Intermediate spaces and the class $L \log^+ L$

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1. Introduction

The role in classical analysis of the Orlicz space $L \log^+ L$ is now well-known ([18]; I, pp. 33, 242, 267; II, p. 159) and, quite recently, O'Neil ([14]) has exhibited certain connections with the theory of interpolation of operators. In this note we show how the modern theory of interpolation methods relates to the space $L \log^+ L$ by describing the intermediate spaces $(L^1, L \log^+ L)_{\theta, q; K}$ and $(L \log^+ L, L^\infty)_{\theta, q; K}$ generated by the K-interpolation method of Peetre ([3], Chap. 3). In particular, we find interesting relationships (Corollaries C, E) between these and the Lorentz spaces. In fact the essential observation in the proofs of our results is that all three interpolated spaces L^1, $L \log^+ L$ and L^∞ are Lorentz Λ-spaces ([12]) which enables us ([15]) to identify the functional norm $K(t; f)$. An application of (some variants of) Hardy's inequality completes the characterization. Our main results are as follows:

THEOREM A. *The intermediate space* $(L^1, L \log^+ L)_{\theta, q; K}$, $0 < \theta < 1$, $1 \leq q \leq \infty$, *consists of all integrable functions* f *on* $[0, 1]$ *for which the norm*

$$\|f\| = \left\{ \int_0^1 [t(\log 1/t)^{\theta - 1/q} f^{**}(t)]^q dt/t \right\}^{1/q}$$

is finite.

COROLLARY B. *When* $0 < \theta < 1$, *we have* $(L^1, L \log^+ L)_{\theta, 1; K} = L (\log^+ L)^\theta$, *with equivalent norms.*

COROLLARY C. *When* $1 \leq q \leq \infty$, *we have* $(L^1, L \log^+ L)_{1/q, q; K} = L^{1q}$, *with equivalent norms.*

Note that Corollary C is an immediate consequence of Theorem A and the definition (3.2) of the Lorentz space L^{1q}. The proof of Corollary B is a little less direct and is postponed until § 7.

THEOREM D. *The intermediate space* $(L \log^+ L,\ L^\infty)_{\theta, q; K}$, $0 < \theta < 1$, $1 \le q \le \infty$, *consists of all integrable functions* f *on* $[0, 1]$ *for which the norm*

$$\|f\| = \left\{ \int_0^1 \left[t^{-\theta} \int_0^{\varphi^{-1}(t)} f^*(s) d\varphi(s) \right]^q dt/t \right\}^{1/q}$$

is finite, where $\varphi(t) = t(1 + \log 1/t)$. *An equivalent (quasi-)norm is*

$$\|f\| = \left\{ \int_0^1 [t^{1-\theta} f^*(\varphi^{-1}(t))]^q dt/t \right\}^{1/q}.$$

Since $\varphi(t)/t$ tends to infinity as $\log 1/t$ (as $t \to 0$), the next corollary follows immediately from the second of the assertions in Theorem D.

COROLLARY E. *When* $1 < p < \infty$, $1 \le q \le \infty$, *the space* $(L \log^+ L,\ L^\infty)_{1-1/p, q; K}$ *is properly contained in the Lorentz space* L^{pq}.

2. The Orlicz spaces $L\,(\log^+ L)^\theta$

For each θ, $0 \le \theta \le 1$, the class of measurable functions[1]) on $[0, 1]$ for which the quantity

$$\int_0^1 |f(x)|\ (\log^+ |f(x)|)^\theta dx \tag{2.1}$$

is finite will be denoted by $L\,(\log^+ L)^\theta$. When $\theta = 0$, $L\,(\log^+ L)^\theta$ is of course the familiar Lebesgue space L^1; when $\theta = 1$, we write $L \log^+ L$ instead of $L\,(\log^+ L)^1$. Thus ([9]), $L\,(\log^+ L)^\theta$ is the Orlicz space generated by the N-function $u \to u\,(\log^+ u)^\theta$ and as such is a Banach space under the usual Orlicz norm $\|\cdot\|_{(L \log L^+)^\theta}$ ([9], p. 67). We shall not make explicit use of this norm. Indeed, all we need to know is the fundamental function $t \to \varphi_1(t)$ of the space $L \log^+ L$ (i.e. the norm in $L \log^+ L$ of the characteristic function $\chi_{[0,t]}$ of the interval $[0, t]$, $0 \le t \le 1$), and a simple computation ([9], p. 72) shows this to be

$$\varphi_1(t) = t(1 - \log t), \quad 0 \le t \le 1. \tag{2.2}$$

[1]) As usual, functions that coincide a.e. are identified.

The Orlicz spaces $L(\log^+ L)^\theta$, $0 \leq \theta \leq 1$, are »close» to L^1 in the sense that $L^p \subseteq L(\log^+ L)^\theta \subseteq L^1$, for all $p > 1$; moreover, following [2], one can show that the indices ([1]) of $L(\log^+ L)^\theta$ are all equal to 1.

3. Lorentz L^{pq} spaces

The usual definition of the Lorentz space L^{pq} ([3], [4], [5], [8]) is that it consists of all (classes of) integrable functions f on [0, 1] for which the norm

$$\|f\| = \left\{ \int_0^\infty (t^{1/p} f^{**}(t))^q dt/t \right\}^{1/q} \tag{3.1}$$

is finite. However, since $f^*(t)$ is a priori undefined for $t > 1$, we shall adopt the more natural definition in which the norm (3.1) is replaced by

$$\|f\|_{pq} = \left\{ \int_0^1 (t^{1/p} f^{**}(t))^q dt/t \right\}^{1/q}. \tag{3.2}$$

Now for $t > 1$, $f^{**}(t)$ is simply an appropriate multiple of $1/t$ so when $1 < p \leq \infty$, the norms in (3.1) and (3.2) are equivalent. However, when $p = 1$, the space defined by (3.1) contains only the zero function (if $q < \infty$) whereas the corresponding Lorentz space L^{1q} defined by (3.2) is of much more interest. Indeed, we shall see that L^{11} is equivalent to $L \log^+ L$ while the spaces L^{1q} occur as the »diagonal» intermediate spaces between $L \log^+ L$ and L^1 (Corollary C).

4. The Hardy-Littlewood maximal function

When f is an integrable function on [0, 1] its Hardy-Littlewood maximal function Mf ([6], [7], [16], [17], [18]) is defined by

$$(Mf)(x) = \sup_{0<h<1} (1/2h) \int_{x-h}^{x+h} |f(y)| dy, \quad 0 \leq x \leq 1\,^{1)}. \tag{4.1}$$

The non-increasing rearrangement ([5], [8], [18]) of an integrable function f is denoted by f^* and the averaged rearrangement f^{**} is defined by

$$f^{**}(t) = (1/t) \int_0^t f^*(s) ds, \quad t > 0. \tag{4.2}$$

[1]) Take $f(y) = 0$ outside [0, 1].

The inequalities

$$f^{**}(2t) \leq M(f^*)(t) \leq f^{**}(t), \quad 0 < t \leq 1, \tag{4.3}$$

are easy consequences of the fact that f^* is non-increasing but the estimates

$$(Mf)^*(t) \leq 4f^{**}(t/2); \; f^{**}(t) \leq 6(Mf)^*(t), \quad 0 < t \leq 1, \tag{4.4}$$

due to Herz ([7]), lie much deeper and play a crucial role in the proof of the next theorem.

4.1. THEOREM. *For any number* θ, $0 < \theta \leq 1$, *and any integrable function* f *on* $[0, 1]$ *the following statements are equivalent:*

(a) $f \in L\,(\log^+ L)^\theta$;

(b) $\displaystyle\int_0^1 |f(x)|\,(\log^+ |f(x)|)^\theta dx < \infty$;

(c) $\displaystyle\int_0^1 f^*(t)\,(\log^+ f^*(t))^\theta dt < \infty$;

(d) $\displaystyle\int_0^1 f^*(t)\,(\log 1/t)^\theta dt < \infty$;

(e) $\displaystyle\int_0^1 f^{**}(t)\,(\log 1/t)^{\theta-1} dt < \infty$;

(f) $\displaystyle\int_0^1 (Mf)^*(t)\,(\log 1/t)^{\theta-1} dt < \infty$.

Proof. That (a) and (b) are equivalent is a matter of definition (cf. (2.1)), and (b) and (c) are equivalent because f and f^* are equimeasurable. If (c) holds then (cf. [18], I, p. 34) let $E = \{t{:}\, f^*(t) \leq t^{-1/2}\}$ and $F = \{t{:}\, f^*(t) > t^{-1/2}\}$. We have

$$\int_0^1 f^*(t)\,(\log 1/t)^\theta dt = \int_E + \int_F \leq \text{const.} + 2^\theta \int_0^1 f^*(t)\,(\log f^*(t))^\theta dt < \infty,$$

and so (c) $\Rightarrow$ (d). The converse is a direct consequence of the fact that for every integrable function f, the function $t \to tf^*(t)$ is bounded. The quantities $\theta \int_0^1 f^*(t)\,(\log 1/t)^\theta dt$ and $\int_0^1 f^{**}(t)\,(\log 1/t)^{\theta-1} dt$ are the same (use (4.2) and change the order of integration) so (d) and (e) are equivalent. Finally, the equivalence of (e) and (f) follows directly from (4.4). This completes the proof.

4.2. *Remarks.* (i) When $\theta = 1$, we deduce from parts (a) and (e) of Theorem 4.1 that $f \in L \log^+ L$ if and only if f^{**} is integrable, a result of Hardy and Littlewood ([6], Theorem 11 (iii)). Hence, by (3.1), the spaces $L \log^+ L$ and L^{11} contain the same (classes of) functions. (ii) The equivalence of the parts (a) and (f) shows that the maximal function Mf is integrable if and only if $f \in L \log^+ L$. This somewhat deeper result is due to Stein ([16], [17]) but was later established by Herz ([7]) using the methods indicated here. We shall not need to use part (f) in what follows; the remaining assertions of Theorem 4.1 are then elementary because they do not require (4.4).

5. Lorentz Λ-spaces

When X is a rearrangement-invariant space ([1], [5], [15]) on [0, 1], the fundamental function φ_X of X is defined by $\varphi_X(t) = \|\chi_t\|_X$, $0 \leq t \leq 1$, where χ_t is the characteristic function of the interval $[0, t]$. The Lorentz space $\Lambda(X)$ associated with X consists of all (classes of) integrable functions f on [0, 1] for which the norm

$$\|f\|_{\Lambda(X)} = \int_0^1 f^*(t) d\varphi_X(t) \tag{5.1}$$

is finite (cf. [15]). Observe that $\Lambda(X) = X$ if $X = L^1$ or L^∞; the subsequent theory depends crucially upon the fact that this remains true when $X = L \log^+ L$.

5.1. Theorem. *Up to equivalence of norms we have*

$$\Lambda(L \log^+ L) = L \log^+ L = L^{11}. \tag{5.2}$$

Proof. We have already observed (cf. Remark 4.2) that $L \log^+ L$ and L^{11} are the same. For the other half of (5.2), note that by (2.2) the fundamental function of the space $L \log^+ L$ is $\varphi(t) = t(1 + \log 1/t)$. Thus, from (5.1), we see that an integrable function f belongs to $\Lambda(L \log^+ L)$ if and only if $\int_0^1 f^*(t)(\log 1/t)dt$ is finite, and by Theorem 4.1 this occurs precisely when f belongs to $L \log^+ L$. Thus, the three spaces $\Lambda(L \log^+ L)$, $L \log^+ L$ and L^{11} coincide algebraically. But any two Banach function spaces consisting of the same functions have equivalent norms (cf. [11], § 2) so the proof is complete.

5.2. *Remark.* Theorem 5.1 is already implicit in the more general results of Lorentz ([11]) which describe the precise circumstances under which an Orlicz space can be equal to a Lorentz space; we included the proof in our special case because it is elementary. Roughly speaking, Lorentz' results show that an Orlicz space can coincide with a Λ-space if and only if it is »close» to L^1; in particular, the spaces $L^p(\log^+ L)$, $p > 1$, are not Λ-spaces and so the techniques used below do not extend to the case $p > 1$.

When $(\mathscr{A}, \mathscr{B})$ is a compatible couple of Banach spaces the K-functional norm of Peetre ([3], Chap. 3) is defined on $\mathscr{A} + \mathscr{B}$ by

$$K(t; f) = K(t; f; \mathscr{A}; \mathscr{B}) = \inf (\|g\|_{\mathscr{A}} + t\|h\|_{\mathscr{B}}), \quad 0 < t < \infty,$$

where the infimum is taken over all possible representations $f = g + h$ of f with $g \in \mathscr{A}$ and $h \in \mathscr{B}$. We shall need to use the elementary inequality

$$K(t; f) \leq \max (1, t/s) K(s; f), \quad 0 < s, t < \infty. \tag{5.3}$$

The importance of the identity (5.2) is that it allows us to regard the couple $(L^1, L \log^+ L)$ (or $(L \log^+ L, L^\infty)$ as a pair of Lorentz Λ-spaces and in this case it is possible to compute $K(t; f)$. Indeed, if X and Y are rearrangement-invariant spaces then

$$K(t; f; \Lambda(X), \Lambda(Y)) = \int_0^1 f^*(s) d \{\min (\varphi_X(s), \ t\varphi_Y(s))\}. \tag{5.4}$$

This result is stated without proof in [12], and in a special case in [1]; a proof is supplied in [15]. We apply (5.4) in case $X = L^1$ and $Y = L \log^+ L$.

5.3. THEOREM. *For each* $f \in L^1$ *we have*

$$K(t; f; L^1; L \log^+ L) = \begin{cases} \displaystyle\int_0^{e(t)} f^*(s) ds + t \int_{e(t)}^1 (\log 1/s) f^*(s) ds, & 0 < t \leq 1, \\ \displaystyle\int_0^1 f^*(s) ds, & t > 1, \end{cases} \tag{5.5}$$

where $e(t) = \exp (1 - 1/t)$.

We shall, for convenience, write $K(t; f)$ for $K(t; f; L^1; L \log^+ L)$. Note that when $0 < t \leq 1$, an integration by parts of the second integral gives

$$\int_0^{e(t)} f^*(s) ds + t(1 - t)^{-1} \int_{e(t)}^1 (\log 1/s) f^*(s) ds = t(1 - t)^{-1} \int_{e(t)}^1 f^{**}(s) ds, \tag{5.6}$$

and this leads to the following characterization of $K(t; f)$ in terms of f^{**}.

5.4. THEOREM. *For each* $f \in L^1$ *we have*

$$K(t;f) \sim t(1-t)^{-1} \int_{e(t)}^{1} f^{**}(s)ds, \quad 0 < t < 1; \tag{5.7}$$

more precisely

$$(2e)^{-1}t(1-t)^{-1} \int_{e(t)}^{1} f^{**}(s)ds \leq K(t;f) \leq t(1-t)^{-1} \int_{e(t)}^{1} f^{**}(s)ds. \tag{5.8}$$

Proof. The second inequality follows directly from Theorem 5.3 and the identity (5.6). For the other one we consider two cases. First, if $0 < t \leq 1/2$, then $(1-t)^{-1} \leq 2$ so from (5.5) and (5.6)

$$t(1-t)^{-1} \int_{e(t)}^{1} f^{**}(s)ds \leq \int_{0}^{e(t)} f^{*}(s)ds + 2t \int_{e(t)}^{1} (\log 1/s) f^{*}(s)ds \leq 2K(t;f),$$

and this establishes (5.8) in case $0 < t \leq 1/2$. If now $\frac{1}{2} < t \leq 1$, then with $x = t^{-1}(1-t)$

$$t(1-t)^{-1} \int_{e(t)}^{1} f^{**}(s)ds = x^{-1} \int_{e^{-x}}^{1} f^{**}(s)ds \leq x^{-1}(1-e^{-x}) f^{**}(e^{-x}) \leq f^{**}(e^{-x}) \leq e^{x} f^{**}(1).$$

But (5.5) shows that $f^{**}(1) = K(1;f)$ so using (5.3) we deduce that

$$t(1-t)^{-1} \int_{e(t)}^{1} f^{**}(s)ds \leq e(t)K(1;f) \leq 2eK(\tfrac{1}{2};f) \leq 2eK(t;f), \quad \tfrac{1}{2} < t < 1.$$

This completes the proof.

6. Hardy's inequality

The following integral inequalities, which will be needed later to identify certain intermediate spaces, are collectively referred to as Hardy's inequality ([3], p. 199; [8], p. 256).

THEOREM 6.1 (Hardy). *Let* $\alpha > 0$, $1 \leq q \leq \infty$. *If* $\psi(s)$ *is a non-negative measurable function on* $(0, \infty)$ *then*

$$\left\{ \int_{0}^{\infty} \left(t^{-\alpha} \int_{0}^{t} \psi(s)ds \right)^{q} dt/t \right\}^{1/q} \leq \alpha^{-1} \left\{ \int_{0}^{\infty} (t^{1-\alpha}\psi(t))^{q} dt/t \right\}^{1/q} \tag{6.1}$$

and

$$\left\{\int_0^\infty \left(t^\alpha \int_t^\infty \psi(s)ds\right)^q dt/t\right\}^{1/q} \le \alpha^{-1}\left\{\int_0^\infty (t^{1+\alpha}\psi(t))^q dt/t\right\}^{1/q}. \tag{6.2}$$

In the next theorem we establish a useful variant of Hardy's inequality. For reasons of brevity we shall denote by μ the measure given by

$$d\mu(t) = dt/[t(1-\log t)].$$

THEOREM 6.2. *Let* $\alpha > 0$, $1 \le q \le \infty$, $\psi \ge 0$. *Then*

$$\left\{\int_0^1 \left[(1-\log t)^\alpha \int_0^t \psi(s)ds\right]^q d\mu(t)\right\}^{1/q} \le \alpha^{-1}\left\{\int_0^1 [t(1-\log t)^{1+\alpha}\psi(t)]^q d\mu(t)\right\}^{1/q} \tag{6.3}$$

and

$$\left\{\int_0^1 \left[(1-\log t)^{-\alpha} \int_t^1 \psi(s)ds\right]^q d\mu(t)\right\}^{1/q} \le \alpha^{-1}\left\{\int_0^1 [t(1-\log t)^{1-\alpha}\psi(t)]^q d\mu(t)\right\}^{1/q}. \tag{6.4}$$

Proof. The proof is modeled on that given in [8] for the classical Hardy inequalities (6.1), (6.2) so we shall only sketch the details. If $1 < q < \infty$ and $1/q + 1/q' = 1$ we write

$$\int_0^t \psi(s)ds = \int_0^t [s^{1-1/q}(1-\log s)^\beta \psi(s)][s^{-1/q'}(1-\log s)^{-\beta}]ds$$

and apply Hölder's inequality to obtain (provided $\beta q' < 1$)

$$\left(\int_0^t \psi(s)ds\right)^q \le (\beta q' - 1)^{1-q}(1-\log t)^{q(1-\beta)-1}\left(\int_0^t [s(1-\log s)^\beta \psi(s)]^q ds/s\right).$$

Using this to estimate the left-hand side, say I, of (6.3) we have

$$I \le \left\{(\beta q' - 1)^{1-q}\int_0^1 (1-\log t)^{(\alpha+1-\beta)q-2} dt/t \int_0^t [s(1-\log s)^\beta \psi(s)]^q ds/s\right\}^{1/q}.$$

An interchange in the order of integration gives

$$I \le \{(\beta q' - 1)^{q-1}[(\alpha + 1 - \beta)q - 1]\}^{-1/q}\left\{\int_0^1 [s(1-\log s)^{1+\alpha-1/q}\psi(s)]^q ds/s\right\}^{1/q},$$

provided $(\alpha + 1 - \beta)q > 1$. It remains therefore only to minimize the constant term (subject to $1/q' < \beta < \alpha + 1/q'$), and a simple computation shows that the minimum value is α^{-1} (occurring at $\beta = (\alpha + 1)/q'$). The cases $q = 1$ and $q = \infty$ are easier so we omit the details. The inequality (6.4) can then be established in much the same way.

7. The intermediate spaces $(L^1, L \log^+ L)_{\theta, 1; K}$

Denote by $\|f\|_{\theta, 1; K}$ the norm of a function in the space $(L^1, L \log^+ L)_{\theta, 1; K}$. Then by (5.5)

$$\|f\|_{\theta, 1; K} \equiv \int_0^\infty t^{-\theta} K(t; f) dt/t = \theta^{-1} \int_0^1 f^*(s) ds + \int_0^1 t^{-\theta-1} K(t; f) dt. \tag{7.1}$$

Now, again using (5.5) and the change of variable $u = e(t)$ we have

$$\int_0^1 t^{-\theta-1} K(t; f) dt =$$

$$= \int_0^1 (1 - \log u)^\theta \int_0^u f^*(s) ds \, d\mu(u) + \int_0^1 (1 - \log u)^{\theta-1} \int_u^1 \log 1/s \, f^*(s) ds \, d\mu(u).$$

A change in the order of integration gives

$$\int_0^1 t^{-\theta-1} K(t; f) dt =$$

$$= \theta^{-1} \int_0^1 f^*(s)[(1 - \log s)^\theta - 1] ds + (1 - \theta)^{-1} \int_0^1 f^*(s)(\log 1/s)(1 - \log s)^{\theta-1} ds$$

which, combined with (7.1), shows that

$$\int_0^1 f^*(s)(1 - \log s)^\theta ds \le \|f\|_{\theta, 1; K} \le [\theta(1 - \theta)]^{-1} \int_0^1 f^*(s)(1 - \log s)^\theta ds.$$

Thus a function f belongs to $(L^1, L \log^+ L)_{\theta, 1; K}$ if and only if $\int_0^1 f^*(s)(1 - \log s)^\theta ds$ is finite; it follows immediately from Theorem 4.1 that the spaces $(L^1, L \log^+ L)_{\theta, 1; K}$ and $L (\log^+ L)^\theta$ coincide. This completes the proof of Corollary B.

8. The intermediate spaces $(L^1,\ L\log^+ L)_{\theta,q;K}$

We shall show that the space $X_{\theta,q}$, which by definition consists of all (classes of) measurable functions for which the norm

$$\|f\|_{\theta,q} = \int_0^1 f^*(t)dt + \left\{\int_0^1 [t(1-\log t)^{\theta-1/q} f^{**}(t)]^q dt/t\right\} \tag{8.1}$$

is finite, coincides with the intermediate space $(L^1,\ L\log^+ L)_{\theta,q;K}$. Since the finiteness of the second integral in (8.1) is not affected if $1-\log t$ is replaced by $-\log t$, this is precisely the assertion of Theorem A.

8.1. THEOREM. *Let* $0<\theta<1$, $1\le q\le\infty$. *Then* $(L^1,\ L\log^+ L)_{\theta,q;K} = X_{\theta,q}$, *with equivalent norms.*

Proof. Let us show first that $X_{\theta,q}\subseteq (L^1,\ L\log^+ L)_{\theta,q;K}$. If $f\in X_{\theta,q}$, then by (5.5), (5.8) and Minkowski's inequality

$$\|f\|_{\theta,q;K} \equiv \left\{\int_0^\infty [t^{-\theta}K(t;f)]^q dt/t\right\}^{1/q}$$
$$\le \left\{\int_0^1 \left[t^{\theta-1}(1-t)^{-1}\int_{e(t)}^1 f^{**}(s)ds\right]^q dt/t\right\}^{1/q} + \left\{\int_1^\infty \left[t^{-\theta}\int_0^1 f^*(s)ds\right]^q dt/t\right\}^{1/q}$$

so, if I denotes the first term, we have

$$\|f\|_{\theta,q;K} \le I + (\theta q)^{-1/q}\int_0^1 f^*(s)ds. \tag{8.2}$$

Now the change of variable $u=e(t)$ gives

$$I = \left\{\int_0^1 \left[(1-\log u)^{\theta}(-\log u)^{-1}\int_u^1 f^{**}(s)ds\right]^q d\mu(u)\right\}^{1/q}$$
$$\le 2\left\{\int_0^{e^{-1}} \left[(1-\log u)^{\theta-1}\int_u^1 f^{**}(s)ds\right]^q d\mu(u)\right\}^{1/q} \tag{8.3}$$
$$+\, 2\left\{\int_{e^{-1}}^1 \left[(-\log u)^{-1}\int_u^1 f^{**}(s)ds\right]^q e\,du\right\}^{1/q} = J_1 + J_2,\ \text{say.}$$

We use the Hardy inequality (6.4) to obtain the estimate

$$J_1 \leq 2(1-\theta)^{-1}\left\{\int_0^1 [u(1-\log u)^\theta f^{**}(u)]^q d\mu(u)\right\}^{1/q} \tag{8.4}$$

while for the second term we use the inequality $\log u \leq u - 1$ to obtain

$$J_2 \leq (e-1)^{1/q} \sup_{e^{-1} \leq u \leq 1} \left\{(-\log u)^{-1} \int_u^1 f^{**}(s)ds\right\}$$

$$\leq (e-1)^{1/q} \sup_{e^{-1} \leq u \leq 1} \left\{(1-u)^{-1} \int_u^1 f^{**}(s)ds\right\} \leq (e-1)^{1/q} f^{**}(e^{-1}).$$

Therefore

$$J_2 \leq (e-1)^{1/q} e \int_0^{e^{-1}} f^*(s)ds \leq e^2 \int_0^1 f^*(s)ds. \tag{8.5}$$

Collecting the estimates (8.3), (8.4) and (8.5) we deduce from (8.1) and (8.2) that

$$\|f\|_{\theta, q; K} \leq c_{\theta, q}\left(\int_0^1 f^*(t)dt + \left\{\int_0^1 [t(1-\log t)^\theta f^{**}(t)]^q d\mu(t)\right\}^{1/q}\right) = c_{\theta, q}\|f\|_{\theta, q}, \tag{8.6}$$

which establishes the desired inclusion.

It remains only to show that if $\|f\|_{\theta, q; K}$ is finite then so is $\|f\|_{\theta, q}$. But, again from (5.8),

$$\|f\|_{\theta, q; K} \geq c\left\{\int_0^1 \left[(1-\log u)^{\theta-1} \int_u^1 f^{**}(s)ds\right]^q d\mu(u)\right\}^{1/q} \tag{8.7}$$

and

$$\int_u^1 f^{**}(s)ds = \int_u^1 s^{-1}ds \int_0^s f^*(t)dt \geq \int_u^1 s^{-1}ds \int_0^u f^*(t)dt = -u(\log u) f^{**}(u)$$

so from (8.7) we have

$$\|f\|_{\theta, q; K} \geq c\left\{\int_0^1 [(1-\log u)^{\theta-1} u(-\log u) f^{**}(u)]^q d\mu(u)\right\}. \tag{8.8}$$

Thus if $\|f\|_{\theta, q; K}$ is finite, so is the integral in (8.8) and hence, using once more the fact that $(1-\log u)$ and $-\log u$ are asymptotically the same as $u \to 0$, so is $\|f\|_{\theta, 1}$. This completes the proof of Theorem 8.1 and therefore, as we remarked above, of Theorem A.

9. The intermediate spaces $(L\log^+ L,\ L^\infty)_{\theta, q; K}$

Recall that the fundamental function of the space $L \log^+ L$ is $\varphi(t) = t(1 - \log t)$ so from (5.4) we have

$$K(t; f; L\log^+ L,\ L^\infty) = \int_0^{\varphi^{-1}(t)} f^*(s) d\varphi(s) \tag{9.1}$$

(cf. the expression for the K-norm of the pair (L^p, L^∞) given in [10]; see also [13]). Hence

$$\|f\|_{\theta, q; K} = \left\{\int_0^\infty \left[t^{-\theta} \int_0^{\varphi^{-1}(t)} f^*(s)d\varphi(s)\right]^q dt/t\right\}^{1/q}, \tag{9.2}$$

the first of the assertions of Theorem D. For the second we note that the change of variable $u = \varphi(s)$ in (9.2) gives

$$\|f\|_{\theta, q; K} = \left\{\int_0^\infty \left[t^{-1} \int_0^{t} F^*(u)du\right]^q dt/t\right\}^{1/q},$$

where $F^*(u) = f^*(\varphi^{-1}(u))$, so from Hardy's inequality (6.1) we have

$$\|f\|_{\theta, q; K} \leq \theta^{-1}\left\{\int_0^\infty [t^{1-\theta} f^*(\varphi^{-1}(t))]^q dt/t\right\}^{1/q}. \tag{9.3}$$

Finally, it is easy to check that $tf^*(\varphi^{-1}(t)) \leq \int_0^{\varphi^{-1}(t)} f^*(s)d\varphi(s)$ so from (9.2)

$$\left\{\int_0^\infty [t^{1-\theta} f^*(\varphi^{-1}(t))]^q dt/t\right\}^{1/q} \leq \|f\|_{\theta, q; K}. \tag{9.4}$$

The second assertion of Theorem D now follows from (9.3) and (9.4).

Remarks. (i) From the hypotheses $T: L^1 \to$ weak-L^1, $T: L^\infty \to L^\infty$ and Marcinkiewicz' theorem ([5], [8]), it follows that $T: L^{pq} \to L^{pq}$, $1 < p < \infty$, $1 \leq q \leq \infty$. Under the weaker (cf. [14]) hypotheses $T: L\log^+ L \to L^1$, $T: L^\infty \to L^\infty$, it is no longer true that $T: L^{pq} \to L^{pq}$ (take $Tf = f \circ \varphi^{-1}$); in this case, the interpolation theorem corresponding to Theorem D shows that if $T: L\log^+ L \to L^1$, $T: L^\infty \to L^\infty$, then $T: (L\log^+ L,\ L^\infty)_{1-1/p, q, K} \to L^{pq}$.

(ii) Because of the stability theorem, the intermediate spaces between pairs (L^{pq}, L^{rs}) of Lorentz spaces depend only upon p and r (and they are again Lorentz spaces, cf. [10], p. 160) whenever $1 < p, r < \infty$. Thus the intermediate spaces depend only upon the indices ([1]) of L^{pq} and L^{rs}. We see from Corollary E that this is not true in general since the spaces $L \log^+ L$ and L^1 have the same indices and yet the intermediate spaces for the couples $(L \log^+ L,\ L^\infty)$ and (L^1, L^∞) are different.

(iii) From the point of view of incorporating classical results such as those of Marcinkiewicz and O'Neil ([14]) into the theory of interpolation methods it would be interesting to have analogues of Theorems A and D where $L \log^+ L$ is replaced by weak-L^1. The difficulty is not so much that weak-L^1 is not normable (cf. [10]) but that weak-L^1 is not a Λ-space, so the methods presented here do not apply. We hope to return to this question at a later time.

Note added in proof (july 73): Some results pertaining to Remark (iii) above are announced in the author's paper »Estimates for weak-type operators», which is to appear in Bull. Amer. Math. Soc. 79 (5) (1973).

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