon)/\varepsilon$. Assuming that ν has been chosen in this way, we will construct a sequence of functions ϱ_n, $n = 1, 2, \ldots$, such that each ϱ_n has support in E^0 and $\partial\varrho_n/\partial\bar{z} \to g\psi$ in the $L^{1+\nu}(E, dA)$ norm. Then by our choice of ν,

$$\lim_{n\to\infty} \int_{E^0} F\,\frac{\partial\varrho_n}{\partial\bar{z}}\,dA = \int_{E^0} Fg\psi dA.$$

On the other hand, integrating by parts

$$\int_{E^0} F\,\frac{\partial\varrho_n}{\partial\bar{z}}\,dA = -\int_{E^0} \frac{\partial F}{\partial\bar{z}}\,\varrho_n dA = 0$$

for all n, since F is analytic in E^0. Hence $\int_{E^0} Fg\psi dA = 0$ as claimed.

In order to construct the ϱ_n there are two cases to be considered. First is the case where $g\psi \in L^{1+\nu}(E, dA)$, $1 + \nu > 2$. Here $\widehat{g\psi}$ is continuous and can therefore be written in the form $\widehat{g\psi} = (k_1 - k_2) + i(k_3 - k_4)$, where the functions k_j are continuous, non-negative and, zero on ∂E. In addition, each k_j and its first order partial derivatives belong to $L^{1+\nu}(E, dA)$ (see [9, p. 316]). Setting $\varrho_n^j = -\pi^{-1}\sup(k_j - 1/n, 0)$, we obtain functions whose supports are obviously in E^0 and which satisfy

$$\lim_{n\to\infty} \int_{E^0} \left|\frac{\partial\varrho_n^j}{\partial\bar{z}} - \frac{\partial k_j}{\partial\bar{z}}\right|^{1+\nu} dA = 0, \quad j = 1, 2, 3, 4.$$

For a proof of the latter assertion we again refer the reader to [9, p. 317]. The desired sequence ϱ_n, $n = 1, 2, \ldots$ is obtained by taking $\varrho_n = (\varrho_n^1 - \varrho_n^2) + i(\varrho_n^3 - \varrho_n^4)$.

When $g\psi \in L^{1+\nu}(E, dA)$, $1 < 1 + \nu \leq 2$, construction of the ϱ_n is a more delicate matter. The simplest and most direct procedure is due to Bers [1, p. 3] and Hedberg [16, lemma 10] and is based on an idea of Ahlfors. It runs as follows: Let k be the piecewise linear function $k(t) = 0$ for $t \leq 1$, $k(t) = 1$ for $t \geq 2$ and, $k(t) = t - 1$ for $1 \leq t \leq 2$. For each $z \in \mathbf{C}$ let $\delta(z)$ denote the distance from z to $\mathbf{C}\setminus E^0$. Put $k_n(t) = k(n/\log\log 1/t)$, $\omega_n(z) = k_n(\delta(z))$ and $\varrho_n = -\pi^{-1}\widehat{g\psi}\,\omega_n$. Evidently, ϱ_n has support in E^0 and

$$\frac{\partial \varrho_n}{\partial \bar{z}} = g\psi\omega_n - \frac{1}{\pi}\,\widehat{g\psi}\,\frac{\partial \omega_n}{\partial \bar{z}}.$$

These derivatives exist almost everywhere because $\delta(z)$ satisfies a Lipschitz condition of order one. Since $0 \leq \omega_n \leq 1$ and $\omega_n \to 1$ almost everywhere (as $n \to \infty$), it follows from Lebesgue's dominated convergence theorem that $g\psi\omega_n \to g\psi$ in the norm of $L^{1+\nu}(E, dA)$. Thus, the sequence ϱ_n, $n = 1, 2, \ldots$, will have the required properties provided

$$\lim_{n\to\infty} \int_{E^0} \left| \widehat{g\psi}\,\frac{\partial \omega_n}{\partial \bar{z}} \right|^{1+\nu} dA = \frac{1}{2^{1+\nu}} \lim_{n\to\infty} \int_{E^0} |\widehat{g\psi}\,\mathrm{grad}\,\omega_n|^{1+\nu} dA = 0.$$

This is where lemma 3 comes in. One verifies that,

$$\int_{E^0} |\widehat{g\psi}(z)\,\mathrm{grad}\,\omega_n(z)|^{1+\nu} dA \leq$$
$$\int_{E^0} \frac{C}{n^\nu}\left\{ (g\psi)^*(z)^{1+\nu} + \frac{|\mathrm{grad}\,\omega_n(z)|}{\delta(z)} \int_{|\zeta - z| < 4\delta(z)} |g\psi(\zeta)|^{1+\nu} dA_\zeta \right\} dA_z. \tag{V}$$

(see [16, lemma 10]). According to the Hardy maximal theorem (see [26, p. 5]), the first term on the right side of (V) is dominated by $\int_{E^0} C' n^{-\nu}\,|g\psi|^{1+\nu} dA$, for some constant C' independent of n. Since $|\mathrm{grad}\,\omega_n(z)| = k_n'(\delta(z))$ almost everywhere in E^0 and since k_n' is a decreasing function on its support, a similar estimate is valid for the second term:

$$\begin{aligned}
&\frac{C}{n^\nu} \int_{E^0} \frac{|\mathrm{grad}\,\omega_n(z)|}{\delta(z)}\,dA_z \int_{|\zeta - z| < 4\delta(z)} |g\psi(\zeta)|^{1+\nu} dA_\zeta \\
&\leq \frac{C_1}{n^\nu} \int_{E^0} dA_z \int_{|u| < 4\delta(z)} \frac{k_n'(|u|)}{|u|}\,|g\psi(z+u)|^{1+\nu} dA_u \\
&\leq \frac{C_2}{n^\nu} \int_{E^0} \frac{k_n'(|u|)}{|u|}\,dA_u \int_{E^0} |g\psi(z)|^{1+\nu} dA_z \\
&\leq \frac{C_3}{n^\nu} \int_{E^0} |g\psi(z)|^{1+\nu} dA_z,
\end{aligned}$$

where C and C_j, $j = 1, 2, 3$, are constants independent of n. It follows that $\lim_{n\to\infty} \int_{E^0} |\widehat{g\psi} \operatorname{grad} \omega_n|^{1+\nu} dA = 0$ and this completes the proof. Q.E.D.

Ideally, one would like the conclusion of Theorem 2 to be: $H^p(E, \psi dA) = L^p_a(E, \psi dA)$. Unfortunately, that need not be true. Take, for example, $E = \{z: |z| \leq 1\}$ and $\psi(z) = |\exp (z + 1)/(z - 1)|$. Then,

$$\psi \in L^\infty(E, dA), \int_E \log \psi dA > -\infty \quad \text{and} \quad H^p(E, \psi dA)$$

has no bounded evaluations off E^0. Nevertheless, Keldysh was able to show, [20, p. 134], that $H^2(E, \psi dA) \neq L^2_a(E, \psi dA)$ and his proof extends to all p. In cases like this where ψ is the modulus of a non-vanishing analytic function on E^0 it is possible to obtain, with slightly stronger hypothesis on ψ, results which assert that $H^p(E, \psi dA) = L^p_a(E, \psi dA)$ for all p. Of course one must assume that $H^p(E, \psi dA)$ has no bounded evaluations off E^0 for any p. If this assumption is made for a fixed p, only, we are unable to prove that $H^p(E, \psi dA) = L^p_a(E, \psi dA)$, even for that p. The next theorem is typical of this situation. It was partially anticipated by Hedberg [15, p. 117] in the case of a Caratheodory domain.

THEOREM 3. *Let E be compact. Let α be a function defined on E with the property that α is analytic and nowhere zero in E^0 and*

$$\int_E (|\alpha|^{-\varepsilon} + |\alpha|^{1+\varepsilon}) dA < \infty$$

for some $\varepsilon > 0$. Then $H^p(E, |\alpha| dA) = L^p_a(E, |\alpha| dA)$ for every p if and only if $H^p(E, dA) = L^p_a(E, dA)$ for every p.

Remark. We have already noted the existence of non-Caratheodory sets E such that $H^p(E, dA) = L^p_a(E, dA)$ for all p. We shall discuss this phenomenon in greater detail following the proof of Theorem 3.

Proof of Theorem 3. Assume first that $H^p(E, dA) = L^p_a(E, dA)$ for all p, $1 \leq p < \infty$. Then E^0 is simply connected and $H^p(E, dA)$ has no bounded evaluations associated to points outside E^0. Likewise, $H^p(E, |\alpha| dA)$ has no bounded evaluations outside E^0. Indeed, if $H^p(E, |\alpha| dA)$ has a bounded evaluation at ζ then, taking $s' = 1 + \varepsilon$ and $s = s'/(s' - 1)$,

$$\begin{aligned}
|f(\zeta)| &\leq k \left[\int |f|^p |\alpha| dA \right]^{1/p} \\
&\leq k \left[\left\{ \int |f|^{ps} dA \right\}^{1/s} \left\{ \int |\alpha|^{s'} dA \right\}^{1/s'} \right]^{1/p} \\
&\leq k' \left[\int |f|^{ps} dA \right]^{1/ps}
\end{aligned}$$

for every polynomial f. Hence, $\zeta \in E^0$. We shall prove that the absence of bounded evaluations in $\mathbf{C} \setminus E^0$ for every $H^p(E, |\alpha| dA)$ implies $H^p(E, |\alpha| dA) = L^p_a(E, |\alpha| dA)$ for all p.

Since the weight $|\alpha|$ satisfies the appropriate hypothesis, we can argue as in the proof of Theorem 2 that every function $g \in L^q(E, |\alpha| dA)$ which annihilates $H^p(E, |\alpha| dA)$, $1/p + 1/q = 1$, vanishes almost everywhere on ∂E. Thus, to prove that $H^p(E, |\alpha| dA) = L^p_a(E, |\alpha| dA)$ it is enough to show that $H^p(E^0, |\alpha| dA) = L^p_a(E^0, |\alpha| dA)$. Now, in view of the fact that E^0 is simply connected and α has no zeros there, it is possible to define in E^0 an analytic branch of α^λ for each real number λ. If we show that $\alpha^{-1/p} \in H^p(E^0, |\alpha| dA)$ it will follow that $H^p(E^0, |\alpha| dA) = L^p_a(E^0, |\alpha| dA)$ (see [20, p. 132], [15, p. 117]). For suppose $f \in L^p_a(E^0, |\alpha| dA)$. Then $\alpha^{1/p} f \in L^p_a(E^0, dA)$ and by hypothesis there is a sequence of polynomials Q_j satisfying

$$0 = \lim_{j \to \infty} \int_{E^0} |\alpha^{1/p} f - Q_j|^p dA$$

$$= \lim_{j \to \infty} \int_{E^0} |f - Q\alpha^{-1/p}|^p |\alpha| dA.$$

Assuming that $\alpha^{-1/p} \in H^p(E^0, |\alpha| dA)$, each $Q_j \alpha^{-1/p}$ also belongs to $H^p(E^0, |\alpha| dA)$ and so $f \in H^p(E^0, |\alpha| dA)$.

To prove that $\alpha^{-1/p} \in H^p(E^0, |\alpha| dA)$ we shall argue inductively. The idea is due to H. S. Shapiro [23, p. 327] and was adopted by Hedberg in [15, p. 118]. The initial step is to observe that $\alpha^{-\delta/p}$ belongs to $L^p_a(E^0, |\alpha| dA) \cap L^{p(1+\varepsilon)/\varepsilon}(E^0, dA)$ for some δ, $0 < \delta < 1$. Hence, by Theorem 2, $\alpha^{-\delta/p} \in H^p(E^0, |\alpha| dA)$. Next, set $\delta = 1 - \lambda$ and consider the weight $|\alpha|^\lambda$. Since $H^p(E, |\alpha| dA)$ has no bounded evaluations outside E^0, it follows from Hölder's inequality that the same is true of $H^p(E, |\alpha|^\lambda dA)$. Furthermore, $|\alpha|^\lambda \in L^{1+\varepsilon'}(E, dA)$ and

$$\alpha^{-\lambda(1-\lambda)/p} \in L^p_a(E, |\alpha|^\lambda dA) \cap L^{p(1+\varepsilon')/\varepsilon'}(E, dA),$$

where $1 + \varepsilon' = (1 + \varepsilon)/\lambda$. Therefore, theorem 2 applies to the measure $|\alpha|^\lambda dA$ and guarantees the existence of polynomials Q_j such that

$$0 = \lim_{j \to \infty} \int_{E^0} |\alpha^{-\lambda(1-\lambda)/p} - Q_j|^p |\alpha|^\lambda dA$$

$$= \lim_{j \to \infty} \int_{E^0} |\alpha^{-(1-\lambda^2)/p} - Q_j \alpha^{-(1-\lambda)/p}|^p |\alpha| dA.$$

Thus, $\alpha^{-(1-\lambda^2)/p}$ belongs to $H^p(E^0, |\alpha| dA)$, since $\alpha^{-(1-\lambda)/p}$ does. Repeating this argument with $\alpha^{-(1-\lambda)/p}$ playing the same role as before and $|\alpha|^\lambda$ replaced successively by $|\alpha|^{\lambda^2}, |\alpha|^{\lambda^3}, \ldots,$ we find that $\alpha^{-(1-\lambda^k)/p} \in H^p(E^0, |\alpha| dA)$ for

$k = 1, 2, \ldots$ Then because $\alpha^{-(1-\lambda^k)/p} \to \alpha^{-1/p}$ pointwise on E^0 as $k \to \infty$ and $|\alpha|^{-(1-\lambda^k)/p} \leq 1 + |\alpha|^{-1/p}$ for all k, the dominated convergence theorem implies

$$\lim_{k\to\infty} \int_{E^0} |\alpha^{-(1-\lambda^k)/p} - \alpha^{-1/p}|^p |\alpha| dA = 0.$$

Therefore, $\alpha^{-1/p} \in H^p(E^0, |\alpha| dA)$ and so $H^p(E^0, |\alpha| dA) = L^p_a(E^0, |\alpha| dA)$. This establishes the theorem in one direction.

To obtain the other half of the theorem assume that $H^p(E, |\alpha| dA) = L^p_a(E, |\alpha| dA)$ for all p. Note that $H^p(E, |\alpha| dA)$ can have no bounded evaluations outside E^0 for any p. Then, if $H^p(E, dA)$ has a bounded evaluation at the point ζ (p fixed) we see, by applying Hölder's inequality to the measure $|\alpha| dA$, that

$$\begin{aligned} |f(\zeta)| &\leq k \left[\int |f|^p |\alpha|^{-1} \cdot |\alpha| dA \right]^{1/p} \leq k \left[\left\{ \int |f|^{ps} |\alpha| dA \right\}^{1/s} \left\{ \int |\alpha|^{1-s'} dA \right\}^{1/s'} \right]^{1/p} \\ &\leq k' \left[\int |f|^{ps} |\alpha| dA \right]^{1/ps} \end{aligned}$$

for every polynomial f, provided $s' \leq 1 + \varepsilon$ and $1/s + 1/s' = 1$. Hence, $\zeta \in E^0$. That is, $H^p(E, dA)$ has no bounded evaluations outside E^0. It follows from Theorem 2 that $H^p(E, dA) = L^p_a(E, dA)$ for every p. Q.E.D.

We remarked in the proof of Theorem 3 that a requirement for $H^p(E, dA) = L^p_a(E, dA)$ is that the interior of E be simply connected. The simplest non-Caratheodory sets with this property are crescents. A crescent is any set E topologically equivalent to the closed region bounded by two internally tangent circles. Such a set may or may not have the property that $H^p(E, dA) = L^p_a(E, dA)$ (see [20, p. 116]). The »thickness» of E near the multiple boundary point is the determining factor. This was discovered by Keldysh in 1939. Only ten years later and with additional restrictions on the set E was a condition found that is both necessary and sufficient for equality to occur. That was due to the efforts of M. M. Dzrbasyan, who established sufficiency, and A. L. Saginyan, who established necessity. The theorem they obtained is this (see [20, p. 158]):

THEOREM. *Let E be a crescent with multiple boundary point at the origin Denote by $l(r)$ the linear measure of $(|z| = r) \cap E$. Assume that $l(r) = e^{-h(r)}$ and $rh'(r) \uparrow \infty$ as $r \downarrow 0$. Then in order for $H^p(E, dA) = L^p_a(E, dA)$ for all p it is necessary and sufficient that*

$$\int_0 \log l(r) dr = -\infty. \tag{VI}$$

Recently, V. P. Havin and V. G. Mazja, [14], have considered the question of polynomial completeness in $L^p_a(E, dA)$ for sets more general than crescents.

The setting is as follows: Let X be a compact set with connected complement and let Ω be a Jordan domain with class C^2 boundary whose closure is contained in X. Assume that $\partial\Omega$ passes through the origin. In order that $H^p(X\setminus\Omega, dA) = L^p_a(X\setminus\Omega, dA)$ Havin and Mazja prove that (VI) is again sufficient. They do not require that $l(r)$ be monotone or that $X\setminus\Omega$ be a crescent. It appears, however, that the completeness of the polynomials in $L^p_a(X\setminus\Omega, dA)$ depends in a more essential way on the function $\delta(z) = \operatorname{dist}(z, \mathbf{C}\setminus X)$ than on $l(r)$. More specifically, the following is true:

THEOREM 4. *Let X and Ω be as above with one exception. Assume only that $\partial\Omega$ is class C^1 and that $\mathbf{n}(z)$, the unit exterior normal to $\partial\Omega$ at z, satisfies a Lipschitz condition $|\mathbf{n}(z_1) - \mathbf{n}(z_2)| \leq C|z_1 - z_2|$. Let $\delta(z) = \operatorname{dist}(z, \mathbf{C}\setminus X)$. Then, in order that $H^p(X\setminus\Omega, dA) = L^p_a(X\setminus\Omega, dA)$, for any p, it is necessary and sufficient that*

$$\int_{\partial\Omega} \log \delta(z)|dz| = -\infty.$$

Remark 1. The regularity conditions imposed on $\partial\Omega$ are precisely what is needed to ensure that when x is sufficiently close to $\partial\Omega$ there is a unique point of $\partial\Omega$ nearest to x. We shall designate by $\operatorname{Unp}(\partial\Omega)$ the set of all points x having a unique nearest point in $\partial\Omega$. The supremum of all numbers r such that $\{x: |x-a| < r\} \subset \operatorname{Unp}(\partial\Omega)$ whenever $a \in \partial\Omega$ is called the reach of $\partial\Omega$. This terminology is due to Federer, [11, p. 432]. We shall adhere to it in the ensuing discussion.

Remark 2. As an immediate consequence of Theorem 4 there is the interesting fact: If $H^1(X\setminus\Omega, dA) = L^1_a(X\setminus\Omega, dA)$ then $H^p(X\setminus\Omega, dA) = L^p_a(X\setminus\Omega, dA)$ for every p. It is conceivable that this is the case without any restrictions on $\partial\Omega$. But, for the present, that remains an unsettled question.

Proof of Theorem 4 — *Necessity*. The proof of necessity is based on an idea of Saginjan [20, p. 121]. Under the assumption that $\int_{\partial\Omega} \log \delta(z)|dz| > -\infty$, we will construct a Jordan curve Γ^* lying in $X\setminus\Omega$ and having the following properties:

(i) Ω lies inside Γ^*

(ii) Any sequence of polynomials bounded in the $L^p(X\setminus\Omega, dA)$ norm forms a normal family inside Γ^*.

Should $H^p(X\setminus\Omega, dA) = L^p_a(X\setminus\Omega, dA)$ this has the implication that every function in $L^p_a(X\setminus\Omega, dA)$ admits an analytic extension to Ω. Of course, that cannot happen and one must therefore conclude that $H^p(X\setminus\Omega, dA) \neq L^p_a(X\setminus\Omega, dA)$.

For our purpose it will be convenient to replace $\delta(z)$ by a regularized distance function $\Delta(z)$ which has essentially the same profile as $\delta(z)$ but is smooth in X^0. The existence of such a function is guaranteed by a theorem of Calderon and Zygmund (see Stein [26, Th. 2, p. 171]). It is defined at all points of the plane and has the properties:

(a) $C_1\delta(z) \leq \Delta(z) \leq C_2\delta(z)$ for all $z \in \mathbf{C}$;

(b) $\Delta(z)$ is C^∞ in X^0 and $\left|\dfrac{\partial^\alpha \Delta}{\partial x^\alpha}(z)\right| \leq B_\alpha \delta(z)^{1-|\alpha|}$.

Here C_1, C_2 and B_α are positive constants. In (b) the letter α signifies an ordered pair of positive integers (α_1, α_2) and $\partial^\alpha/\partial x^\alpha$ is the corresponding partial derivative of order $|\alpha| = \alpha_1 + \alpha_2$. For any integer $m \geq 3$ one can easily verify that $\Delta^m(z)$ is an $(m-2)$-smooth function throughout the plane. The construction of Γ^* relies heavily on this fact.

In order to carry out that construction choose a parametric representation $\gamma(t)$, $0 \leq t \leq 1$, for $\partial\Omega$ whose first derivative $\gamma'(t)$ is nowhere zero and Lipschitz in t. Let $\varrho(z) = \operatorname{dist}(z, \partial\Omega)$. Extend $\mathbf{n}(z)$ outside $\partial\Omega$ by setting $\mathbf{n}(z) = -\operatorname{grad}\varrho(z)$ if $z \in \Omega$ and $\mathbf{n}(z) = \operatorname{grad}\varrho(z)$ if $z \notin \Omega \cup \partial\Omega$. Thus extended, the function $\mathbf{n}$ is defined almost everywhere, has unit modulus and, since $\partial\Omega$ has positive reach, it is Lipschitz and everywhere defined in the vicinity thereof. Verification of the last assertion will be made in the couse of proving sufficiency. By convolving $\mathbf{n}$ with a suitable real valued C^∞ function τ and setting $\mathbf{N} = \mathbf{n} * \tau/|\mathbf{n} * \tau|$, we obtain a field of unit vectors $\mathbf{N}(z)$ which, in a neighborhood of $\partial\Omega$, is C^∞ and has the property $\mathbf{n}(z) \cdot \mathbf{N}(z) \geq 1/2$. The notation here denotes the usual dot product and the bound on $\mathbf{n}(z) \cdot \mathbf{N}(z)$ implies that, at each $z \in \partial\Omega$, the vectors $\mathbf{n}(z)$ and $\mathbf{N}(z)$ make an angle of not more than $\pi/6$ radians. Since the field $\mathbf{N}$ is Lipschitz and transverse along $\partial\Omega$, the vectors $\varepsilon\mathbf{N}(z)$, attached to $\partial\Omega$ at z, fill out a tubular neighborhood T around $\partial\Omega$ in a one-to-one manner, provided ε is sufficiently small (see Whitehead [30, Theorem 1.5, p. 157]). Likewise, for small ε, the normal field $\varepsilon\mathbf{n}(z)$, $z \in \partial\Omega$, fills out a tube T' in similar fashion. I am indebted to Dennis Pepe for bringing the theorem of Whitehead to my attention and for suggesting that it could be used in this context. By choosing ε properly, we can arrange that, for each $z \in \partial\Omega$,

(c) The vectors $\varepsilon\Delta^4(z)\mathbf{N}(z)$ and $\varepsilon\Delta^4(z)\mathbf{n}(z)$ lie entirely within $T \cap T'$;

(d) $\varepsilon\Delta^4(z) \leq \delta(z)/2$.

Thus, the curve Γ^* parameterized by

$$\gamma^*(t) = \gamma(t) + \varepsilon\Delta^4(\gamma(t))\mathbf{N}(\gamma(t)), \quad 0 \leq t \leq 1,$$

is a simple closed Jordan curve lying in $X \diagdown \Omega$ and satisfying property (i). As defined, Γ^* is class C^1 and, by taking ε smaller if necessary, it has a nonvanishing Lipschitz normal.

To complete the proof of necessity it remains to verify that Γ^* has property (ii). Assume, therefore, that f_j, $j = 1, 2, \ldots$, is a sequence of polynomials which is bounded in the $L^p(X \diagdown \Omega, dA)$ norm. By virtue of (c) and (d) the disk with center at $\gamma^*(t)$ and radius $\varepsilon\Delta^4(\gamma(t))/3$ is contained in $X \diagdown \Omega$ for all t. Thus, by the area mean value theorem,

$$|f_j(\gamma^*(t))| \leq \frac{K}{\Delta^{8/p}(\gamma(t))} \left\{ \int\limits_{X \setminus \Omega} |f_j|^p dA \right\}^{1/p} \leq \frac{K'}{\Delta^{8/p}(\gamma(t))}, \quad j = 1, 2, \ldots, \tag{VII}$$

and the constants K and K' depend only on p. We shall use the hypothesis $\int_{\partial\Omega} \log \delta(z)|dz| > -\infty$ to construct a bounded, nowhere vanishing, analytic function h in the domain Ω^* bounded by Γ^* having the following property: For almost all t, $|h|$ takes the boundary value $\Delta^{8/p}(\gamma(t))$ at $\gamma^*(t)$. Then, according to (VII), $|f_j h| \leq K'$, $j = 1, 2, \ldots$, almost everywhere with respect to arc length along Γ^*. It follows that $|f_j h| \leq K'$ everywhere in Ω^* and that the sequence $f_j h$, $j = 1, 2, \ldots$, is a normal family there. Hence f_j, $j = 1, 2, \ldots$, is also a normal family in Ω^* and so Γ^* has property (ii).

The function h is obtained as follows: Set $\lambda(\gamma^*(t)) = \Delta^{8/p}(\gamma(t))$ along Γ^*. Because $|\gamma^{*\prime}(t)| \leq K|\gamma'(t)|$, there is a constant K_1 such that

$$\int\limits_{\Gamma^*} \log \lambda(z)|dz| = \int\limits_0^1 \log \lambda(\gamma^*(t))|\gamma^{*\prime}(t)|dt \geq K_1 \int\limits_0^1 \log \delta(\gamma(t))|\gamma'(t)|dt =$$

$$= K_1 \int\limits_{\Gamma} \log \delta(z)|dz| > -\infty.$$

Suppose now that ϕ is the Riemann map from $|w| < 1$ onto Ω^*. Since Γ^* has a Lipschitz normal, it is known (see Kellogg [18, p. 123]) that ϕ extends to a continuously differentiable function on $|w| \leq 1$ and that there are positive constants C_1 and C_2 for which $0 < C_1 \leq |\phi'(w)| \leq C_2 < \infty$ everywhere on $|w| = 1$. In view of this, it follows that

$$\int\limits_{|w|=1} \log \lambda(\phi(w))|dw| > -\infty.$$

Hence, by a theorem of Szegö (see [17, p. 53]), there exists a bounded, nowhere vanishing, analytic function $\tilde{h}$ on $|w| < 1$ whose boundary values are almost everywhere equal in modulus to the composite function $\lambda \circ \phi$ on $|w| = 1$. Setting $h = \tilde{h} \circ \phi^{-1}$ we obtain the required function on Ω^* and the proof of necessity is complete. Q.E.D.

Proof of Theorem 4 — sufficiency. Assume that $\int_{\partial\Omega} \log \delta(z)|dz| = -\infty$. Fix p and let $k \in L^q(X \setminus \Omega, dA)$, $1/p + 1/q = 1$, have the property that $\int fk dA = 0$ for every $f \in H^p(X \setminus \Omega, dA)$. Thus, $\hat{k} = 0$ in $\mathbf{C} \setminus X$. To prove that $H^p(X \setminus \Omega, dA) = L^p_a(X \setminus \Omega, dA)$ it suffices, by the argument in Theorem 2, to show that $\hat{k} = 0$ in Ω. Furthermore, since the polynomials are complete in $L^s_a(X \setminus \Omega, dA)$ for $s \leq s_0$ whenever they are complete in $L^{s_0}_a(X \setminus \Omega, dA)$, we can assume that p is large, say $p \geq 2$. On the other hand, the proof can be significantly shortened when $1 \leq p < 2$ and for that reason this case is treated separately.

Suppose, therefore, that $1 \leq p < 2$. Then $k \in L^q(X \setminus \Omega, dA)$, $q > 2$, and $\hat{k}$ satisfies a Hölder condition

$$|\hat{k}(z_1) - \hat{k}(z_2)| \leq C|z_1 - z_2|^\mu,$$

where $0 < \mu < 1$ and z_1, z_2 are any two complex numbers. If $q = \infty$ any $\mu < 1$ will work; otherwise, we take $\mu = (q-2)/q$ (see [28, p. 38]). Hence $|\hat{k}(z)| \leq C\delta(z)^\mu$ and

$$\int_{\partial\Omega} \log |\hat{k}(z)||dz| \leq C' + \mu \int_{\partial\Omega} \log \delta(z)|dz| = -\infty.$$

If ϕ is the Riemann map from $|w| \leq 1$ onto $\Omega \cup \partial\Omega$ the composite function $\hat{k} \circ \phi$ is continuous on $|w| \leq 1$, analytic in $|w| < 1$ and

$$\int_{\partial\Omega} \log |\hat{k}(z)||dz| = \int_{|w|=1} \log |\hat{k}(\phi(w))||\phi'(w)||dw| = -\infty.$$

The analyticity of $\hat{k} \circ \phi$ on $|w| < 1$ is a consequence of the analyticity of $\hat{k}$ on Ω. As previously noted, there are constants C_1 and C_2 such that $0 < C_1 \leq |\phi'(w)| \leq C_2 < \infty$ on $|w| = 1$, since $\partial\Omega$ has a Lipschitz normal. Hence

$$\int_{|w|=1} \log |\hat{k}(\phi(w))||dw| = -\infty.$$

It follows from a well known theorem of Jensen, [17, p. 52], that $\hat{k} \circ \phi = 0$ in $|w| < 1$. Therefore, $\hat{k} = 0$ in Ω. This completes the argument when $1 \leq p < 2$.

Suppose now that $2 \leq p < \infty$. Then $k \in L^q(X \setminus \Omega, dA)$, $1 < q \leq 2$, and $\hat{k}$ need not satisfy a Hölder condition. It is not even clear whether $\int_{\partial\Omega} \log |\hat{k}(z)||dz|$ exists or not. To avoid these difficulties we do the following: Fix t_0, $0 < t_0 < \text{reach}(\partial\Omega)$. Let $\varrho(z) = \text{dist}(z, \partial\Omega)$ and let $\Omega_t = \{z \in \Omega: \varrho(z) \geq t\}$. Since $\hat{k}$ is analytic in Ω, the integral $\int_{\partial\Omega_t} \log |\hat{k}(z)||dz|$ exists. We shall rely upon Lemma 3, in lieu of a Hölder condition, to prove that $\int_{\partial\Omega_t} \log |\hat{k}(z)||dz| \to -\infty$ as $t \to 0$. From this we will deduce that $\hat{k} = 0$ in Ω and thereby establish the theorem. The first step is to multiply and divide $|\hat{k}(z)|^q$ by $\delta(z)^\varepsilon$, where $\varepsilon > 0$ has yet to be specified. This yields the identity

$$\int_{\partial\Omega_t} \log |\hat{k}(z)|^q|dz| = \int_{\partial\Omega} \log \delta(z)^\varepsilon|dz| + \int_{\partial\Omega_t} \log \left(\frac{|\hat{k}(z)|^q}{\delta(z)^\varepsilon}\right) |dz|.$$

As $t \to 0$, we shall see that the first integral on the right approaches $-\infty$ and that, for suitable ε, the second is uniformly bounded above for $0 < t < t_0$. The result is $\lim_{t\to 0} \int_{\partial\Omega_t} \log |\hat{k}(z)||dz| = -\infty$ as claimed.

In order to obtain the bound alluded to let us extend $\mathbf{n}(z)$ from $\partial\Omega$ to $\Omega\setminus\Omega_{t_0}$, by setting $\mathbf{n}(z) = -\operatorname{grad} \varrho(z)$. Note that for $z \in \partial\Omega_t$, $t \le t_0$, the vector $\mathbf{n}(z)$ is the unit normal pointing outward from Ω_t. Since

$$\int_{\partial\Omega_t} \log\left(\frac{|\hat{k}(z)|^q}{\delta(z)^\varepsilon}\right) |dz| < \int_{\partial\Omega_t} \frac{|\hat{k}(z)|^q}{\delta(z)^\varepsilon} |dz|,$$

we can direct our efforts toward finding a bound for the larger integral. Furthermore, by the divergence theorem

$$\begin{aligned}\int_{\partial\Omega_t} \frac{|\hat{k}(z)|^q}{\delta(z)^\varepsilon} \mathbf{n}(z)\cdot\mathbf{n}(z)|dz| + \int_{\partial\Omega_{t_0}} \frac{|\hat{k}(z)|^q}{\delta(z)^\varepsilon} \mathbf{n}(z)\cdot(-\mathbf{n}(z))|dz| &= \int_{\Omega_t\setminus\Omega_{t_0}} \operatorname{div}\left(\frac{|\hat{k}(z)|^q}{\delta(z)^\varepsilon}\mathbf{n}(z)\right) dA \\ &= \int_{\Omega_t\setminus\Omega_{t_0}} \frac{|\hat{k}(z)|^q}{\delta(z)^\varepsilon} \operatorname{div} \mathbf{n}(z) dA + \int_{\Omega_t\setminus\Omega_{t_0}} \operatorname{grad}\left(\frac{|\hat{k}(z)|^q}{\delta(z)^\varepsilon}\right)\cdot \mathbf{n}(z) dA \end{aligned} \tag{VIII}$$

and it therefore suffices to estimate the size of the latter two integrals in (VIII). The divergence theorem is valid in this situation because the vector field $|\hat{k}(z)|^q\delta(z)^{-\varepsilon}\mathbf{n}(z)$ is Lipschitz on $\Omega_t\setminus\Omega_{t_0}$. To verify this it is sufficient to prove that $\mathbf{n}$ is Lipschitz and that is best understood by considering the map $\xi\colon \Omega\setminus\Omega_{t_0} \to \partial\Omega$ which associates to a point $x \in \Omega\setminus\Omega_{t_0}$ the unique point of $\partial\Omega$ nearest to x. By a theorem of Federer, [11, p. 434], $|\xi(x) - \xi(y)| \le C|x - y|$ whenever $x, y \in \Omega\setminus\Omega_{t_0}$. Coupled with the assumption that $\mathbf{n}$ is Lipschitz on $\partial\Omega$, this implies

$$|\mathbf{n}(x) - \mathbf{n}(y)| = |\mathbf{n}(\xi(x)) - \mathbf{n}(\xi(y))| \le C'|\xi(x) - \xi(y)| \le C''|x - y|$$

provided $x, y \in \Omega\setminus\Omega_{t_0}$. This has the added implication that $|\operatorname{div} \mathbf{n}(z)| \le 2C''$ almost everywhere in $\Omega\setminus\Omega_{t_0}$. It follows from (VIII), the chain rule and these remarks that there exist constants C_1 and C_2 for which

$$\left|\int_{\Omega_t\setminus\Omega_{t_0}} \operatorname{div}\left(\frac{|\hat{k}(z)|^q}{\delta(z)^\varepsilon}\mathbf{n}(z)\right) dA\right| \le C_1 \int_{\Omega_t\setminus\Omega_{t_0}} \frac{|\hat{k}(z)|^q}{\delta(z)^{1+\varepsilon}} dA + C_2 \int_{\Omega_t\setminus\Omega_{t_0}} \frac{|\hat{k}(z)|^{q-1}}{\delta(z)^\varepsilon} |\operatorname{grad} |\hat{k}(z)||dA.$$

If $1 + \varepsilon < q$ it is easily seen, with the aid of Lemma 3, that the first term on the right is bounded by a constant which does not depend on t, $0 < t < t_0$. The computation is similar to a corresponding one in the proof of Theorem 2. By Hölder's inequality the second term does not exceed

$$C_2 \left\{ \int\limits_{\Omega_t \setminus \Omega_{t_0}} \frac{|\hat{k}(z)|^q}{\delta(z)^{p\varepsilon}}\, dA \right\}^{1/p} \left\{ \int\limits_{\Omega_t \setminus \Omega_{t_0}} |\mathrm{grad}\, |\hat{k}(z)||^q dA \right\}^{1/q}. \tag{IX}$$

If $p\varepsilon < q$ it follows once again from Lemma 3 that the first factor in (IX) admits a bound which is independent of t. Likewise, by the Calderon-Zygmund theory for singular integrals, [5] (see also [28, p.p. 68—70]), the second factor is less than $C_3\{\int |k|^q dA\}^{1/q}$. Thus, if ε is chosen so that $\varepsilon < q - 1 = q/p$ then

$$\int\limits_{\partial\Omega_t} \log \left(\frac{|\hat{k}(z)|^q}{\delta(z)^\varepsilon} \right) |dz| \leq M,$$

where M is a constant which does not depend on t, $0 < t < t_0$. Because $\xi\colon \partial\Omega_t \to \partial\Omega$ is a bi-Lipschitzian map for each $t \leq t_0$ and since the Lipschitz constants are independent of t, it is easy to see that $\lim_{t\to 0} \int_{\partial\Omega_t} \log \delta(z)^\varepsilon |dz| = -\infty$. Therefore, it follows as asserted earlier that $\lim_{t\to 0} \int_{\partial\Omega_t} \log |\hat{k}(z)||dz| = -\infty$.

At this point we would like to conclude that $\hat{k}$ vanishes identically in Ω. To see that this is indeed the case fix a point $x_0 \in \Omega_{t_0}$. Note that $x_0 \in \Omega_t$ for all $t \leq t_0$. For each t let ψ_t be the conformal map of $\Omega_t \cup \partial\Omega_t$ onto the closed unit disk $|w| \leq 1$ such that $\psi_t(x_0) = 0$ and $\psi_t'(x_0) > 0$. Since $\partial\Omega_t$ has a Lipschitz normal, it is possible to find constants μ_1 and μ_2 with the property that $0 < \mu_1 \leq |\psi_t'(z)| \leq \mu_2 < \infty$ for all $z \in \partial\Omega_t$. A priori, μ_1 and μ_2 will depend on t. But, once again since $\xi\colon \partial\Omega_t \to \partial\Omega$ is bi-Lipschitzian for $t \leq t_0$ and since the Lipschitz constants are independent of t, we can arrange that μ_1 and μ_2 are also independent of t. This follows from a theorem of Warschawski [29, Theorem III*, p. 327] (see also [29, Theorem V, p. 336]). Because $\int_{\partial\Omega_t} \log |\hat{k}(z)||dz| \to -\infty$ and $\int_{\partial\Omega_t} \log^+ |\hat{k}(z)||dz|$ is bounded as $t \to 0$, we can conclude that

$$\lim_{t\to 0} \int\limits_{\partial\Omega_t} \log |\hat{k}(z)||\psi_t'(z)||dz| = -\infty.$$

Now, $\log |\hat{k}(z)|$ is subharmonic in Ω and $(2\pi)^{-1}|\psi_t'(z)||dz|$ is the harmonic measure on $\partial\Omega_t$ which represents x_0 and so, as a function of t, $\int_{\partial\Omega_t} \log |\hat{k}(z)||\psi_t'(z)||dz|$ is monotone nondecreasing as $t \downarrow 0$ (see [10, p.p. 9 & 172]). Hence

$$\int\limits_{\partial\Omega_t} \log |\hat{k}(z)||\psi_t(z)||dz| = -\infty \quad \text{for every } t \leq t_0.$$

Consequently, $\int_{\partial\Omega_t} \log |\hat{k}(z)||dz| = -\infty$ and we can argue as in part one of this proof that $\hat{k} = 0$ in Ω. Q.E.D.

Remark. It has been called to our attention by J.-P. Ferrier that a compact X satisfying the conditions of Wermer's example (see Section 2) can also be obtained by taking X to be a thick Jordan arc lying in a »thin» crescent E with the property that the bounded component of $\mathbf{C}\setminus X$ contains the bounded component of $\mathbf{C}\setminus E$. A crescent E is said to be thin if $H^p(E, dA) = L^p_a(E, dA)$ for all p. The easiest way to obtain such a crescent, however, is to remove from the annulus $W = \{z: 1/2 \leq |z| \leq 1\}$ a sequence of wedges

$$W_j = \{z: 1/2 < |z| < r_j < 1, \ |\arg z| < \pi/2^j\}$$

such that if $E = W\setminus\bigcup_{j=1}^{\infty} W_j$ then $H^p(E, dA)$ fails to have a bounded evaluation at the origin (see [20, p. 116]). This construction is similar to and of the same order of difficulty as the one employed by Wermer.

Acknowledgment

I would like to express my gratitude to Professors Ronald Gariepy, W.D. Pepe and James H. Wells for many helpful conversations. I am also grateful to Professor John Wermer for supplying the example in Section 2.

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Received December 16, 1971

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