On fundamental solutions supported by a convex cone

ARNE ENQVIST

1. Introduction

Let $P(D)$ be a partial differential operator in $\mathbf{R}^n$ with constant coefficients and Γ a closed convex cone in $\mathbf{R}^n$. Thus we assume that $x, y \in \Gamma$ and $s, t \geq 0$ implies that $sx + ty \in \Gamma$. The problem discussed here is to decide when $P(D)$ has a fundamental solution with support in Γ.

When Γ is a proper cone, that is, when Γ contains no straight line, this condition means precisely that $P(D)$ is hyperbolic with respect to the supporting planes of Γ which meet Γ only at the origin (Gårding [4], see also Hörmander [5, Theorem 5.6.2]). In the other extreme case where Γ is a half space sufficient conditions were given long ago by Petrowsky (see Gelfand — Shilov [2]), and a complete answer to the question was obtained by Hörmander [6].

In general the intersection $\Gamma \cap (-\Gamma) = W$ is a linear subspace and $x \in \Gamma$ implies $x + y \in \Gamma$ for every $y \in W$. This shows that Γ is the inverse image in $\mathbf{R}^n$ of the image V of Γ in $\mathbf{R}^n/W$ under the quotient map. It is clear that V is a proper cone. We shall use the notations $n' = \dim W$, $n'' = n - n'$ and coordinates $x = (x', x'')$ such that W is defined by $x'' = 0$. Also for $n' > 0$ and $n'' > 1$ sufficient conditions for the existence of a fundamental solution of $P(D)$ with support in Γ, analogous to those of Petrowsky for $n'' = 1$, have been given by Gindikin [3]. We shall extend these in the direction suggested by the technique used by Hörmander [6]. However, when $n'' > 1$ there are polynomials such that $P(\zeta', D'')$ is not hyperbolic for any ζ'. This introduces a new difficulty and in consequence of this the result is far from complete.

In Section 2 we investigate the general necessary conditions. The methods used are very close to those of Hörmander [6]. In the hyperbolic case the principal part plays a very important role. (See L. Svensson [9].) Here the principal part does not give so much information about the polynomial, and we have not been able to find any substitute. However, in Section 3 we study some stability properties of the necessary conditions which allow us to carry them over to various polynomials related to the behavior of P at infinity.

In Section 4 we reduce the existence theorems to a priori estimates. To be able to prove these estimates, we are forced to assume that $P(\zeta', D'')$ in some way can be written as a product of hyperbolic polynomials. In Section 5 we give general sufficient conditions of that type and in Section 6 we investigate polynomials that are partially homogeneous.

When $\deg P = 2$, we solve our problem completely. (Section 7.) We have also tried to find new methods to prove the existence of fundamental solutions. In Section 8 we present a constructive method in the case of two variables.

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2. General necessary conditions

Let $P(D)$ be a partial differential operator in $\mathbf{R}^n$ with constant coefficients and let V be a proper, closed and convex cone in $\mathbf{R}^{n''}$. Set $\Gamma = \mathbf{R}^{n'} \times V$, $x = (x', x'') \in \mathbf{R}^{n'} \times \mathbf{R}^{n''}$ and $|x| = \max_{1 \leq j \leq n} |x_j|$. For every real number A we define the set V_A^* as follows

$$V_A^* = \{\eta'' \in \mathbf{R}^{n''};\ \langle \eta'', x'' \rangle \geq A,\ |x''| = 1,\ x'' \in V\}.$$

THEOREM 2.1. *Assume that $P(D)$ has a fundamental solution with support in Γ. Then the following condition is satisfied.*

(2.1) *There is a constant C such that the following is true: Let $\zeta' \to \zeta''(\zeta')$ be any analytic function such that $P(\zeta', \zeta''(\zeta')) = 0$, $\operatorname{Im} \zeta''(\zeta') \in -V_A^*$ and $|(\zeta', \zeta''(\zeta'))| \leq M$ for all $\zeta' \in \Omega(\xi_0', R) = \{\zeta' \in \mathbf{C}^{n'};\ |\zeta' - \xi_0'| < R\}$ where $\xi_0' \in \mathbf{R}^{n'}$ and $R > 0$. Then $\min(R, A) \leq C \log(2 + M)$.*

We need some lemmas for the proof.

LEMMA 2.2. *If $P(D)$ has a fundamental solution with support in Γ, then for every compact neighbourhood K of zero there are constants C and μ such that*

$$|u(0)| \leq C \sum_{|\alpha| \leq \mu} \sup_{-\Gamma} |D^\alpha P(D) u|, \quad u \in C_0^\infty(K). \tag{2.2}$$

Proof. If $P(D)E = \delta$, $\operatorname{supp} E \subset \Gamma$ then $u(0) = \check{E}(P(D)u)$. Since the support of $\check{E}$ is a subset of $-\Gamma$, and Γ is regular in the sense of Whitney, we conclude that (2.2) holds for arbitrary K. (See Schwartz [8, p. 98].)

LEMMA 2.3. *Let $P(D)$ satisfy condition* (2.2) *and let $\chi \in C_0^\infty(K)$ be equal to* 1 *in a neighbourhood of zero. Set* $K' = (-\Gamma) \cap \operatorname{supp}(d\chi)$. *Then there exist constants* C *and* μ, *such that for all* $u \in C^\infty(\mathbf{R}^n)$ *with* $P(D)u = 0$

$$|u(0)| \leq C \sum_{|\alpha| \leq \mu} \sup_{K'} |D^\alpha u|. \tag{2.3}$$

Proof. We apply (2.2) to the function $v = \chi u$, noting that $P(D)v = \sum_{\alpha \neq 0} (P^{\alpha}(D)u)D^\alpha \chi/\alpha!$.

LEMMA 2.4. *Let* $\chi_N(t) = N$ *when* $|t| \leq 1/2N$ *and* $\chi_N(t) = 0$ *otherwise,* $t \in \mathbf{R}$. *If* φ^N *is the convolution of* N *factors* χ_N *we have*

(i) $\operatorname{supp} \varphi^N \subset (-\frac{1}{2}, \frac{1}{2})$; $\varphi^N \geq 0$, $\int \varphi^N dt = 1$,

(ii) $d^k\varphi^N(t)/dt^k$ *is a measure with total mass* $\leq (2N)^k$ *when* $0 \leq k \leq N$.

Proof. See the proof of Lemma 2.2 in Hörmander [6].

In the following lemma we use the notation

$$\Phi_R^N(\xi') = R^{-n'}\varphi^N(\xi_1/R) \ldots \varphi^N(\xi_{n'}/R).$$

LEMMA 2.5. *Let* $\xi_0' \in \mathbf{R}^{n'}$, *and let* F *be a function that is analytic in the polydisc* $\Omega = \Omega(\xi_0', R)$. *Set*

$$u^N(x') = \int e^{i\langle x', \xi'\rangle} F(\xi')\, \Phi_R^N(\xi' - \xi_0')d\xi'.$$

Then

$$|x'|^k|u^N(x')| \leq (4N/R)^k \sup_{\Omega} |F(\zeta')|, \ 0 \leq k \leq N. \tag{2.4}$$

Proof. See the proof of Lemma 2.3 in Hörmander [6].

Proof of Theorem 2.1. Consider

$$u^N(x) = \int e^{i\langle x', \xi'\rangle + i\langle x'', \zeta''(\xi')\rangle} \Phi_R^N(\xi' - \xi_0')d\xi'.$$

It is clear that $P(D)u^N = 0$ and that $u^N(0) = 1$. Let K' be the set in Lemma 2.3. We have to estimate u^N and its derivatives in K'. By hypothesis $\langle x'', \operatorname{Im} \zeta''(\zeta')\rangle \geq 0$ for $x \in K'$ and $\zeta' \in \Omega(\xi_0', R)$, and furthermore there is a constant $\delta > 0$, such that $|x'| > \delta$ or $\langle x'', \operatorname{Im} \zeta''(\zeta')\rangle \geq A\delta$ for all $\zeta' \in \Omega(\xi_0', R)$. Set $K_1 = \{x \in K' ; |x'| > \delta\}$ and $K_2 = \{x \in K' ; \langle x'', \operatorname{Im} \zeta''(\zeta')\rangle \geq \delta A$ for all $\zeta' \in \Omega(\xi_0', R)\}$. Using Lemma 2.5 with $k = N$ we obtain

$$|u^N(x)| \leq (4N/\delta R)^N, \ \ x \in K_1.$$

Differentiation of u^N gives factors, which are coordinates of ζ' or $\zeta''(\zeta')$. Since these are bounded by M we obtain

$$\sum_{|\alpha|\leq\mu} \sup_{K_1} |D^\alpha u^N| \leq C(1+M)^\mu(4N/\delta R)^N.$$

Similarly we obtain

$$\sum_{|\alpha|\leq\mu} \sup_{K_2} |D^\alpha u^N| \leq C(1+M)^\mu e^{-\delta A}.$$

If we use this in Lemma 2.3 and observe that $u^N(0) = 1$ it follows that

$$1 \leq C(1+M)^\mu((4N/\delta R)^N + e^{-\delta A}). \tag{2.5}$$

We choose the integer N so that $(\delta R)/(8e) \leq N \leq (\delta R)/(4e)$ which is possible if $R > (8e)/\delta$. Then we obtain that

$$1 \leq C(1+M)^\mu e^{-C_1 \min(R,A)}, \text{ where } C_1 = \min(1, \delta/(8e)),$$

and this completes the proof.

The conditions in Theorem 2.1 are hard to apply since they involve rather general polydiscs in $P^{-1}(0)$. However, when the function $\zeta' \to \zeta''(\zeta')$ is algebraic of bounded degree and Γ is semialgebraic, it is possible to sharpen these conditions. (A semialgebraic set is a set that can be defined by finitely many real polynomial equations and inequalities.)

THEOREM 2.6. *Let q_0 be an integer and assume that $P(D)$ has a fundamental solution with support in the semialgebraic, closed and convex cone Γ. Then there are constants A_0 and R_0 such that if the function $\mathbf{C}^{n'} \ni \zeta' \to \zeta''(\zeta') \in \mathbf{C}^{n''}$ is analytic and algebraic of order $\leq q_0$ in $\Omega(\xi_0', R_0)$ ($\xi_0' \in \mathbf{R}^{n'}$) and if $P(\zeta', \zeta''(\zeta')) = 0$ in $\Omega(\xi_0', R_0)$, then there is at least one point $\zeta' \in \Omega(\xi_0', R_0)$ such that* $\operatorname{Im} \zeta''(\zeta') \notin -V_{A_0}^*$.

We need some preliminaries before the proof. The function $\zeta' \to \zeta''(\zeta')$ in Theorem 2.6 shall satisfy the following conditions:

(2.6) $\zeta''(\zeta')$ is analytic in $\Omega(\xi_0', R)$. There are polynomials $p_j = p_j(\zeta', \tau)$, $j = n'+1, \ldots, n$ with $\deg p_j \leq q_0$, such that $p_j(\zeta', \zeta_j(\zeta')) = 0$ in $\Omega(\xi_0', R)$ and $a_j(\zeta')\Delta_j(\zeta') \neq 0$ in $\Omega(\xi_0', R)$. Here a_j and Δ_j denote respectively the coefficient of the term of highest degree and the discriminant of p_j as a polynomial in τ.

Let U_t be the set of all $(\xi_0', R, A) \in \mathbf{R}^{n'+2}$ with $|\xi_0'| < t$, $R < t$, such that there exists a function $\zeta''(\zeta')$, satisfying (2.6) with $|\zeta''(\zeta')| < t$, $\operatorname{Im} \zeta''(\zeta') \in -V_A^*$ and $P(\zeta', \zeta''(\zeta')) = 0$ for all ζ' in $\Omega(\xi_0', R)$.

LEMMA 2.7. *The function*

$$f(t) = \sup \{ s\ ;\ \exists (\xi_0', R, A) \in U_t,\ 0 \leq s \leq A,\ 0 \leq s \leq R \}$$

is algebraic for large t.

Proof. If the condition $(\xi_0', R, A) \in U_t$ can be built up from equations and inequalities involving polynomials in real variables, then the lemma follows from the Tarski — Seidenberg theorem. However, the condition $(\xi_0', R, A) \in U_t$ can be expressed in the following way:

> $|\xi_0'| < t$, $R < t$. In addition there are polynomials p_j with $\deg p_j \leq q_0$ and $|a_j(\zeta')\Delta_j(\zeta')|^2 \neq 0$ when $|\zeta' - \xi_0'| < R$, such that for some τ_{j0} with $|p_j(\xi_0', \tau_{j0})|^2 = 0$, $j = n' + 1, \ldots, n$, we have $\operatorname{Im} \tau = \operatorname{Im} (\tau_{n'+1}, \ldots, \tau_n) \in - V_A^*$, $|P(\zeta', \tau)|^2 = 0$ and $|\tau| \leq t$ for all ζ' with $|\zeta' - \xi_0'| \leq R$, if τ_j is the value for $\theta = \zeta'$ of the unique continuous solution σ of $p_j(\theta, \sigma) = 0$ defined on the line segment between ξ_0' and ζ', such that $\sigma = \tau_{j0}$ for $\theta = \xi_0'$.

By Lemma A.9 in Hörmander [6] this condition can be expressed in the required algebraic form.

Proof of Theorem 2.6. Theorem 2.1 shows that $f(t) \leq C \log (2 + t)$. Since f is increasing and algebraic for large t, $\lim_{t \to \infty} f(t)$ exists. Let $A_0 > \lim_{t \to \infty} f(t)$ and set $R_0 = A_0/\gamma$, where γ is the constant we get in Lemma A2 in [6] for $\nu = n'$ and $M \geq \deg (\prod_{n'+1}^{n} (a_j \Delta_j))$, whenever p_j are irreducible polynomials of degree $\leq q_0$. This proves the theorem, for a polydisc $\Omega(\xi_0', R)$ with this radius contains one with radius A_0 and real centre where a_j and Δ_j do not vanish.

Corollary 2.8. *If* $P(D)$ *has a fundamental solution with support in the closed, convex and semialgebraic cone* Γ, *then* P *satisfies the following condition*:

(2.7) *There are constants* A_0 *and* R_0 *such that if* $\xi_0' \in \mathbf{R}^{n'}$, $\zeta'', \eta'' \in \mathbf{C}^{n''}$, $\eta'' \neq 0$ *and* $\Omega(\xi_0', R_0) \ni \zeta' \to \tau(\zeta') \in \mathbf{C}$ *is an analytic function satisfying the equation* $P(\zeta', \zeta'' + \tau(\zeta')\eta'') = 0$ *when* $\zeta' \in \Omega(\xi_0', R_0)$, *then there is at least one point* $\zeta' \in \Omega(\xi_0', R_0)$ *such that* $\operatorname{Im} (\zeta'' + \tau(\zeta')\eta'') \notin - V_{A_0}^*$.

Remark 2.9. Assume that $P(D)E = \delta$, $\operatorname{supp} E \subset \Gamma = \{x \in \mathbf{R}^n\ ;\ x_{n'+1} \geq \sum_{n'+2}^{n} |x_j|\}$. If $\hat{E}(\zeta_{n'+2}, \ldots, \zeta_n) \in \mathcal{D}'(\mathbf{R}^{n'+1})$ is the Fourier transform of E with respect to $x_{n'+2}, \ldots, x_n$, we obtain that $P(D_1, \ldots, D_{n'+1}, \zeta_{n'+2}, \ldots, \zeta_n)\hat{E}(\zeta_{n'+2}, \ldots, \zeta_n) = \delta(x_1, \ldots, x_{n'+1})$ and $\hat{E}$ is an analytic function of $\zeta_{n'+2}, \ldots, \zeta_n$ with values in $\mathcal{D}'(\mathbf{R}^{n'+1})$. Now it follows from Trèves [10] (see also Appendix) that the operator $P(D_1, \ldots, D_{n'+1}, \zeta_{n'+2}, \ldots, \zeta_n)$ has constant strength. It is not clear if this follows from the conditions in Theorem 2.1.

Remark 2.10. In the proof of Theorem 2.1 (and Theorem 2.6) we have only used the existence of a local fundamental solution, i.e., a distribution E such that $P(D)E = \delta$ in a neighbourhood of zero and $\operatorname{supp} E \subset \Gamma$.

We finish this section by proving a theorem which shows that there is no restriction in assuming that Γ has interior points. Thus, we shall assume this later on.

THEOREM 2.11. *The operator $P(D)$ has a fundamental solution with support in the hyperplane $\langle x, N \rangle = 0$ if and only if $P(\zeta + tN) = P(\zeta)$ for all $t \in \mathbf{R}$, $\zeta \in \mathbf{C}^n$.*

Proof. We can assume that $N = (0, \dots, 0, 1)$ and $P(D) = \sum_0^m a_j(D')D_n^j$. Let E be the fundamental solution. Locally we can write $E = \sum_0^r E_j(x') \otimes D_n^j\, \delta(x_n)$. From $P(D)E = \delta$ we obtain that

$$\sum_{+k=i} a_j(D')E_k = \begin{cases} \delta(x') & \text{for } i = 0 \\ 0 & \text{for } i > 0. \end{cases}$$

Hence, $a_0(D')E_0 = \delta(x')$ and, if $m \geq 1$, $a_m(D')E_r = 0$, $a_m(D')E_{r-1} + a_{m-1}(D')E_r = 0, \dots$. By induction it follows that $a_m^{j+1}(D')E_{r-j} = 0$. Thus $0 = (a_m(D'))^{r+1}a_0(D')E_0 = (a_m(D'))^{r+1}\delta(x')$, which implies that $a_m = 0$ for $m \geq 1$.

3. Stability of the necessary conditions

We shall here prove a theorem which gives some information about P at infinity.

THEOREM 3.1. *Let $P(D)$ satisfy (2.1) with respect to $\Gamma = \mathbf{R}^{n'} \times V$. Assume that $\xi_j \in \mathbf{R}^n$, $s_j, t_j \in \mathbf{R}$, $a_j \in \mathbf{C}$, where s_j and $t_j \to \infty$, $j \to \infty$ and that there is a constant N such that $|\xi_j| \leq t_j^N$, $t_j \leq s_j^N$, $s_j \leq t_j^N$ for all j. Furthermore, assume that*

$$Q_j(\xi) = a_j P(\xi_j' + s_j\xi', \xi_j'' + t_j\xi'') \to Q_0(\xi), j \to \infty.$$

If $\xi' \in \mathbf{R}^{n'}$ and $Q_0(\xi', \xi'') \neq 0$ for some $\xi'' \in \mathbf{R}^{n''}$, then $Q_0(\xi', \zeta'') \neq 0$ for all $\zeta'' \in \mathbf{C}^{n''}$ with $\operatorname{Im} \zeta'' \in -\operatorname{int} V^*$.

Proof. Let $\xi' \in \mathbf{R}^{n'}$ be such that $\deg_{\xi''} Q_0(\xi_0', \xi'') \geq 1$ and assume that $\zeta'' \in \mathbf{C}^{n''}$, $\operatorname{Im} \zeta'' \in -\operatorname{int} V^*$ and $Q_0(\xi_0', \zeta'') = 0$.

Let $\tau \in \mathbf{C}$ and take $N'' \in \mathbf{R}^{n''}$ such that $\deg_\tau Q_0(\xi_0', \zeta'' + \tau N'') \geq 1$ and let $b_j(\xi')$ be the coefficient of the term of highest degree with respect to τ of the polynomial $Q_j(\xi', \zeta'' + \tau N'')$. Now, consider $Q_j(\xi', \zeta'' + \tau N'')$ as a polynomial of (ξ', τ) and write it as a product of irreducible factors. Set $d_j =$ the product of the discriminants of these factors, considered as polynomials of τ. We have that $\Delta_j = \Delta_j(\zeta') = b_j(\zeta')d_j(\zeta') \not\equiv 0$ is a polynomial of ζ' and if $\tau = \tau(\zeta')$ is a continuous solution of the equation $Q_j(\zeta', \zeta'' + \tau N'') = 0$, then $\tau(\zeta')$ is analytic when $\Delta_j(\zeta') \neq 0$.

Let $B \subset \mathbf{C}^{n'}$ be a ball with centre $\xi_0' \in \mathbf{R}^{n'}$ and radius $r > 0$. Then by Lemma A2 in [6] there is a ball $B_0 \subset B$ with centre $\xi' \in \mathbf{R}^{n'}$ and radius $\gamma r > 0$, such that $\Delta_0(\zeta') \neq 0$ in B_0. Furthermore, for every j there is a ball $B_j \subset B_0$ with centre $\xi_j' \in \mathbf{R}^{n'}$ and radius $\gamma^2 r > 0$ such that $\Delta_j(\zeta') \neq 0$ in B_j. This implies that there is a ball $\tilde{B} \subset B_0$ with real centre and radius $\gamma^2 r/2 = \gamma_1 r > 0$ and a subsequence Δ_{j_k} of Δ_j such that $\Delta_{j_k}(\zeta') \neq 0$ in $\tilde{B}$ for all k. In order not to complicate the notations we assume that this is true for the whole sequence, i.e., $\Delta_j(\zeta') \neq 0$ in $\tilde{B}$ for all $j \geq 0$.

Now, consider the solutions of the equation $Q_0(\zeta', \zeta'' + \tau N'') = 0$, $\zeta' \in \tilde{B}$. These are analytic in $\tilde{B}$ and if the radius $r > 0$ above is small enough, there is a constant $\varepsilon > 0$ such that some of the solutions, say τ_0, satisfies the condition $\operatorname{Im}(\zeta'' + \tau_0(\zeta')N'') \in - V_\varepsilon^*$ for all $\zeta' \in \tilde{B}$. If $\tau_0(\zeta')$ has the multiplicity μ when $\zeta' \in \tilde{B}$ and if $U \subset \mathbf{C}$ is a small neighbourhood of zero then Rouché's theorem shows that the equation $Q_j(\zeta', \zeta'' + \tau N'') = 0$, $\zeta' \in \tilde{B}$ has exactly μ solutions in $\tau_0(\zeta') + U$ for large j. Let τ_j be one of these. Then $\tau_j \to \tau_0$ uniformly on $\tilde{B}$ when $j \to \infty$. This implies that

$$\operatorname{Im}(\zeta'' + \tau_j(\zeta')N'') \in - V_{\varepsilon/2}^* \text{ for all } \zeta' \in \tilde{B}$$

if j is large. Thus, for large j

$$P(\xi_j' + s_j\zeta', \ \xi_j'' + t_j(\zeta'' + \tau_j(\zeta')N'')) = 0, \text{ for all } \zeta' \in \tilde{B}$$

and

$$\operatorname{Im}(\xi_j'' + t_j(\zeta'' + \tau_j(\zeta')N'')) \in - V_{t_j\varepsilon/2}^* \text{ for all } \zeta' \in \tilde{B}.$$

It follows that (2.1) is not valid since then we would have

$$\min(t_j\varepsilon/2, \ s_j\gamma_1 r) \leq C \ \log(2 + |\xi_j| + C_1(s_j + t_j)) =$$
$$= O(\min(\log t_j, \ \log s_j)), \ j \to \infty,$$

which implies that

$$\min(\varepsilon, r) = 0.$$

This completes the proof.

Corollary 3.2. *Let $P(D)$ have a fundamental solution with support in $\Gamma = \mathbf{R}^{n'} \times V$ and let p be the principal part of P. Then for every $\xi' \in \mathbf{R}^{n'}$ we have that $p(\xi', D'')$ is either hyperbolic with respect to V or zero.*

4. Reduction of existence theorems to a priori estimates

In this section we shall assume that P satisfies the necessary conditions of Theorem 2.1. Let $k \in \mathcal{K}(\mathbf{R}^n)$ so that for some constants C and N

$$k(\xi+\eta) \leq k(\xi)(1+C\,|\eta|)^N.$$

(See p. 34 in [5].) If $1 \leq p \leq \infty$ we shall use the norm

$$\|u\|_{p,k} = \left((2\pi)^{-n} \int |k(\xi)\hat{u}(\xi)|^p d\xi\right)^{1/p}, \quad u \in \mathcal{S}(\mathbf{R}^n).$$

As before, let $\Gamma = \mathbf{R}^{n'} \times V$ be a convex cone with interior points. (Cf. Theorem 2.11.) Then we are interested in the quotient norms defined by

$$\|u\|_{p,k}^{\Gamma_-} = \inf\{\|v\|_{p,k};\ v = u \ \text{ on } \ \Gamma_- = -\Gamma,\ v \in \mathcal{S}(\mathbf{R}^n)\}, \quad u \in \mathcal{S}(\mathbf{R}^n).$$

THEOREM 4.1. *Let P be a polynomial and a an element of $\mathcal{K}(\mathbf{R}^n)$ such that for every compact set K there is a constant C for which*

$$\|u\|_{1,1}^{\Gamma_-} \leq C\,\|P(D)u\|_{1,1/a}^{\Gamma_-}, \quad \text{for all } \ u \in C_0^\infty(K).$$

Let $1 < p_j \leq \infty$ and $k_j \in \mathcal{K}(\mathbf{R}^n)$, $j = 1, 2, \ldots$. *Then for every* $f \in B_{p_j,k_j}^{\mathrm{loc}}(\mathbf{R}^n)$, $j = 1, 2, \ldots,$ *with* $\operatorname{supp} f \subset \Gamma$ *there is a solution* u *to* $P(D)u = f$ *such that* $\operatorname{supp} u \subset \Gamma$ and $u \in B_{p_j, ak_j}^{\mathrm{loc}}(\mathbf{R}^n)$, $j = 1, 2, \ldots$.

The proof is rather long so we first prove the following local version.

THEOREM 4.2. *Let $a \in \mathcal{K}(\mathbf{R}^n)$, $\Omega \subset\subset \mathbf{R}^n$ and let P be a polynomial for which there is a constant C such that*

$$\|u\|_{1,1}^{\Gamma_-} \leq C\,\|P(D)u\|_{1,1/a}^{\Gamma_-} \quad \text{for all } \ u \in C_0^\infty(-\Omega).$$

Then there is a $u \in B_{\infty,a}$ with $\operatorname{supp} u \subset \Gamma$ such that $P(D)u = \delta$ in Ω.

Proof. The equation $P(D)u = \delta$ in Ω means that $\check{u}(P(D)v) = v(0)$ for all $v \in C_0^\infty(-\Omega)$. We have that

$$|v(0)| \leq \|v\|_{1,1}^{\Gamma_-} \leq C\,\|P(D)v\|_{1,1/a}^{\Gamma_-}, \quad v \in C_0^\infty(-\Omega).$$

Then the linear form

$$P(D)v \to v(0) \quad \text{on} \quad P(D)C_0^\infty(-\Omega)$$

can be extended by the Hahn-Banach theorem to a linear form $\check{u}$ on $C_0^\infty(\mathbf{R}^n)$ such that

$$|\check{u}(w)| \leq C\|w\|_{1,1/a}^{\Gamma_-}$$

Thus u is a distribution with support in Γ, such that $P(D)u = \delta$ on Ω nap $u \in B_{\infty,a}$.

Using local fundamental solutions of the type introduced in Theorem 4.2 we can prove the following approximation theorem.

THEOREM 4.3. *Let* $\Omega_1 \subset \Omega_2$ *be bounded open sets in* $\mathbf{R}^n$ *and let* P *satisfy the conditions of Theorem* 4.2 *for every* $\Omega \subset\subset \mathbf{R}^n$. *Set* $\Gamma^\circ = \operatorname{int} \Gamma$ *and denote by* N_j *the set of solutions* $u \in C^\infty(\Omega_j)$ *of the equation* $P(D)u = 0$, *such that* $\operatorname{supp} u \subset \Omega_j \cap \Gamma$. *Furthermore, let* N_j *have the topology induced by* $C^\infty(\Omega_j)$. *If*

$$\mu \in \mathcal{E}'(\mathbf{R}^n),\ \Gamma^\circ \cap \operatorname{supp} \mu \subset \bar{\Omega}_2,\ \ \Gamma^\circ \cap \operatorname{supp} P(-D)\mu \subset\subset \Omega_1 \Rightarrow \Gamma^\circ \cap \operatorname{supp} \mu \subset\subset \Omega_1,$$

then the restriction to Ω_1 *of the elements in* N_2 *form a dense subset* N_2' *of* N_1.

Proof. If we prove that every $\nu \in \mathcal{E}'(\Omega_1)$ which is orthogonal to N_2' is also orthogonal to N_1, the statement will follow from the Hahn-Banach theorem.

Let $\Omega_3 \subset \Omega_4$ be open bounded sets such that $\Omega_2 \subset\subset \Omega_3$ and $\complement\, \Omega_4 \pm \Omega_3 \subset \complement\, \bar{\Omega}_3$. Furthermore, let E be a local fundamental solution in Ω_4, i.e. $P(D)E = \delta$ in Ω_4, $\operatorname{supp} E \subset \Gamma$. Set $u = \varphi * E$ where $\varphi \in C_0^\infty(\Gamma \cap (\complement\, \Omega_2) \cap \Omega_3)$. Then $P(D)u = 0$ in Ω_2 and $\operatorname{supp} u \subset \Gamma$. Thus

$$0 = \nu(\varphi * E) = \varphi * E * \check{\nu}(0) = (\check{E} * \nu)(\varphi),\quad \varphi \in C_0^\infty(\Gamma \cap (\complement \Omega_2) \cap \Omega_3).$$

Let $\chi \in C_0^\infty(\Omega_3)$ be 1 in a neighbourhood of $\bar{\Omega}_2$ and set $\mu = \chi(\check{E} * \nu)$. Then

$$\Gamma^\circ \cap \operatorname{supp} \mu \subset \bar{\Omega}_2,\quad \Gamma^\circ \cap \operatorname{supp} P(-D)\mu \subset\subset \Omega_1.$$

Thus by hypothesis

$$\Gamma^\circ \cap \operatorname{supp} \mu \subset\subset \Omega_1.$$

Hence we can choose $\psi \in C_0^\infty(\Omega_1)$ such that $\psi = 1$ in a neighbourhood of $\Gamma^\circ \cap \operatorname{supp} \mu$. If we set $\mu_1 = \psi\mu = \psi(\check{E} * \nu)$ and $\nu_1 = P(-D)\mu_1$ we obtain that $\mu_1 \in \mathcal{E}'(\Omega_1)$, $\nu - \nu_1 \in \mathcal{E}'(\Omega_1)$ and $\Gamma^\circ \cap \operatorname{supp} (\nu - \nu_1) = \emptyset$, which implies $(\nu - \nu_1)(u) = 0$ and $\nu_1(u) = \mu_1(P(D)u) = 0$ for all $u \in N_1$. Thus $\nu(u) = 0$.

LEMMA 4.4. *There is a set* $\Omega \subset \mathbf{R}^n$ *such that if* $\Omega_\nu = \nu\Omega$ *then the pair of sets* Ω_ν *and* $\Omega_{\nu+2}$ *satisfies the hypotheses in Theorem* 4.3 *for all* $\nu \geq 1$.

Proof. Let p be the principal part of P and let $\omega > 0$. Set $N_0 = (0, N_0'')$, $N_1 = (N_1', \omega N_1''), \ldots, N_n = (N_n', \omega N_n'')$, where $N_0'', \ldots, N_n'' \in \operatorname{int} V^*$ and $p(N_j', \xi'') \not\equiv 0$ for $j = 1, 2, \ldots, n$. Furthermore, we choose the vectors N_j so that every $\xi \in \mathbf{R}^n$ can be written as a linear combination of $-N_0, N_1, \ldots, N_n$ with non-negative coefficients. (Observe that this condition is independent of $\omega > 0$.) Set

$$\Omega = \{x \in \mathbf{R}^n;\ \langle x, N_j\rangle < 1,\ j = 1, 2, \ldots, n,\ \langle x, N_0\rangle > -1\}$$

and $\Omega_\nu = \nu\Omega$. We shall prove that these sets will do if $\omega > 0$ is large enough.

It is clear that Ω is a bounded neighbourhood of zero. We now want to show that

$$\mu \in \mathcal{E}'(\mathbf{R}^n),\quad \Gamma^\circ \cap \operatorname{supp} \mu \subset \bar{\Omega}_{\nu+2},\quad \Gamma^\circ \cap \operatorname{supp} P(-D)\mu \subset\subset \Omega_\nu \Rightarrow$$
$$\Rightarrow \Gamma^\circ \cap \operatorname{supp} \mu \subset\subset \Omega_\nu .$$

Set $U_j(s) = \{x \in \mathbf{R}^n ;\ x'' \in \operatorname{int} V,\ \langle x, N_j\rangle > s\}$, $j = 1, 2, \ldots, n$. Then the statement above is a consequence of the following:

$$\mu \in \mathcal{E}'(\mathbf{R}^n),\ \ \mu = 0 \ \text{ in }\ U_j(s),\ \ P(-D)\mu = 0 \ \text{ in }\ U_j(t) \Rightarrow \mu = 0 \ \text{ in }\ U_j(t)$$
$$(j = 1, 2, \ldots, n).$$

From Theorem 5.3.3 in Hörmander [5] we now get that it is sufficient to prove that every characteristic plane that intersects $U_j(t)$ also meets $U_j(s)$. A plane that does not meet $U_j(s)$ has a normal that lies in the dual cone U_j^* of $U_j(0)$. However, U_j^* is the convex hull of $\{hN_j ; h \geq 0\} \cup \Gamma^*$, since this is closed. Thus we have to prove that if $N = (0, N'') \in \Gamma^*$ then $p(hN_j + N) \neq 0$ for all $h > 0$. (Note that a plane with normal in Γ^* is defined by an equation in the x'' variables, so it meets $U_j(s)$ if and only if it meets $U_j(t)$.) However

(4.1) $\quad p(hN_j + N) = h^m p(N_j', \omega N_j'' + h^{-1}N'') \neq 0$ for large ω and $h > 0$.

To see this we first observe that $h^{-1}N'' \in V^*$, $N_j'' \in V^*$ and $p(N_j', D'')$ is hyperbolic with respect to V by Remark 2.10 and Theorem 3.1. It follows from Theorem 1.3 in [9] that also $p(N_j', -iD'')$ is hyperbolic with respect to V, which by Theorem 5.5.4 in [5] implies (4.1).

Proof of Theorem 4.1. Let Ω_ν be the sets defined in Lemma 4.4 and let $\varphi_\nu \in C_0^\infty(\Omega_\nu)$ be 1 in a neighbourhood of $\Omega_{\nu-1}$. Then it will be sufficient to prove that there exist $u_\nu \in \bigcap_1^\infty B_{p_j, ak_j}^{\mathrm{loc}}$ such that

$$\|\varphi_\mu(u_{\nu+1} - u_\nu)\|_{p_j, ak_j} \leq 2^{-\nu},\ \mu \leq \nu,\ j \leq \nu,$$
$$P(D)u_{\nu+1} = f \ \text{ in }\ \Omega_{\nu+2} \ \text{ and }\ \operatorname{supp} u_{\nu+1} \subset \Gamma.$$

In fact, for such u_ν we obtain that $u_\nu \to u$ in $B_{p_j, ak_j}^{\mathrm{loc}}$ as $\nu \to \infty$, where $P(D)u = f$ and $\operatorname{supp} u \subset \Gamma$.

Set $u_1 = E * (\varphi_3 f)$, where E is a local fundamental solution in a large neighbourhood of zero. Then $u_1 \in \bigcap_1^\infty B_{p_j, ak_j}$, $P(D)u_1 = f$ in Ω_2 and $\operatorname{supp} u_1 \subset \Gamma$. When $u_1, \ldots, u_\nu$ are chosen, we want to construct $u_{\nu+1}$. Then there is a distribution $v_0 \in \bigcap_1^\infty B_{p_j, ak_j}$ such that $P(D)v_0 = \varphi_{\nu+3} f = f$ in $\Omega_{\nu+2}$, $\operatorname{supp} v_0 \subset \Gamma$. If $v = v_0 - u_\nu$, then $P(D)v = P(D)v_0 - P(D)u_\nu = 0$ in $\Omega_{\nu+1}$. Choose $\psi \in C_0^\infty(\Gamma)$ so that $\operatorname{supp} \psi \subset -\Omega_1$ and $\|\varphi_\mu(\psi * v - v)\|_{p_j, ak_j} \leq 2^{-\nu-1}$ for $\mu \leq \nu$, $j \leq \nu$. It follows from Theorem 4.3 that there is a C^∞ function w such that $P(D)w = 0$ in $\Omega_{\nu+2}$, $\operatorname{supp} w \subset \Gamma$ and $\|\varphi_\mu(w - \psi * v)\|_{p_j, ak_j} \leq 2^{-\nu-1}$ for $\mu \leq \nu$, $j \leq \nu$. Set $u_{\nu+1} = v_0 - w$. Then

$$\|\varphi_\mu(u_{\nu+1} - u_\nu)\|_{p_j, \alpha k_j} \leq 2^{-\nu}, \quad \mu \leq \nu, \quad j \leq \nu, \quad P(D)u_{\nu+1} = f \quad \text{in} \quad \Omega_{\nu+2}$$

and

$$\operatorname{supp} u_{\nu+1} \subset \Gamma.$$

This proves the theorem.

5. Sufficient conditions

If the operator $P(\zeta', D'')$ is hyperbolic with respect to V for sufficiently many $\zeta' \in \mathbf{C}^{n'}$, then we can prove the existence of a fundamental solution with support in $\Gamma = \mathbf{R}^{n'} \times V$. The following definition helps us to describe sets which suffice.

Definition 5.1. Let $0 < \delta < 1$ and let $B \subset \mathbf{C}^{n'}$ be a ball with radius R and centre ζ_0'. We say that a subset S of B is of *type* N_δ if for an arbitrary, logarithmically plurisubharmonic function $g \geq 0$ the following inequality is true

$$g(\zeta_0') \leq (\sup_B g)^{1-\delta} (\sup_S g)^\delta$$

(Cf. Lemma 3.2 in [6].)

THEOREM 5.2. *Let P be a polynomial in $n = n' + n''$ variables and let $N = (0, N'')$, where $N'' \in V^* = V_0^*$. Furthermore, let $a(\zeta') \not\equiv 0$ be the coefficient of the term of highest degree of $P(\zeta + \tau N)$ as a polynomial in τ. Then the operator $P(D)$ has a fundamental solution $E \in B^{\mathrm{loc}}_{\infty, \tilde{a}}$ with* $\operatorname{supp} E \subset \Gamma = \mathbf{R}^{n'} \times V$ *if P satisfies the following condition*:

(5.1) *There are constants $R > 0$, A and δ, $0 < \delta < 1$, such that in every ball B with radius R and real centre ξ_0', there is a subset $S_{\xi_0'}$ of type N_δ for which it is true that $\zeta' \in S_{\xi_0'}$, $\operatorname{Im} \zeta'' \in - V_A^* \Rightarrow P(\zeta) \neq 0$.*

It is sufficient to prove the theorem for $A = -1$. (Cf. page 349 in [6].) For $1 \leq p \leq \infty$, $k \in \mathcal{K}(\mathbf{R}^{n''})$ we set

$$\|u\|^{V_-}_{p,k} = \inf \{\|v\|_{p,k}\,;\ v = u \ \text{ on } \ V_- = -V, \ v \in \mathcal{S}(\mathbf{R}^{n''})\}, \quad u \in \mathcal{S}(\mathbf{R}^{n''}),$$

where

$$\|v\|_{p,k} = \left((2\pi)^{-n''} \int (|\hat{v}(\xi'')|\, k(\xi''))^p d\xi'' \right)^{1/p}.$$

LEMMA 5.3. *Let Q be a polynomial in n'' variables with $Q(\zeta'') \neq 0$ when $\operatorname{Im} \zeta'' \in - V_{-1}^*$ and let $k \in \mathcal{K}(\mathbf{R}^{n''})$. Then there is a constant C depending only on A, n'' and* $\deg Q$ *such that*

$$\|u\|^{V_-}_{p, \tilde{Q}k} \leq C \|Q(D'')u\|^{V_-}_{p,k} \quad \text{for all} \quad u \in C_0^\infty(\mathbf{R}^{n''}).$$

Proof. Set

$$\check{E}(u) = (2\pi)^{-n''} \int \hat{u}(\xi'')/Q(\xi'')\, d\xi''.$$

Then $\operatorname{supp} E \subset V$, $Q(D'')E = \delta$ and

$$|\check{E}(u)| \leq C \int |\hat{u}(\xi'')|/\tilde{Q}(\xi'')\, d\xi'',$$

where C only depends on A, n'' and $\deg Q$, that is $\|E\|_{\infty, \tilde{Q}k} \leq C$. If $g = Q(D'')u$ on V_-, then $E * g = u$ on V_- so that

$$\|u\|^{V_-}_{p, \tilde{Q}k} \leq \|E * g\|_{p, \tilde{Q}k} \leq C\, \|g\|_{p, k}$$

which shows that

$$\|u\|^{V_-}_{p, \tilde{Q}k} \leq C\, \|Q(D'')u\|^{V_-}_{p, k} \quad \text{for all} \quad u \in C_0^\infty(\mathbf{R}^{n''}).$$

LEMMA 5.4. *If* $h(\zeta', x'')$ *is analytic in* ζ' *and* $k \in \mathcal{K}(\mathbf{R}^{n''})$ *then* $\|h(\zeta', \cdot)\|^{V_-}_{p, k}$ *is logarithmically plurisubharmonic.*

Proof. $\|h(\zeta', \cdot)\|^{V_-}_{p, k}$ is the norm of an analytic function with values in a Banach space, hence it is logarithmically plurisubharmonic.

LEMMA 5.5. *If the polynomial* P *satisfies condition* (5.1) *with* $A = -1$ *and* $K \subset\subset \mathbf{R}^n$, *then there is a constant* C *such that*

$$\sup_{\mathbf{R}^{n'}} \|\hat{u}(\xi', \cdot)\|^{V_-}_{p, \tilde{a}k} \leq C \sup_{\mathbf{R}^{n'}} \|P(\xi', D'')\hat{u}(\xi', \cdot)\|^{V_-}_{p, k}$$

for all $u \in C_0^\infty(K)$, $k \in \mathcal{K}(\mathbf{R}^{n''})$. *Here* $\hat{u}(\xi', \cdot)$ *is the Fourier transform of* u *with respect to* x'.

Proof. For $\zeta' \in S_{\xi_0'}$ we have

$$|a(\zeta')|\, \|\hat{u}(\zeta', \cdot)\|^{V_-}_{p, k} \leq C\, \|P(\zeta', D'')\hat{u}(\zeta', \cdot)\|^{V_-}_{p, k} \leq C\, G$$

where $G = \sup_{\mathbf{R}^{n'}} \|P(\xi', D'')\hat{u}(\xi', \cdot)\|^{V_-}_{p, k}$. The first estimate follows from Lemma 5.3 and the second from the fact that the function

$$\zeta' \to \|P(\zeta', D'')\hat{u}(\zeta', \cdot)\|^{V_-}_{p, k}$$

is logarithmically plurisubharmonic and of exponential type. The properties of the set $S_{\xi_0'}$ now show that

$$|a(\xi_0')|\, \|\hat{u}(\xi_0', \cdot)\|^{V_-}_{p, k} \leq C\, (\sup_{|\zeta' - \xi_0'| \leq R} |a(\zeta')|\, \|\hat{u}(\zeta', \cdot)\|^{V_-}_{p, k})^{1-\delta} G^\delta \leq$$

$$\leq C\, (\sup_{\mathbf{R}^{n'}} |a(\xi')|\, \|\hat{u}(\xi', \cdot)\|^{V_-}_{p, k})^{1-\delta} G^\delta, \quad 0 < \delta < 1,$$

which implies that

$$|a(\xi')|\,\|\hat{u}(\xi',\cdot)\|^{V-}_{p,k} \le CG, \quad \xi' \in \mathbf{R}^{n'}.$$

However, by Lemma A 1 in [6] there is a $\theta \in A$ such that

$$\tilde{a}(\xi') \le C\,\tilde{a}(\xi' + z\theta) \le C'\,|a(\xi' + z\theta)|, \quad z \in \mathbf{C}, \quad |z| = 1.$$

Thus

$$\tilde{a}(\xi')\,\|\hat{u}(\xi' + z\theta,\cdot)\|^{V-}_{p,k} \le C\,G.$$

Since $\|u(\xi' + z\theta,\cdot)\|^{V-}_{p,k}$ is subharmonic as a function of z we now obtain the required estimate.

LEMMA 5.6. *Let P be a polynomial and $b \in \mathcal{K}(\mathbf{R}^{n''})$. If for every compact set K, there is a constant C_1 such that*

$$\sup_{\mathbf{R}^{n'}} \|\hat{u}(\xi',\cdot)\|^{V-}_{p,bk} \le C_1 \sup_{\mathbf{R}^{n'}} \|P(\xi', D'')\hat{u}(\xi',\cdot)\|^{V-}_{p,k}, \tag{5.2}$$

$$\textit{for all}\quad u \in C_0^\infty(K), \quad k \in \mathcal{K}(\mathbf{R}^{n''}),$$

then for every $k \in \mathcal{K}(\mathbf{R}^n)$ there is a constant C such that

$$\|u\|^{\Gamma-}_{p,bk} \le C\,\|P(D)u\|^{\Gamma-}_{p,k} \quad \textit{for all}\quad u \in C_0^\infty(K). \tag{5.3}$$

Proof. See the proof of Theorem 3.10 in Hörmander [6].

Proof of Theorem 5.2. The theorem follows immediately from Lemma 5.5, Lemma 5.6 and Theorem 4.1.

6. Partially homogeneous operators

Let P be homogeneous in the ξ'' variables, that is

$$P(\xi) = \sum_{|\alpha|=m} a_\alpha(\xi')\xi''^{\alpha}.$$

If $P(D)$ has a fundamental solution with support in $\Lambda = \{x \in \mathbf{R}^n\,;\, x_{n'+1} \ge c\sum_{n'+2}^{n}|x_j|\}$, $c > 0$, then by Remark 2.9 we know that the operator $P(D_1, \ldots, D_{n'+1}, \xi_{n'+2}, \ldots, \xi_n)$ has constant strength. This implies that $b(\xi') = a_{(m,0,\ldots,0)}(\xi')$ is stronger than a_α for all α, i.e. there is a constant C such that

$$\tilde{a}_\alpha(\xi') \le C\,\tilde{b}(\xi') \quad \text{for all } \alpha \text{ and all } \xi' \in \mathbf{R}^{n'}. \tag{6.1}$$

Set

$\Sigma = \Sigma_P = \{Q\,;\, Q(\xi'') = \lim_{j\to\infty}(P(\xi_j', \xi'')/b(\xi_j'))$ for some sequence $\xi_j' \in \mathbf{R}^{n'}$ such that $d(\xi_j', b^{-1}(0)) \to \infty$ as $j \to \infty\}$, where $d(\xi'_j, b^{-1}(0))$ is the distance in $\mathbf{C}^{n'}$ from ξ'_j to the zeros of b. We assume here that b is not a constant, for this would imply that P is a polynomial in ξ'' which is a trivial case. The set Σ is a compact subset of the set of all polynomials of degree m. Furthermore, we have

THEOREM 6.1. *If the polynomial* $P(\xi) = \sum_{|\alpha|=m} a_\alpha(\xi')\xi''^\alpha$ *satisfies* (2.1) *and* a_α *is weaker than* b *for every* α, *then* $Q(D'')$ *is hyperbolic with respect to* V *for every* Q *in* Σ_P.

Proof. (Cf. the proof of Theorem 2.2 in Hörmander [7].) If $P(\xi'_j, \xi'')/b(\xi'_j) \to Q(\xi'')$ as $j \to \infty$, then we can assume that every coordinate of ξ'_j has fixed sign. In order not to complicate the notations we assume that all these coordinates are non-negative. It follows from the Tarski-Seidenberg theorem that if $Q(\xi'') = \Sigma q_\alpha \xi''^\alpha$ then

$$\inf\{\eta_1 ; \Sigma\, |a_\alpha(\eta')/b(\eta') - q_\alpha|^2 \leqq 1/t,\, d(\eta', b^{-1}(0)) \geq t,\, \eta_1 \geq 0, \ldots, \eta_{n'} \geq 0\} \tag{6.2}$$

is an algebraic function of t for large t. By repeated use of the same theorem we get that the infimum of non-negative η_2, when the infimum in (6.2) is attained, is an algebraic function of t for large t, and so on. For large t we have the Puiseux series expansion

$$\eta'(t) = \sum_{-\infty}^{k_1} \theta_j t^{j/k}$$

and $P(\eta'(t), \xi'')/b(\eta'(t)) \to Q(\xi'')$ as $t \to \infty$. Let $\varepsilon > 0$ and consider

$$P(\eta'(t) + t^\varepsilon \xi', \xi'')/b(\eta'(t)) = \sum_\alpha (\sum_\beta a_\alpha^{(\beta)}(\eta'(t)) t^{\varepsilon|\beta|} \xi'^\beta/\beta!)\xi''^\alpha/b(\eta'(t)).$$

Since a_α is weaker than b and $d(\eta'(t), b^{-1}(0)) \geq t$ it follows from Theorem 3.3.2 and Lemma 4.1.1 in [5] that $a_\alpha^{(\beta)}(\eta'(t))/b(\eta'(t)) = O(t^{-|\beta|})$, $t \to \infty$, if $\beta \neq 0$. Thus, if $0 < \varepsilon < 1$ we have

$$P(\eta'(t) + t^\varepsilon \xi', \xi'')/b(\eta'(t)) =$$
$$= P(\eta'(t) + t^\varepsilon \xi', t\xi'')/t^m b(\eta'(t)) \to Q(\xi''), \quad t \to \infty.$$

Now the theorem follows from Theorem 3.1.

We are going to study two special cases of partially homogeneous operators. First we will consider the case $n'' = 2$ and later on operators, such that $Q(D'')$ is *strictly* hyperbolic for all $Q \in \Sigma_P$. Thus, let $n'' = 2$ and $P(\xi) = \sum_{|\alpha|=m} a_\alpha(\xi')\xi''^\alpha$. First we reformulate the necessary conditions.

THEOREM 6.2. *Let* $P(\xi) = \Sigma\, a_\alpha(\xi')\xi_{n'+1}^{\alpha_1}\xi_{n'+2}^{\alpha_2}$, *where* $b = a_{(m,0)}$ *is stronger than* a_α *for all* α. *Assume that* P *satisfies the necessary conditions of Theorem* 2.6 *with* *respect to* $\mathbf{R}^{n'} \times W$ *and let* $V = \{x'' \in \mathbf{R}^2 ; x_{n'+1} \geq c_0^{-1} |x_{n'+2}|\}$ *be a cone* $\subset W$ *such that every* $Q \in \Sigma$ *is hyperbolic with respect to* V. *Then for every* $c > c_0$ *there are constants* R_0 *and* t *such that if* $P(\zeta', \tau(\zeta'), 1) = 0$ *in a ball* B *with real centre* ξ' *and radius* R_0, *where* τ *is analytic, then we have* $\operatorname{Im} \tau(\zeta') = 0$ *for some* $\zeta' \in B$. *Furthermore,* $|\tau(\theta')| \leqq c$ *if* $d(\theta', b^{-1}(0)) \geq t$ *and* $\theta' \in B$.

Proof. Assume that $d(\xi_\nu', b^{-1}(0)) \to \infty$ and $P(\xi_\nu', \xi'')/b(\xi_\nu') \to Q(\xi'') \in \Sigma$ as $\nu \to \infty$. If $P(\xi_\nu' + \zeta', \tau_\nu(\zeta'), 1) = 0$ for $|\zeta'| \leqq R_0$, then we obtain from Rouché's theorem (cf. the proof of Theorem 3.1) that there is a subsequence of $\tau_\nu(\zeta')$, which converges to λ as $\nu \to \infty$, uniformly for $|\zeta'| \leqq R_0$, where $Q(\lambda, 1) = 0$, so that $|\lambda| \leqq c_0$. This proves the last statement. Write P as a product

$$P(\zeta) = b(\zeta') \prod_1^m (\zeta_{n'+1} - \tau_j(\zeta')\zeta_{n'+2}).$$

From Theorem 2.6 we now obtain that there are constants A and R_0 such that for every $\xi_{n'+2} \in \mathbf{R}$ there is a point ζ' with $|\zeta' - \xi'| < R_0$ such that $\operatorname{Im} \xi_{n'+2}\tau(\zeta') \geq A$. If we let $\xi_{n'+2} \to \pm \infty$ we see that there is a point ζ' with $|\zeta' - \xi'| \leq R_0$ such that $\operatorname{Im} \tau(\zeta') = 0$.

THEOREM 6.3. *Let* $P(D)$ *satisfy the conclusions of Theorem* 6.2 *and set* $W = \{x'' \in \mathbf{R}^2 ; x_{n'+1} \geq c^{-1} |x_{n'+2}|\}$, *where* c *is the constant in Theorem* 6.2. *Then the operator* $P(D)$ *has a fundamental solution* $E \in B^{\text{loc}}_{\infty, \tilde{b}}$ *with support in* $\Lambda = \mathbf{R}^{n'} \times W$.

The proof is similar to the proof of Theorem 1.1 in Hörmander [6]. Write $P(\zeta', \zeta_{n'+1}, 1)$ as a product of irreducible factors and let Δ be the product of their discriminants, when they are regarded as polynomials of $\zeta_{n'+1}$. Set $R = (b(\zeta')\Delta(\zeta'))^M$ where M is a large integer and set for $v \in C_0^\infty(\mathbf{R}^n)$

$$N_\nu(v, \zeta') = \max |R(\zeta')b(\zeta')| \, \| \prod_1^\nu (D_{n'+1} - \tau_j(\zeta')D_{n'+2})\hat{v}(\zeta', \cdot)\|_{p, \tilde{k}}^W,$$

where the maximum is taken over all labellings of the zeros τ_j. ($k \in \mathcal{K}(\mathbf{R}^{n''})$.)

LEMMA 6.4. *The function* $N_\nu(v, \zeta')$ *is logarithmically plurisubharmonic.*

Proof. See the proof of Lemma 3.6 in Hörmander [6].

LEMMA 6.5. *Let* $v \in C_0^\infty(\mathbf{R}^n)$ *with* $v(x) = 0$ *for* $|x'| > H$. *Then*

$$N_\nu(v, \zeta') \leq e^{H|\operatorname{Im} \zeta'|} \sup_{\mathbf{R}^{n'}} N_\nu(v, \xi'), \quad \nu = 0, 1, \ldots, m.$$

Proof. See the proof of Lemma 3.7 in Hörmander [6].

LEMMA 6.6. *Let $P(D)$ be as in Theorem 6.2. Then for every compact set K there is a constant C such that*

$$\sup_{\mathbf{R}^{n'}} N_\nu(v, \xi') \leq C \sup_{\mathbf{R}^{n'}} N_{\nu+1}(v, \xi'), \quad v \in C_0^\infty(K), \quad \nu = 0, 1, \ldots, m-1.$$

Proof. By Lemma A1 in Hörmander [6] we first choose a finite set $A \subset \mathbf{R}^{n'}$ for polynomials of degree $\leq \deg P$. Let $B = R_0 + 3$. For every $\xi' \in \mathbf{R}^{n'}$ and $B > 0$ we can find a $\theta \in A$ such that the distance from $\xi' + z\theta$ to the zeros of R is at least B when $|z| = B$. Let

$$\Theta^\pm = \{\xi' + z\theta \; ; \; |z| = B, \ \operatorname{Im} z \gtreqless 0\}$$

and denote by $\Omega_j^\pm$, $j = 0, 1, 2$, the points at a distance $\leq R_0 + j$ from $\Theta^\pm$. In what follows we shall work in the sets Ω_j^+; the same arguments can be applied in the sets Ω_j^-.

From Lemma 5.3 above and Lemma 3.2 in Hörmander [6] we obtain the following estimates

$$\sup_{\Omega_0^+} |R(\zeta')b(\zeta')| \, \| \prod_1^\nu (D_{n'+1} - \tau_j(\zeta')D_{n'+2})\hat{v}(\zeta', \cdot)\|_{p,\tilde{k}}^{W} \leq$$

$$\leq (\sup_{\Omega_2^+} N_\nu(v, \zeta'))^{1-\delta} \Big(\sup_{\substack{\zeta' \in \Omega_1^+ \\ \operatorname{Im} \tau_{\nu+1}(\zeta') = 0}} |R(\zeta')b(\zeta')| \, \| \prod_1^\nu (D_{n'+1} - \tau_j(\zeta')D_{n'+2})\hat{v}(\zeta', \cdot)\|_{p,\tilde{k}}^{W} \Big)^\delta \leq$$

$$\leq (\sup_{\Omega_2^+} N_\nu(v, \zeta'))^{1-\delta} \Big(\sup_{\substack{\zeta' \in \Omega_1^+ \\ \operatorname{Im} \tau_{\nu+1}(\zeta') = 0}} |R(\zeta')b(\zeta')| \, \| \prod_1^{\nu+1} (D_{n'+1} - \tau_j(\zeta')D_{n'+2})\hat{v}(\zeta', \cdot)\|_{p,\tilde{k}}^{W} \Big)^\delta \leq$$

$$\leq (\sup_{\Omega_2^+} N_\nu(v, \zeta'))^{1-\delta} (\sup_{\Omega_1^+} N_{\nu+1}(v, \zeta'))^\delta \leq C \, (\sup_{\mathbf{R}^{n'}} N_\nu(v, \xi'))^{1-\delta} (\sup_{\mathbf{R}^{n'}} N_{\nu+1}(v, \xi'))^\delta.$$

Here $0 < \delta < 1$; the first inequality follows from Lemma 3.2 in Hörmander [6], the second is a consequence of Lemma 5.3, and the fourth follows from Lemma 6.5. If we now take the maximum of the left side over all labellings of the zeros we obtain that

$$\sup_{\Omega_0^+} N_\nu(v, \zeta') \leq C \, (\sup_{\mathbf{R}^{n'}} N_\nu(v, \xi'))^{1-\delta} (\sup_{\mathbf{R}^{n'}} N_{\nu+1}(v, \xi'))^\delta.$$

The same estimate holds for Ω_0^- and since the function $N_\nu(v, \zeta')$ is plurisubharmonic, we can use the maximum principle for the function $N_\nu(v, \xi' + z\theta)$. We obtain that

$$\sup_{\mathbf{R}^{n'}} N_\nu(v, \xi') \leq C \, (\sup_{\mathbf{R}^{n'}} N_\nu(v, \xi'))^{1-\delta} (\sup_{\mathbf{R}^{n'}} N_{\nu+1}(v, \xi'))^\delta,$$

i.e.,

$$\sup_{\mathbf{R}^{n'}} N_\nu(v, \xi') \leq C \sup_{\mathbf{R}^{n'}} N_{\nu+1}(v, \xi').$$

LEMMA 6.7. *Let $P(D)$ be as in Theorem 6.2. Then for every compact set K there is a constant C such that*

$$\sup_{\mathbf{R}^{n'}} \tilde{R}(\xi')\tilde{b}(\xi') \,\|\hat{v}(\xi', \cdot)\|^{W}_{p, \tilde{k}} \leq C \sup_{\mathbf{R}^{n'}} \tilde{R}(\xi') \,\|P(\xi', D'')\hat{v}(\xi', \cdot)\,\|^{W}_{p, \tilde{k}}, \quad \text{if} \quad v \in C_0^\infty(K).$$

Proof. Repeated use of Lemma 6.6 gives that

$$\begin{aligned}\sup_{\mathbf{R}^{n'}} |R(\xi')b(\xi')| \,\|\hat{v}(\xi', \cdot)\|^{W}_{p, \tilde{k}} &= \sup_{\mathbf{R}^{n'}} N_0(v, \xi') \leq \\ \leq C \sup_{\mathbf{R}^{n'}} N_m(v, \xi') &= C \sup_{\mathbf{R}^{n'}} |R(\xi')| \,\|P(\xi', D'')\hat{v}(\xi', \cdot)\|^{W}_{p, \tilde{k}} \leq \\ &\leq C \sup_{\mathbf{R}^{n'}} \tilde{R}\,(\xi') \,\|P(\xi', D'')\hat{v}(\xi', \cdot)\|^{W}_{p, \tilde{k}} = C\, G.\end{aligned}$$

By Lemma 6.5 we have

$$|R(\zeta')b(\zeta')| \,\|\hat{v}(\zeta', \cdot)\|^{W}_{p, \tilde{k}} \leq C\, G \quad \text{for} \quad |\operatorname{Im} \zeta'| \leq \text{const.}$$

Let A be the set we get from Lemma A1 in Hörmander [6] when we apply it to polynomials of degree $\leq \deg(Rb)$. For every ξ' there is a $\theta \in A$ such that

$$\tilde{R}(\xi')\tilde{b}(\xi') \leq C\, \tilde{R}(\xi' + z\theta)\tilde{b}(\xi' + z\theta) \leq C' \,|R(\xi' + z\theta)b(\xi' + z\theta)|$$

for $|z| = 1$. This gives that

$$\tilde{R}(\xi')\tilde{b}(\xi') \,\|\hat{v}(\xi' + z\theta, \cdot)\|^{W}_{p, \tilde{k}} \leq C\, G, \quad |z| = 1,$$

which implies that

$$\tilde{R}(\xi')\tilde{b}(\xi') \,\|\hat{v}(\xi', \cdot)\|^{W}_{p, \tilde{k}} \leq C\, G,$$

where C is independent of ξ'.

Proof of Theorem 6.3. Lemma 6.7 and Lemma 5.6 show that for every compact set K and every $k \in \mathcal{K}(\mathbf{R}^n)$ there is a constant C such that

$$\|u\|^{A-}_{p, \tilde{R}\tilde{b}k} \leq C \,\|P(D)u\|^{A-}_{p, \tilde{R}k}, \quad u \in C_0^\infty(K),$$

so the theorem follows from Theorem 4.1.

We have not been able to prove any analogues of Theorems 6.2 and 6.3 for arbitrary n'', so we must put some extra conditions on the polynomials when $n'' \geq 3$. Therefore, we shall now study polynomials satisfying the following condition:

(S) $P(\xi) = \sum_{|\alpha|=m} a_\alpha(\xi')\xi''^\alpha$ has *real* coefficients,
$b = a_{(m,0,\ldots,0)}$ is stronger than a_α for all α and $Q(D'')$ is *strictly* hyperbolic with respect to V for every $Q \in \Sigma$.

Let W be a convex cone such that $\operatorname{int} W \supset V \setminus \{0\}$ and set $\Lambda = \mathbf{R}^{n'} \times W$. Then we can prove

THEOREM 6.8. *If P satisfies condition (S) then the operator $P(D)$ has a fundamental solution $E \in B^{\mathrm{loc}}_{\infty, \tilde{P}}$ with support in Λ.*

For technical reasons we will consider $Q_0(\xi) = P(\xi', \xi'' - iN'')$ where $N'' \in \operatorname{int} W^*$.

THEOREM 6.9. *Let $K \subset \mathbf{R}^n$ be a compact set and $k \in \mathcal{K}(\mathbf{R}^n)$. If Q_0 is the polynomial introduced above, then there is a constant C such that*

$$\|u\|^{\Lambda-}_{1, \tilde{Q}_0 k} \leq C \, \|Q_0(D)u\|^{\Lambda-}_{1,k}, \quad u \in C_0^\infty(K).$$

Proof of Theorem 6.8. Set $k(\xi) = 1/\tilde{Q}_0(\xi)$. Then it follows from Theorem 6.9 and Theorem 4.1 that $Q_0(D)$ has a fundamental solution $E_0 \in B^{\mathrm{loc}}_{\infty, \tilde{Q}_0}$ with support in Λ. This implies that $P(D)$ has a fundamental solution $E \in B^{\mathrm{loc}}_{\infty, \tilde{P}}$ with support in Λ. (Cf. page 349 in Hörmander [6].)

LEMMA 6.10. *Assume that $Q(D'')$ is strictly hyperbolic with respect to V for every Q in Σ. Then there is a neighbourhood of Σ, in the set of all real polynomials of degree m, such that all polynomials in this neighbourhood are strictly hyperbolic with respect to the cone W if* $\operatorname{int} W \supset V \setminus \{0\}$.

Proof. We have to prove that there is a neighbourhood of Σ, in the set of all real homogeneous polynomials of degree m, such that every polynomial Q in this neighbourhood satisfies the following conditions:

$Q(N'') \neq 0$ for all $N'' \in W^* \setminus \{0\}$.

The equation $Q(\xi'' + \tau N'') = 0$, where $N'' \in W^*$, $|N''| = 1$, $\xi'' \in \mathbf{R}^{n''}$, $|\xi''| = 1$ and $N'' \perp \xi''$, has only real simple zeros.

However, this is obvious for every polynomial in Σ satisfies these conditions and Σ is compact. (Cf. Definition 5.5.1 in Hörmander [5].)

Let P satisfy condition (S) and set $Q_\alpha(\xi) = \frac{1}{2}(P(\xi', \xi'' - iN'') + P^{(\alpha)}(\xi', \xi'' - iN''))$ where $N'' \in \operatorname{int} W^*$ and $\alpha = (\alpha', 0)$. The first step in the proof of Theorem 6.9 is the following lemma.

LEMMA 6.11. *There is a constant* C *such that*

$$\|\hat{u}(\xi', \cdot)\|^{W^-}_{1, \tilde{Q}_0 k} \leq C \sum_{\alpha=(\alpha', 0)} \|Q_\alpha(\xi', D'')\hat{u}(\xi', \cdot)\|^{W^-}_{1, k}$$

for all $\xi' \in \mathbf{R}^{n'}$, $u \in C_0^\infty(\mathbf{R}^n)$ *and* $k \in \mathcal{K}(\mathbf{R}^{n''})$.

To prove this we need a lemma due to L. Hörmander.

LEMMA 6.12. *Let* $\Omega \subset \mathbf{C}^n$ *be a convex set and let* $Q_1, \ldots, Q_m$ *be polynomials of degree* $< N$, *such that* $Q_j(\zeta) \neq 0$, $j = 1, \ldots, m$, *when* $\zeta \in \Omega$. *Then there are functions* $g_1, \ldots, g_m$, *that are analytic in* Ω, *such that*

$$1 = \Sigma\, Q_j(\zeta) g_j(\zeta), \quad \zeta \in \Omega,$$

and

$$|g_j(\zeta)| \leq C / \sum_1^m |Q_j(\zeta)|, \quad \zeta \in \Omega.$$

Here C *is a constant that depends only on* m *and* N.

Proof. The variation of the argument of Q_j is at most $\pi(N-1)$ when $\zeta \in \Omega$, for if $\zeta_1, \zeta_2 \in \Omega$ then $Q_j(t\zeta_1 + (1-t)\zeta_2) = a\Pi(t - t_i)$, where $t_i \notin [0, 1]$. Since Ω is simply connected we can choose an analytic branch of $Q_j^{1/N}$ in Ω. We obtain that the variation of the argument of $Q_j^{1/N}$ is $\leq \pi(N-1)/N < \pi$. Thus there are constants $a_j \in \mathbf{C}$, $|a_j| = 1$, such that $|\arg(a_j Q_j^{1/N})| \leq \pi(N-1)/2N < \pi/2$ when $\zeta \in \Omega$. This implies that there is a constant $c > 0$ such that

$$c|Q_j^{1/N}| \leq \operatorname{Re}(a_j Q_j^{1/N}) \leq |Q_j^{1/N}|, \quad \zeta \in \Omega.$$

Set $q = \sum_1^m a_j Q_j^{1/N}$. Then

$$c\,\Sigma\,|Q_j|^{1/N} \leq \operatorname{Re} q \leq |q| \leq \Sigma\,|Q_j|^{1/N} \quad \text{and} \quad q^{mN} = \Big(\sum_1^m a_j Q_j^{1/N}\Big)^{Nm} = \sum_1^m Q_j h_j. \tag{6.3}$$

Thus

$$1 = \sum_1^m Q_j(\zeta) g_j(\zeta), \quad \zeta \in \Omega \quad \text{where} \quad g_j(\zeta) = h_j(\zeta) q(\zeta)^{-mN}.$$

The function h_j is a sum of terms of the form $\Pi(a_k Q_k^{1/N})$, where the products consist of $Nm - N$ factors. Thus, according to (6.3)

$$|h_j| \leq C_1 |q|^{Nm-N},$$

which shows that

$$|g_j(\zeta)| \leq C_2 |q(\zeta)|^{-N} \leq C / \Big(\sum_1^m |Q_j|\Big),$$

where the constant C depends only on N and m.

Proof of Lemma 6.11. We are going to apply Lemma 6.12 to the polynomials Q_α. Since $(a_\beta^{(\alpha)}(\xi')/b(\xi')) \to 0$ if $|\alpha| \neq 0$ and $d(\xi', b^{-1}(0)) \to \infty$, we obtain from Lemma 6.10 that there exists a constant $t > 0$ such that $Q_\alpha(\xi', \zeta'') \neq 0$ if $\operatorname{Im} \zeta'' \in N'' - - \operatorname{int} W^*$ and $d(\xi', b^{-1}(0)) \geq t$. Lemma 6.12 shows that if $d(\xi', b^{-1}(0)) \geq t$ then there are analytic functions $g_{\alpha, \xi'}$, such that

$$1 = \Sigma\, Q_\alpha(\xi', \zeta'')g_{\alpha, \xi'}(\zeta''), \quad \operatorname{Im} \zeta'' \in -W^*,$$

and

$$|g_{\alpha, \xi'}(\zeta'')| \leq C/\Sigma\, |Q_\alpha(\xi', \zeta'')| \leqq C_1/\tilde{Q}_0(\xi', \zeta''), \quad \operatorname{Im} \zeta'' \in -W^*.$$

(Cf. the proof of Theorem 5.5.7 in [5].) Set

$$\check{E}_{\alpha, \xi'}(v) = (2\pi)^{-n''} \int \hat{v}(\xi'')g_{\alpha, \xi'}(\xi'')d\xi'', \quad v \in C_0^\infty(\mathbf{R}^{n''}).$$

Then we see immediately that $E_{\alpha, \xi'} \in B_{\infty, \tilde{Q}_0}$, with $\|E_{\alpha, \xi'}\|_{\infty, \tilde{Q}_0}$ bounded by a constant that does not depend on ξ'. By changing the integration contour we obtain that $\operatorname{supp} E_{\alpha, \xi'} \subset W$. Moreover,

$$\hat{u}(\xi', x'') = \sum_{\alpha = (\alpha', 0)} E_{\alpha, \xi'} * (Q_\alpha(\xi', D'')\hat{u}(\xi', \cdot)), \quad u \in C_0^\infty(\mathbf{R}^n).$$

If $h_{\alpha, \xi'} \in \mathcal{S}$ and $h_{\alpha, \xi'}(x'') = Q_\alpha(\xi', D'')\hat{u}(\xi', x'')$ on W_- then $h_{\xi'}(x'') = \Sigma(E_{\alpha, \xi'} * h_{\alpha, \xi'})(x'') = \hat{u}(\xi', x'')$ on W_-. This implies that

$$\|\hat{u}(\xi', \cdot)\|_{1, \tilde{Q}_0 k}^{W_-} \leq \|h_{\xi'}\|_{1, \tilde{Q}_0 k} \leq \Sigma \| E_{\alpha, \xi'} * h_{\alpha, \xi'}\|_{1, \tilde{Q}_0 k} \leq C\, \Sigma\, \|h_{\alpha, \xi'}\|_{1, k}.$$

Thus

$$\|\hat{u}(\xi', \cdot)\|_{1, \tilde{Q}_0 k}^{W_-} \leq C \sum_{\alpha = (\alpha', 0)} \|Q_\alpha(\xi', D'')\hat{u}(\xi', \cdot)\|_{1, k}^{W_-}$$

for all $\xi' \in \mathbf{R}^{n'}$ with $d(\xi', b^{-1}(0)) \geq t$. When $\xi' \in \mathbf{R}^{n'}$ is arbitrary, then by Lemma A2 in Hörmander [6] there is a point $\theta' = \theta'(\xi') \in \mathbf{R}^{n'}$ with $|\theta'| \leqq t/\gamma$ and $d(\xi' + \theta', b^{-1}(0)) \geq t$. This implies that

$$\|\hat{u}(\xi', \cdot)\|_{1, \tilde{Q}_0(\xi'+\theta', \cdot)k}^{W_-} \leq C\, \Sigma\, \|Q_\alpha(\xi' + \theta', D'')\hat{u}(\xi', \cdot)\|_{1, k}^{W_-}.$$

Thus

$$\|\hat{u}(\xi', \cdot)\|_{1, \tilde{Q}_0 k}^{W_-} \leq C \sum_{\alpha = (\alpha', 0)} \|Q_\alpha(\xi', D'')\hat{u}(\xi', \cdot)\|_{1, k}^{W_-}$$

for all $\xi' \in \mathbf{R}^{n'}$, $u \in C_0^\infty(\mathbf{R}^n)$ and $k \in \mathcal{K}(\mathbf{R}^{n''})$.

Now we want to find a bound for the right-hand side of the estimate in Lemma 6.11. Let t be the constant introduced in the proof of that lemma. Thus we have $Q_\alpha(\xi', \zeta'') \neq 0$ for all α if $d(\xi', b^{-1}(0)) \geq t$ and $\operatorname{Im} \zeta'' \in N'' - \operatorname{int} W^*$. Furthermore, let $0 \not\equiv \chi \in C_0^\infty(\{x' \in \mathbf{R}^{n'}; |x'| \leq 1\})$ and set $\chi_{\theta'}(\xi') = \chi(\xi' - \theta')$. If $\zeta' \in \mathbf{C}^{n'}$

we define $\chi_{\theta'}(\zeta')$ by setting $\chi_{\theta'}(\zeta') = \chi_{\theta'}(\mathrm{Re}\, \zeta')$. When $\theta' \in T = \{\xi' \in \mathbf{R}^{n'};\ d(\xi', b^{-1}(0)) \geq t + 2\}$ and $\beta = (\beta', 0)$, $|\beta| = 1$ we define

$$\check{E}(v) = \check{E}(\alpha, \beta, \theta'; v) = (2\pi)^{-n'} \int \chi_{\theta'}(\xi') Q_\alpha^{(\beta)}(\xi)\hat{v}(\xi)/Q_\alpha(\xi) d\xi,\ v \in C_0^\infty(\mathbf{R}^n).$$

LEMMA 6.13. *Let E be defined as above. Then $E \in B_{\infty,1}^{\mathrm{loc}}$ and $\mathrm{supp}\, E \subset \Lambda = \mathbf{R}^{n'} \times W$. Furthermore, if $\varphi \in C_0^\infty(\mathbf{R}^{n'})$ then $\|\varphi E\|_{\infty,1}$ is bounded uniformly with respect to θ'.*

Proof. By changing the integration contour in the ξ'' variables, we obtain immediately that $\mathrm{supp}\, E \subset \Lambda$.

Let K be a compact set and assume that $\beta = (1, 0, \ldots, 0)$. If $v \in C_0^\infty(K)$ then

$$\check{E}(v) = (2\pi)^{-n'} \int \sum_j \chi_{\theta'}(\xi')\hat{v}(\xi)/(\xi_1 - \tau_j(\xi^0)) d\xi =$$

$$= (2\pi)^{-n'} \int d\xi^0 \sum_j \int_{\gamma_j} \chi_{\theta'}(\zeta_1, \xi^0)\hat{v}(\zeta_1, \xi^0)/(\zeta_1 - \tau_j(\xi^0)) d\zeta_1 +$$

$$+ (2\pi)^{-n'} \int d\xi^0 \sum_j (2i)\, \mathrm{sgn}\, (\mathrm{Im}\, \tau_j(\xi^0)) \iint_{\Omega_j} \partial\chi_{\theta'}/\partial\bar{\zeta}_1 \hat{v}(\zeta_1, \xi^0)/(\zeta_1 - \tau_j(\xi^0)) d\xi_1 d\eta_1$$

where $\xi^0 = (\xi_2, \ldots, \xi_n)$ and $\gamma_j = \mathbf{R} + i$ if $\mathrm{Im}\, \tau_j < 0$ and $\gamma_j = \mathbf{R} - i$ if $\mathrm{Im}\, \tau_j > 0$. Furthermore, Ω_j denotes the support of $\partial\chi_{\theta'}/\partial\bar{\zeta}_1$ between γ_j and $\mathbf{R}$. We get the estimate

$$|\check{E}(v)| \leq \int d\xi^0\, \Sigma \int_{\gamma_j} |\hat{v}(\zeta_1, \xi^0)|\, d\zeta_1 +$$

$$+ C \int \sup_{\Omega_j} |\hat{v}(\zeta_1, \xi^0)|\, d\xi^0 \leq C_1 \int |\hat{v}(\xi)|\, d\xi,$$

where C_1 is independent of θ', for $\int |\hat{v}(\xi + i\eta)|\, d\xi \leq e^{A|\eta|} \int |\hat{v}(\xi)|\, d\xi$.

We also need the equivalence between two norms that we are going to use. The following lemma, which is inspired by Beurling [1], proves this.

LEMMA 6.14. *Let $K' \subset \mathbf{R}^{n'}$ be a compact set and let $\psi \in C_0^\infty(\mathbf{R}^{n'})$ be 1 on K'. If $\chi_{\theta'}$ is the function defined above then there is a constant C such that*

$$\sup_{\mathbf{R}^{n'}} \|\hat{u}(\xi', \cdot)\|_{1,\tilde{k}}^{W} \leq C \sup_{\mathbf{R}^{n'}} \sup_{\theta' \in T} \|(\hat{\psi} * (\chi_{\theta'}\hat{u}))(\xi', \cdot)\|_{1,\tilde{k}}^{W}$$

for all $u \in C_0^\infty(K' \times \mathbf{R}^{n''})$ and $k \in \mathcal{K}(\mathbf{R}^{n''})$.

Proof. Let g be an element in the unit ball of the dual space of $C_0^\infty(\mathbf{R}^{n''})$ equipped with the norm $\|\cdot\|_{1,k}^{W}$ and set $v(x') = \langle u(x', \cdot), g\rangle$. If we prove that

$$\sup_{\mathbf{R}^{n'}} |\hat{v}(\xi')| \leq C \sup_{\mathbf{R}^{n'}} \sup_{\theta' \in T} |(\hat{\psi} * (\chi_{\theta'}\hat{v}))(\xi')|, \quad v \in C_0^\infty(K'), \tag{6.4}$$

the statement will follow as g varies.

Assume that (6.4) is false. Then there is a sequence $v_j \in C_0^\infty(K')$ such that

$$\sup_{\mathbf{R}^{n'}} |v_j(\xi')| = 1$$

and

$$|(\hat{\psi} * (\chi_{\theta'}\hat{v}_j))(\xi')| \leq 1/j \text{ for all } \xi' \in \mathbf{R}^{n'} \text{ and } \theta' \in T.$$

Take $\xi_j' \in \mathbf{R}^{n'}$ such that $|\hat{v}_j(\xi_j')| = 1$ and set $\hat{w}_j(\xi') = \hat{v}_j(\xi' + \xi_j')$. Then we have that

$$|\hat{w}_j(\xi')| \leq |\hat{w}_j(0)| = 1 \text{ for all } \xi' \in \mathbf{R}^{n'}$$

and

$$\left| \int \hat{\psi}(\xi' - \eta')\chi(\eta' + \xi_j' - \theta')\hat{w}_j(\eta')d\eta' \right| \leq 1/j \text{ for all } \xi' \in \mathbf{R}^{n'} \text{ and } \theta' \in T.$$

We can now choose a subsequence $\hat{w}_{j_k}$ of $\hat{w}_j$ such that $\hat{w}_{j_k}$ converge uniformly to h on every compact set. Then h is analytic, $|h(\xi')| \leq |h(0)| = 1$ for all $\xi' \in \mathbf{R}^{n'}$. Furthermore, by Lemma A2 in Hörmander [6] we can choose a sequence $\theta_j' \in T$ such that $\xi_j' - \theta_j'$ is bounded. Finally, choose a subsequence of $\xi_{j_k}' - \theta_{j_k}'$ which converges to $-\theta_0'$. Then

$$\int \hat{\psi}(\xi' - \eta')\chi(\eta' - \theta_0')h(\eta')d\eta' = 0 \text{ for all } \xi' \in \mathbf{R}^{n'}$$

This implies that $\psi \cdot \mathcal{F}^{-1}(\chi_{\theta_0'}h) \equiv 0$, so $\chi_{\theta_0'}h \equiv 0$, for $\mathcal{F}^{-1}(\chi_{\theta_0'}h)$ is analytic. However, h is also analytic and $\chi_{\theta_0'} \not\equiv 0$, so $h \equiv 0$, which contradicts that $|h(0)| = 1$. This proves the lemma.

Note that we have only used that T is defined by some polynomial $b \neq 0$, with given degree. Thus, the constant C in the lemma depends only on the degree of the polynomial defining T and not on the polynomial itself.

If $k \in \mathcal{K}(\mathbf{R}^n)$ then there are constants C and N such that $k(\xi + \eta) \leq \leq (1 + C|\xi|)^N k(\eta)$ for all $\xi, \eta \in \mathbf{R}^n$. For given C and N we shall here use the notation $\mathcal{K}(\mathbf{R}^n, C, N)$ for all $k \in \mathcal{K}(\mathbf{R}^n)$ such that $k(\xi + \eta) \leq (1 + C|\xi|)^N k(\eta)$ for all $\xi, \eta \in \mathbf{R}^n$. We shall now use Lemma 6.13 and Lemma 6.14 to prove

LEMMA 6.15. *Let $K \subset \mathbf{R}^n$ be a compact set, $C_0, N \in \mathbf{R}_+$ and let Q_α be as before. Then there is a constant C such that*

$$\|Q_\alpha(D)u\|_{1,k}^{\Lambda_-} \leq C\|Q_0(D)u\|_{1,k}^{\Lambda_-}$$

for all $u \in C_0^\infty(K)$ *and* $k \in \mathcal{K}(\mathbf{R}^{n''}, C_0, N)$.

Proof. Let $u \in C_0^\infty(K)$ and $\psi \in C_0^\infty(\mathbf{R}^n)$, where $\psi = 1$ in a neighbourhood of K. Furthermore, let $\varphi \in C_0^\infty(\mathbf{R}^n)$ be 1 in a neighbourhood U of zero, such that $((\complement U) \pm K) \cap \operatorname{supp} \psi = \emptyset$. If $E = E(\alpha, \beta, \theta')$ is the distribution in Lemma 6.13, $|\beta| = 1$, and $\mathcal{S} \ni g = Q_\alpha(D)u$ in Λ_-, then

$$\psi((\varphi E) * g) = \psi\chi_{\theta'}(D')Q_\alpha^{(\beta)}(D)u \quad \text{in} \quad \Lambda_-.$$

Thus

$$\|\psi\chi_{\theta'}(D')Q_\alpha^{(\beta)}(D)u\|_{1,k}^{\Lambda_-} \leq \|\psi((\varphi E) * g)\|_{1,k} \leq \|\psi\|_{1,M_k} \|\varphi E\|_{\infty,1} \|g\|_{1,k} \leq C\|g\|_{1,k},$$

which implies that

$$\|\psi\chi_{\theta'}(D')Q_\alpha^{(\beta)}(D)u\|_{1,k}^{\Lambda_-} \leq C\|Q_\alpha(D)u\|_{1,k}^{\Lambda_-}.$$

Note that C is independent of $\theta' \in T$. However, with $\hat{Q}_\alpha^{(\beta)} = Q_\alpha^{(\beta)}(\xi', D'')$,

$$\sup_{\xi'} \|(\hat{\psi} * (\chi_{\theta'}\hat{Q}_\alpha^{(\beta)}\hat{u}))(\xi', \cdot)\|_{1,k}^{W_-} \leq C_1 \|\psi\,\chi_{\theta'}(D')Q_\alpha^{(\beta)}(D)u\|_{1,k}^{\Lambda_-} \leq C_2 \|Q_\alpha(D)u\|_{1,k}^{\Lambda_-}.$$

In view of Lemma 6.14, this implies that

$$\sup_{\mathbf{R}^{n'}} \|Q_\alpha^{(\beta)}(\xi', D'')\hat{u}(\xi', \cdot)\|_{1,k}^{W_-} \leq C\|Q_\alpha(D)u\|_{1,k}^{\Lambda_-}$$

i.e.,

$$\sup_{\mathbf{R}^{n'}} \|Q_\alpha^{(\beta)}(\xi', D'')\hat{u}(\xi', \cdot)\|_{1,k}^{W_-} \leq C \int \|Q_\alpha(\xi', D'')\hat{u}(\xi', \cdot)\|_{1,k}^{W_-} d\xi'.$$

By using the technique in the proof of Theorem 3.10 in Hörmander [6] we obtain from this that

$$\|Q_\alpha^{(\beta)}(D)u\|_{1,k}^{\Lambda_-} \leq C\|Q_\alpha(D)u\|_{1,k}^{\Lambda_-}.$$

Hence

$$\begin{aligned}\|Q_{\alpha+\beta}(D)u\|_{1,k}^{\Lambda_-} &= \tfrac{1}{2}\|(Q_0(D) + 2Q_\alpha^{(\beta)}(D) - Q_0^{(\beta)}(D))u\|_{1,k}^{\Lambda_-} \leq \\ &\leq \|Q_0(D)u\|_{1,k}^{\Lambda_-} + \|Q_\alpha^{(\beta)}(D)u\|_{1,k}^{\Lambda_-} + \|Q_0^{(\beta)}(D)u\|_{1,k}^{\Lambda_-} \leq \\ &\leq C(\|Q_\alpha(D)u\|_{1,k}^{\Lambda_-} + \|Q_0(D)u\|_{1,k}^{\Lambda_-}).\end{aligned}$$

This proves the lemma, for $\alpha = (\alpha', 0)$ and $\beta = (\beta', 0)$, $|\beta| = 1$ are arbitrary.

Proof of Theorem 6.9. By Lemma 6.11 we have

$$\|\hat{u}(\xi', \cdot)\|_{1,\tilde{Q}_0 k_{\eta'}}^{W_-} \leq C\,\Sigma\,\|Q_\alpha(\xi', D'')\hat{u}(\xi', \cdot)\|_{1,k_{\eta'}}^{W_-} \leq C\,\Sigma\,\|Q_\alpha(D)u\|_{1,k_{\eta'}}^{\Lambda_-},$$

where $k_{\eta'}(\xi'') = k(\eta', \xi'')$. However, there are constants C_0 and N such that $k_{\eta'} \in \mathcal{K}(\mathbf{R}^{n''}, C_0, N)$ for all η'. Thus Lemma 6.15 implies that for all η'

$$\|\hat{u}(\xi', \cdot)\|^{W^-}_{1, \tilde{Q}_0 k_{\eta'}} \leq C \, \|Q_0(D)u\|^{\Lambda^-}_{1, k_{\eta'}} = C \int \|Q_0(\xi', D'')\hat{u}(\xi', \cdot)\|^{W^-}_{1, \bar{k}_{\eta'}} \, d\xi'.$$

By using the technique in the proof of Theorem 3.10 in [6] we obtain from this that

$$\|u\|^{\Lambda^-}_{1, \tilde{Q}_0 k} \leq C \, \|Q_0(D)u\|^{\Lambda^-}_{1, k}, \quad u \in C_0^\infty(K).$$

We shall now investigate in detail the necessary conditions for polynomials P satisfying the following condition:

(6.5) $P(\xi) = \sum_{|\alpha| = m} a_\alpha(\xi')\xi''^\alpha$, where $b(\xi') = a_{(m, 0, \ldots, 0)}(\xi')$ is real and stronger than a_α for all α. Furthermore, assume that $Q(\xi'')$ is strictly hyperbolic with respect to every vector in the interior of the proper, closed and convex cone V^* for all $Q \in \Sigma$, and that $N'' = (1, 0, \ldots, 0) \in$ int V^*.

Note that we can make b real by multiplying the polynomial by the complex conjugate of b. Observe also that, since Σ is compact, there is a smallest, proper, closed and convex cone $V^* \ni N''$ satisfying the condition in (6.5).

THEOREM 6.16. *Let P be a polynomial satisfying* (6.5). *Then the following conditions are equivalent*:

(i) *$P(D)$ is an evolution operator with respect to every half space containing $\mathbf{R}^{n'} \times (V \setminus \{0\})$ in its interior.*

(ii) $\operatorname{Im} P(\xi)$ *is dominated by $P(\xi)$.*

(iii) *P satisfies the necessary conditions of Theorem* 2.6 *with respect to $\Gamma =$ $= \mathbf{R}^{n'} \times W$ for every cone W such that* int $W \supset V \setminus \{0\}$.

We are going to prove the theorem after some lemmas.

LEMMA 6.17. *Let a and b be polynomials such that $a(\xi_j)/b(\xi_j) \to 0$, $j \to \infty$, for all sequences $\xi_j \in \mathbf{R}^n$ such that $d(\xi_j, b^{-1}(0)) \to \infty, j \to \infty$. Then a is dominated by b.*

Proof. Set $\varepsilon(t) = \sup \{ |a(\xi)/b(\xi)| \, ; d(\xi, b^{-1}(0)) \geq t > 0\}$. It follows from the assumption that $\varepsilon(t) \to 0$ as $t \to \infty$. From Lemma A2 in [6] we obtain that there is a constant $\gamma > 0$ such that for every $\xi \in \mathbf{R}^n$ there is a $\theta \in \mathbf{R}^n, \gamma|\theta| < 1$, such that the distance from $\xi + t\theta$ to the zeros of ab is at least t. Now we have that

$$\tilde{a}(\xi, t) \leqq C_1 \tilde{a}(\xi + t\theta, t) \leqq C_2 \, |a(\xi + t\theta)| \leqq C_2 \varepsilon(t) \, |b(\xi + t\theta)| \leqq$$
$$\leqq C_2 \varepsilon(t) \tilde{b}(\xi + t\theta, t) \leqq C_3 \varepsilon(t) \tilde{b}(\xi, t),$$

where the second estimate follows from Lemma 4.1.1 in [5]. Thus

$$\sup_{\mathbf{R}^n} \tilde{a}(\xi, t)/\tilde{b}(\xi, t) \leqq C_3 \, \varepsilon(t) \to 0, \quad t \to \infty,$$

which proves the lemma.

LEMMA 6.18. *Let P satisfy* (6.5) *and assume that P is an evolution operator with respect to the half space $\{x \in \mathbf{R}^n \, ; \, x_{n'+1} \geq 0\}$. Then P satisfies the following condition:*

(6.6) *There are constants R_0, t_0 such that if $\xi_0' \in \mathbf{R}^{n'}$, $d(\xi_0', b^{-1}(0)) \geq t_0$ and $\Omega(\xi_0', R_0) \ni \zeta' \to \tau(\zeta') \in \mathbf{C}$ is an analytic function satisfying the equation $P(\zeta', \xi'' + \tau(\zeta')N'') = 0$ in $\Omega(\xi_0', R_0)$, then there is a point in $\Omega(\xi_0', R_0)$ such that* $\operatorname{Im} \tau(\zeta') = 0$. *Here $N'' = (1, 0, \ldots, 0)$ and ξ'' is orthogonal to N''.*

Proof. Set $R_0 = A_1$, where A_1 is the constant we obtain from Theorem 1.1 in [6]. If τ is the function in (6.6) then we can extend it to an analytic function $\tau(\zeta', \zeta'')$ such that $P(\zeta', \zeta'' + \tau(\zeta', \zeta'')N'') = 0$ for $\zeta' \in \Omega = \Omega(\xi_0', R_0)$ and ζ'' near ξ'' (t_0 large). However, the function $\tau(\zeta', \zeta'')$ is homogeneous of degree 1 with respect to ζ'', so we can extend τ by homogeneity and if t is large we see that $\tau(\zeta', t\xi'' + z'')$ is defined and analytic for all $\zeta' \in \Omega$ and all $z'' = (0, z_{n'+2}, \ldots, z_n)$ with $|z''| \leqq R_0$. Now we obtain from the assumptions that for every large t there are $\zeta_t' \in \Omega$ and z_t'' with $|z_t''| \leq R_0$ such that $\operatorname{Im} \tau(\zeta_t', t\xi'' + z_t'') \geq 0$. This implies that $\operatorname{Im} \tau(\zeta_t', \xi'' + z_t''/t) \geq 0$ and as $t \to \infty$ we obtain that there is a point $\zeta' \in \Omega$ such that $\operatorname{Im} \tau(\zeta', \xi'') \geq 0$. In the same way we obtain that there is a point $\zeta' \in \Omega$ where $\operatorname{Im} \tau(\zeta', -\xi'') \geq 0$. Thus, there is also a point $\zeta' \in \Omega$ where $\operatorname{Im} \tau(\zeta', \xi'') = 0$.

Now we can prove that condition (i) of Theorem 6.16 implies (ii).

LEMMA 6.19. *Let P be an evolution operator satisfying* (6.5). *Then* $\operatorname{Im} P(\xi)$ *is dominated by* $P(\xi)$.

Proof. Let $P(\xi) = P_1(\xi) + iP_2(\xi) = \sum_{|\alpha|=m} c_\alpha(\xi')\xi''^\alpha + i \sum_{|\alpha|=m} d_\alpha(\xi')\xi''^\alpha$, where c_α and d_α are real. We obtain from Lemma 6.17 that d_α is dominated by $b = a_{(m,0,\ldots,0)} = c_{(m,0,\ldots,0)}$ for all α. We shall here use the notation $\sim$ (see page 35 in [5]) with respect to the ξ' variables only. Set

$$k_1(\xi') = \sum_\alpha (\tilde{d}_\alpha(\xi') + \sum_{=1}^{n'} \widetilde{(\partial_j a_\alpha)}(\xi'))$$

and

$$k_2(\xi) = \sum_1^{n'} (\widetilde{\partial_j P})(\xi).$$

In order to prove that P_2 is dominated by P it is sufficient to prove that there is a constant C such that

$$\tilde{b}(\xi')\tilde{P}_2(\xi) \leqq C(\tilde{P}(\xi)\, k_1(\xi') + \tilde{b}(\xi')\, k_2(\xi))$$

for all $\xi' \in \mathbf{R}^{n'}$ and $\xi'' \in \mathbf{R}^{n''}$ with $|\xi''| = 1$. However, this will follow if we prove that there are constants C_0 and t_0 such that

$$|b(\xi')\, P_2(\xi)| \leqq C_0(\tilde{P}(\xi)\, k_1(\xi') + \tilde{b}(\xi')\, k_2(\xi))$$

for all $\xi' \in \mathbf{R}^{n'}$ with $d(\xi', b^{-1}(0)) \geq t_0$ and all ξ'' with $|\xi''| = 1$. We are going to prove this by contradiction. Thus, suppose that this is false. Then there exists a sequence $\xi_\nu \in \mathbf{R}^n$ such that

$$|b(\xi_\nu')\, P_2(\xi_\nu)| \geqq \nu(\tilde{P}(\xi_\nu)\, k_1(\xi_\nu') + \tilde{b}(\xi_\nu')\, k_2(\xi_\nu)), \tag{6.7}$$
$$|\xi_\nu''| = 1,$$
$$d(\xi_\nu', b^{-1}(0)) \geq \nu.$$

We can assume that $P(\xi_\nu', \xi'')/b(\xi_\nu') \to Q(\xi'')$ and $\xi_\nu'' \to \xi_0''$ as $\nu \to \infty$. If we divide (6.7) by $b(\xi_\nu')\, k_1(\xi_\nu')$ and let $\nu \to \infty$, then we obtain that $Q(\xi_0'') = 0$. Now, consider the equation $P(\xi_\nu' + \zeta', \xi_\nu'' + \tau N'') = 0$ for $|\zeta'| \leqq R_0$, where R_0 is the constant we obtain from Lemma 6.18. If ν is large then this equation has an analytic solution τ_ν such that $\sup_{|\zeta'| \leqq R_0} |\tau_\nu(\zeta')| \to 0$ as $\nu \to \infty$. From Lemma 6.18 we obtain that for every ν there is a point ζ_ν', $|\zeta_\nu'| \leqq R_0$, such that $\operatorname{Im} \tau_\nu(\zeta_\nu') = 0$. Thus,

$$P(\xi_\nu' + \zeta_\nu', \xi_\nu'') + \tau_\nu(\zeta_\nu')\, A_\nu(\zeta_\nu') = 0$$

and

$$\operatorname{Im} P(\xi_\nu' + \zeta_\nu', \xi_\nu'') + \tau_\nu(\zeta_\nu') \operatorname{Im} A_\nu(\zeta_\nu') = 0,$$

where

$$A_\nu(\zeta') = \sum_1^m (\partial^j P/\partial \xi_{n'+1}^j)(\xi_\nu' + \zeta', \xi_\nu'')(\tau_\nu(\zeta'))^{j-1}/j!.$$

This implies that

$$A_\nu(\zeta_\nu') \ \operatorname{Im} P(\xi_\nu' + \zeta_\nu', \xi_\nu'') = P(\xi_\nu' + \zeta_\nu', \xi_\nu'') \ \operatorname{Im} A_\nu(\zeta_\nu').$$

However, $A_\nu(\zeta_\nu')/b(\xi_\nu') \to (\partial Q/\partial \xi_{n'+1})(\xi_0'') \neq 0$ as $\nu \to \infty$, which implies that $|A_\nu(\zeta_\nu')| \geq c_0 |b(\xi_\nu')|$, $c_0 > 0$, if ν is large. Furthermore, $|\operatorname{Im} A_\nu(\zeta_\nu')| \leq C_1 k_1(\xi_\nu')$ and $|P_2(\xi_\nu) - \operatorname{Im} P(\xi_\nu' + \zeta_\nu', \xi_\nu'')| \leq C_2 k_2(\xi_\nu)$. Thus, for some constant C we have

$$|b(\xi_\nu')\, P_2(\xi_\nu)| \leq C(\tilde{b}(\xi_\nu')\, k_2(\xi_\nu) + \tilde{P}(\xi_\nu)\, k_1(\xi_\nu'))$$

if ν is large. This contradicts (6.7) and thus the lemma is proved.

LEMMA 6.20. *Let $f = (f_{n'+1}, \ldots, f_n) : \mathbf{C}^{n'} \to \mathbf{C}^{n''}$ be analytic in a ball with radius R and centre ξ' and assume that there is a constant $C_1 > 0$ such that $0 < |\operatorname{Im} f(\zeta')| \leqq C_1 \operatorname{Im} f_{n'+1}(\zeta')$ for all $\zeta' \in B$. Then there is a constant C_2 such that*

$$|f(\zeta') - f(\xi')| \leqq \mu C_2 \operatorname{Im} f_{n'+1}(\xi')$$

for all $\zeta' \in \mathbf{C}^{n'}$ with $|\zeta' - \xi'| \leqq \mu R,\ \ 0 < \mu \leqq 1/2$.

Proof. Set $g_1(z) = C_1 f_{n'+1}(\xi' + z\zeta')$ and $g_2(z) = f_j(\xi' + z\zeta')$ for $z \in \mathbf{C}$, $|\zeta'| \leqq R$ and $j \geq n' + 2$. For $|z| \leqq \mu \leqq 1/2$ we obtain that

$$|g_j'(z)| = \left| i/\pi \int_0^{2\pi} \operatorname{Im} g_j(e^{i\theta}) e^{i\theta}/(e^{i\theta} - z)^2 \, d\theta \right| \leqq$$

$$\leqq 1/\pi \int_0^{2\pi} \operatorname{Im} g_1(e^{i\theta})/(1-\mu)^2 \, d\theta = 2 \operatorname{Im} g_1(0)/(1-\mu)^2, \quad j = 1, 2.$$

Thus $|g_j(\mu) - g_j(0)| \leqq 2\mu \operatorname{Im} g_1(0)/(1-\mu)^2$, which implies that $|f(\xi' + \zeta') - f(\xi')| \leqq \mu C_2 \operatorname{Im} f_{n'+1}(\xi')$ for $|\zeta'| \leqq \mu R$.

LEMMA 6.21. *Let P satisfy* (6.5) *and the following condition:*

> *For every $\nu > 0$ there are $\xi_\nu' \in \mathbf{R}^{n'}$ and analytic functions $f^{(\nu)} : \Omega(\xi_\nu', \nu) \to \mathbf{C}^{n''}$ such that $b(\zeta') \neq 0$, $P(\zeta', f^{(\nu)}(\zeta')) = 0$ and $\operatorname{Im} f^{(\nu)}(\zeta') \in -\operatorname{int} W^*$ for all $\zeta' \in \Omega(\xi_\nu', \nu)$.*

Then P also satisfies the following condition:

> *There is a vector $0 \neq N'' \in W^*$ such that for every $R > 0$ there are $\xi \in \mathbf{R}^n$ and an analytic function $\tau(\zeta')$ such that $P(\zeta', \xi'' + \tau(\zeta')N'') = 0$ and $\operatorname{Im} \tau(\zeta') < 0$ for all $\zeta' \in \Omega(\xi', R)$.*

Proof. We can assume that $P(\xi_\nu', \xi'')/b(\xi_\nu') \to Q(\xi'') \in \Sigma$ as $\nu \to \infty$ and that $|\zeta_\nu''| = 1$ where $\zeta_\nu'' = \xi_\nu'' + i\eta_\nu'' = f^{(\nu)}(\xi_\nu')$. Further, we can assume that $\zeta_\nu'' \to \zeta_0''$ and $\eta_\nu''/|\eta_\nu''| \to -N''$ as $\nu \to \infty$. Then $Q(\zeta_0'') = 0$ and since $\operatorname{Im} \zeta_0'' \in -W^*$ we conclude that $\zeta_0'' = \xi_0'' \in \mathbf{R}^{n''}$.

Now, let $R > 0$ and consider the equation $P(\zeta', \zeta'' + \tau N'') = 0$ for $\zeta' \in \Omega(\xi_\nu', R)$ and $|\zeta'' - \xi_0''| < \varepsilon$. If $\varepsilon > 0$ is small and ν is large then this equation has a unique analytic solution $\tau_\nu(\zeta)$ such that $\tau_\nu(\xi_\nu', \zeta_\nu'') = 0$. To see this we observe that $Q(\xi'')$ is strictly hyperbolic with respect to N''. If $\zeta_\nu' \in \Omega(\xi_\nu', R)$ then $P(\zeta_\nu', \xi'')/b(\zeta_\nu') \to Q(\xi'')$ as $\nu \to \infty$, and $Q(\xi_0'' + \tau N'') = 0$ has $\tau = 0$ as a simple zero, so the implicit function theorem can be applied. Furthermore, this theorem implies that $\partial\tau_\nu/\partial\zeta_j$ is bounded for every j if ν is large.

From Lemma 6.20 we obtain that $|f^{(\nu)}(\zeta') - \zeta_\nu''| \leq C(R/\nu)\, |\eta_\nu''|$ if $\zeta' \in \Omega(\xi_\nu', R)$. If $\zeta' \in \Omega(\xi_\nu', R)$ we have $0 = P(\zeta', f^{(\nu)}(\zeta')) = P(\zeta', f^{(\nu)}(\zeta') + i|\eta_\nu''|\, N'' - - i|\eta_\nu''|\, N'') = 0$, so that $\tau_\nu(\zeta', f^{(\nu)}(\zeta') + i|\eta_\nu''|\, N'') = - i|\eta_\nu''|$ if ν is large. From this we obtain that $|\tau_\nu(\zeta', \xi_\nu'') + i|\eta''_\nu|\,| \leq C|\, f^{(\nu)}(\zeta') + i|\eta_\nu''|\, N'' - \xi_\nu''| \leq C(|f^{(\nu)}(\zeta') - - \zeta_\nu''| + |\eta_\nu'' + |\eta_\nu''|\, N''|) \leq \frac{1}{2}\, |\eta_\nu''|$ for all $\zeta' \in \Omega(\xi_\nu', R)$ if ν is large. Now, the lemma follows if we take ν large and set $\tau(\zeta') = \tau_\nu(\zeta', \xi_\nu'')$ and $\xi = \xi_\nu$.

Proof of Theorem 6.16. It is trivial that (iii) implies (i) and Lemma 6.21 proves that if (iii) is false then (i) is also false. Thus (i) and (iii) are equivalent. From Lemma 6.10 we obtain that $(\mathrm{Re}\, P)(D)$ is an evolution operator and then it follows from Theorem 4.1 in [6] that (i) follows from (ii). Finally Lemma 6.19 proves that (i) implies (ii).

Example 6.22. Let $P(\xi) = \sum_2^n a_j(\xi_1)\xi_j$, where a_j, $j = 2, \ldots, n$, are real and $\deg a_2 > \deg a_j$, $j \geq 3$. Then P satisfies condition (S) with respect to $V = = \{x'' \in \mathbf{R}^{n''}\,;\, x_2 \geq 0,\ x_3 = \ldots = x_n = 0\}$ so $P(D)$ has a fundamental solution in $\Lambda = \mathbf{R} \times W$ if $\mathrm{int}\, W \supset V \setminus \{0\}$. (Cf. Theorem 2.11.) Furthermore, if $q(\xi) = = \sum_3^n c_j(\xi_1)\xi_j$ is real and deg $c_j < \deg a_2$ then $P + iq$ satisfies the necessary conditions of Theorem 2.6 with respect to W if and only if q is dominated by P.

Remark 6.23. Set $Q_\alpha(\xi) = (P(\xi', \xi'' - iN'') + P^{(\alpha)}(\xi', \xi'' - iN''))/2$ where $\alpha = = (\alpha', 0)$. In the proofs of Theorem 6.8 — Lemma 6.15 we have only used that there is a constant t such that $Q_\alpha(\xi', \zeta'') \neq 0$ for all ξ' with $d(\xi', b^{-1}(0)) \geq t$ and all ζ'' with $\mathrm{Im}\, \zeta'' \in - W^*$. Thus Theorem 6.8 is true also for $P(\xi) = = \sum_{n'+1}^n a_j(\xi')\xi_j + a_0(\xi')$, where a_j is real for every j and $a_{n'+1}$ is stronger than $a_{n'+2}, \ldots, a_n$.

We have not been able to prove the existence of fundamental solutions with support in $\Lambda = \mathbf{R}^{n'} \times W$ for all polynomials satisfying the conditions of Theorem 6.16. However, the next theorem shows the existence of local fundamental solutions. (Cf. Remark 2.10.)

THEOREM 6.24. *Let $P = P_1 + iP_2$ satisfy the conditions of Theorem* 6.16 *and let W be a closed, convex cone such that* $\mathrm{int}\, W \supset V \setminus \{0\}$. *Then there is a distribution $E \in B_{\infty, \tilde{P}}^{\mathrm{loc}}$ such that* $\mathrm{supp}\, E \subset \Lambda = \mathbf{R}^{n'} \times W$ *and* $P(D)E = \delta$ *in a neighbourhood of zero.*

Proof. Set $Q_\varepsilon(\xi) = P_1(\xi/\varepsilon)$, $\varepsilon > 0$. Inspection of the proofs of Theorem 6.8 — Lemma 6.15 shows that Theorem 6.8 is true for Q_ε with a constant that is independent of ε, $0 < \varepsilon \leq 1$. (Cf. Remark 6.23.) Thus, if $K \subset \mathbf{R}^n$ is compact then there is a constant C such that

$$\|u\|_{1,1}^{A-} \leqq C\|Q_\varepsilon(D)u\|_{1,1/\tilde{Q}_\varepsilon}^{A-}$$

for all $u \in C_0^\infty(K)$ and all ε, $0 < \varepsilon \leqq 1$. If we replace u by $u(\varepsilon x)$ where $u \in C_0^\infty(\varepsilon K)$ then it follows that

$$\|u\|_{1,1}^{A-} \leqq C\,\|P_1(D)u\|_{1,k_\varepsilon}^{A-}, \quad u \in C_0^\infty(\varepsilon K),$$

where $k_\varepsilon(\xi) = (\sum_\alpha |P_1^{(\alpha)}(\xi)|^2\varepsilon^{-2|\alpha|})^{-1/2}$. Thus,

$$\|u\|_{1,1}^{A-} \leqq C\,\|P_1(D)u\|_{1,k_\varepsilon}^{A-} \leqq C\,\|P(D)u\|_{1,k_\varepsilon}^{A-} + C\,\|P_2(D)u\|_{1,k_\varepsilon}^{A-} \leqq$$
$$\leqq C\,\|P(D)u\|_{1,k_\varepsilon}^{A-} + C\sup\,(|P_2(\xi)|\,k_\varepsilon(\xi))\,\|u\|_{1,1}^{A-}.$$

If $\varepsilon > 0$ is small enough then $C\sup\,(|P_2(\xi)|\,k_\varepsilon(\xi)) \leqq 1/2$, so that

$$\|u\|_{1,1}^{A-} \leqq 2C\,\|P(D)u\|_{1,k_\varepsilon}^{A-} \leqq C_1\,\|P(D)u\|_{1,1/\tilde{P}}^{A-}, \quad u \in C_0^\infty(\varepsilon K).$$

Now the theorem follows from Theorem 4.2.

Remark 6.25. The polynomial $P(\xi) = \xi_1^2\xi_2 + \xi_3 - i$ satisfies the Petrowsky condition with respect to $N = (0, 0, -1)$, i.e. $P(\xi + itN) \neq 0$ for $t \geq 0$. However, Petrowsky's fundamental solution E defined by

$$\check{E}(u) = (2\pi)^{-3}\int \hat{u}(\xi)/P(\xi)d\xi, \quad u \in C_0^\infty(\mathbf{R}^3),$$

does not belong to $B_{\infty,\tilde{P}}^{\mathrm{loc}}$.

Proof. If $E \in B_{\infty,\tilde{P}}^{\mathrm{loc}}$ then $D_2E \in B_{\infty,1}^{\mathrm{loc}}$. This implies that for every $\phi \in C_0^\infty(\mathbf{R}^3)$ there is a constant C_ϕ such that

$$\left|\int \phi(\xi - \lambda)\lambda_2/(\lambda_1^2\lambda_2 + \lambda_3 - i)d\lambda\right| \leqq C_\phi \quad \text{for all} \quad \xi \in \mathbf{R}^3.$$

Let $\varepsilon > 0$. Then we obtain that

$$\left|\int \hat{\phi}(\xi - \lambda)(\lambda_2 + t)/(\lambda_1^2(\lambda_2 + t) + \lambda_3 + t\varepsilon^2 - i)d\lambda\right| \leqq C_\phi \quad \text{for all} \quad \xi \in \mathbf{R}^3, \quad t \in \mathbf{R}.$$

However, for fixed $\xi \in \mathbf{R}^3$ we have that

$$|\hat{\phi}(\xi - \lambda)(\lambda_2 + t)/(\lambda_1^2(\lambda_2 + t) + \lambda_3 + t\varepsilon^2 - i)| \leqq C_N/(\varepsilon^2(1 + |\lambda|)^N).$$

When $t \to \infty$ we now obtain that

$$\left|\int \hat{\phi}(\xi - \lambda)/(\lambda_1^2 + \varepsilon^2)d\lambda\right| \leqq C_\phi \quad \text{for all} \quad \xi \in \mathbf{R}^3.$$

If $\phi(x) = \phi_1(x_1)\ \phi_2(x_2, x_3)$ and $\phi_2(0) \neq 0$ then

$$\left|\int \hat{\phi}_1(\xi_1 - \lambda_1)/(\lambda_1^2 + \varepsilon^2)d\lambda_1\right| \leqq C_{\phi_1} \quad \text{for all} \quad \xi_1 \in \mathbf{R}.$$

However, if $\hat{\phi}_1(\xi_1) \neq 0$ then the left side tends to infinity as $\varepsilon \to 0$.

We could use the same technique as in this remark to prove that in general it is not possible to extend Lemma 6.13 to $E(\alpha, \beta, \theta')$ with $|\beta| > 1$.

7. Second order operators

First we shall investigate the necessary conditions further.

THEOREM 7.1. *Let P be a polynomial with $\deg P = 2$ satisfying the necessary conditions of Theorem 2.6. Then $P(D)$ is hyperbolic with respect to some cone contained in Γ or, after multiplication by a constant, a complex translation and a linear transformation of the variables preserving the edge of Γ, $P(\xi)$ can be written in one of the following forms:*

(i) $P(\xi) = \xi_{n'+1} + B_1(\xi') + iB_2(\xi')$, *where B_1 and B_2 are real quadratic forms and B_2 is negative semidefinite.*
(ii) $P(\xi) = (\xi_1 + i\xi_2)\xi_{n'+1} + a\xi_1 + b$, $a, b \in \mathbf{C}$.
(iii) $P(\xi)$ *is independent of* ξ''.
(iv) $P(\xi) = \xi_1\xi_{n'+1} + a\xi_{n'+2} + B(\xi^0) + c$ *where* $a \in \mathbf{R}$, $c \in \mathbf{C}$ *and B is a real polynomial of* $\xi^0 = (\xi_2, \ldots, \xi_{n'})$ with $\deg B \leq 2$.

With our standard notations a linear transformation leaves the edge of Γ invariant if the equation $x'' = 0$ is invariant or equivalently if the equation $\xi' = 0$ in the dual variables of the Fourier transform is invariant.

Proof. Let p be the principal part of P and set $Q(\xi) = \lim_{t\to\infty} t^{-\mu}P(t\xi', t^3\xi'')$ where $\mu = \deg_t P(t\xi', t^3\xi'')$. Theorem 3.1 shows that $p(\xi', D'')$ and $Q(\xi', D'')$ are hyperbolic with respect to V (or zero).

If $\mu = 6$, then $Q(D)$ is hyperbolic with respect to some cone contained in Γ. From this we get that $P(D)$, too, is hyperbolic with respect to some cone contained in Γ. In fact, every supporting plane of Γ which only meets the edge is non-characteristic, and from Hörmander [6] we obtain that, for every half space containing Γ, $P(D)$ has a fundamental solution with support in that half space. Then it follows that $P(D)$ is hyperbolic with respect to some cone contained in Γ. (Theorem 5.4.1 and Theorem 5.6.1 in [5].)

If $\mu = 4$ then Q is of the form $Q(\xi) = \sum_{n'+1}^{n} a_j(\xi')\xi_j$, where $a_j(\xi')$ are linear forms. After multiplication by a constant and a transformation of the variables we can write $a_{n'+1}(\xi') = \xi_1 + i\xi_2$ or $a_{n'+1}(\xi') = \xi_1$. In both cases we obtain from the properties of Q that the linear forms a_j are proportional. Thus, after a transformation of the variables we can write $Q(\xi) = (\xi_1 + i\xi_2)\xi_{n'+1}$ or $Q(\xi) = \xi_1\xi_{n'+1}$.

If $Q(\xi) = (\xi_1 + i\xi_2)\xi_{n'+1}$ then $P(\xi) = (\xi_1 + i\xi_2)\xi_{n'+1} + \sum_{n'+1}^{n} c_j\xi_j + B(\xi')$. Consider ζ' in balls with centre $(t, 0)$, t large, and let $\xi_j \to \pm\infty$ for some $j > n' + 1$. Then we obtain from Theorem 2.6 that $c_j \in \mathbf{R}$, $j > n' + 1$. In the same way we see that $c_j \in i\mathbf{R}$. Thus, after a translation of the variables we can write $P(\xi) = (\xi_1 + i\xi_2)\xi_{n'+1} + B(\xi')$, so that $p(\xi) = (\xi_1 + i\xi_2)\xi_{n'+1} + B_1(\xi')$, where B_1 is a quadratic form. From Theorem 3.1 we now obtain that $\operatorname{Im}(\xi_1 - i\xi_2)B_1(\xi') \leq 0$ for all $\xi' \in \mathbf{R}^{n'}$. This implies that $\xi_1 \operatorname{Im} B_1(\xi') - \xi_2 \operatorname{Re} B_1(\xi') \equiv 0$, i.e. $B_1(\xi') = (\xi_1 + i\xi_2)(\sum_1^{n'} d_j\xi_j)$, with d_j real. After a linear transformation we can write $p(\xi) = (\xi_1 + i\xi_2)\xi_{n'+1}$ so that $P(\xi) = (\xi_1 + i\xi_2)\xi_{n'+1} + \sum_1^{n'} f_j\xi_j + b$. We immediately obtain that $f_j = 0$ for $j \geq 3$, so after a final translation of the variables we can write $P(\xi) = (\xi_1 + i\xi_2)\xi_{n'+1} + a\xi_1 + b$.

If $Q(\xi) = \xi_1\xi_{n'+1}$ then the principal part p is of the form $p(\xi) = \xi_1\xi_{n'+1} + B_2(\xi')$, where B_2 is a quadratic form. From Theorem 3.1 we get that $\operatorname{Im}\xi_1 B_2(\xi') \leq 0$ for all $\xi' \in \mathbf{R}^{n'}$, which implies that $\operatorname{Im} B_2 \equiv 0$. Thus, after a transformation of the variables we can write $p(\xi) = \xi_1\xi_{n'+1} + B_3(\xi^0)$, where $B_3(\xi^0)$ is a real quadratic form of $\xi^0 = (\xi_2, \ldots, \xi_{n'})$. Now we have that

$$P(\xi) = \xi_1\xi_{n'+1} + \sum_{n'+1}^{n} a_j\xi_j + B_3(\xi^0) + \sum_1^{n'} c_j\xi_j + c_0.$$

After a translation and a transformation of the variables we can write

$$P(\xi) = \xi_1\xi_{n'+1} + \sum_{n'+2}^{n} a_j\xi_j + \sum_2^{k} b_j\xi_j^2 + \sum_{k+1}^{n'} c_j\xi_j + c,$$

where $b_j \in \mathbf{R}$. From Theorem 2.6 we now obtain that $a_j \in \mathbf{R}$ and $c_j \in \mathbf{R}$. A f ir a transformation of the variables gives that P can be written in the form

$$P(\xi) = \xi_1\xi_{n'+1} + a\,\xi_{n'+2} + B(\xi^0) + c,$$

where $a \in \mathbf{R}$, $c \in \mathbf{C}$ and B is a real polynomial of degree ≤ 2.

If $\mu = 3$ then $P(\xi)$ is of the form $P(\xi) = c\sum_{n'+1}^{n} a_j\xi_j + B(\xi')$ where $a_j \in \mathbf{R}$. After a multiplication by a constant and an admissible linear transformation of the variables, we can write P in the form $P(\xi) = \xi_{n'+1} + B(\xi')$, where $B(\xi') = \langle A\xi', \xi'\rangle + i\langle\theta', \xi'\rangle$. Here $\theta' \in \mathbf{R}^{n'}$ and $A = A_1 + iA_2$ with A_1, A_2 real and symmetric. Let L be the linear hull of the images of A_1 and A_2. If $\theta' \notin L$ then there is a vector $\xi' \in L^0$ such that $\langle\theta', \xi'\rangle > 0$. Set $\zeta' = t\xi' + z'$, $t \in \mathbf{R}$. Then $B(\zeta') = \langle Az', z'\rangle + i\langle\theta', t\xi'\rangle + i\langle\theta', z'\rangle$. If $|z| \leq R$ and $t \to \infty$ then $\operatorname{Im} B(\zeta') \to \infty$, which contradicts the necessary conditions of Theorem 2.6. Thus there are $\xi_0', \eta_0' \in \mathbf{R}^{n'}$ such that $\theta'/2 = A_1\eta_0' + A_2\xi_0'$. After a translation and a transformation of the variables, we can write $P(\xi) = \xi_{n'+1} + B_1(\xi') + iB_2(\xi')$, where B_1 and B_2 are real quadratic forms. Finally, we obtain from the necessary conditions that B_2 is negative semidefinite.

Finally, if $\mu \leq 2$ then P is independent of ξ''.

THEOREM 7.2. *Let $\Gamma = \mathbf{R}^{n'} \times V$ where V is a proper, closed and convex cone in $\mathbf{R}^{n''}$ and let $V_0 = \{x'' \in \mathbf{R}^{n''}\,;\ x_{n'+1} > 0,\ x_{n'+2} = \ldots = x_n = 0\}$. If P is one of the polynomials in* (i)—(iv) *of Theorem* 7.1, *then P satisfies the necessary conditions of Theorem* 2.6 *with respect to Γ if and only if*

(i) $V_0 \subset V$ *if* $B_2 \not\equiv 0$,
$V_0 \subset V$ *or* $-V_0 \subset V$ *if* $B_2 \equiv 0$.
(ii) $V_0 \subset V$ *or* $-V_0 \subset V$.
(iii) V *is arbitrary.*
(iv) $V_0 \subset V$ *or* $-V_0 \subset V$ *if* $a = 0$.
$V_0 \subset \operatorname{int} V$ *or* $-V_0 \subset \operatorname{int} V$ *if* $a \neq 0$.

Proof. It follows from the properties of $Q(\xi) = \lim_{t\to\infty} t^{-\mu} P(t\xi', t^3\xi'')$, $\mu = \deg_t P(t\xi', t^3\xi'')$, that V must contain V_0 or $-V_0$ (except in the case (iii)). Then the only case that is not quite clear is (iv) with $a \neq 0$.

Assume that $P(\xi) = \xi_1\xi_{n'+1} + \xi_{n'+2} + B(\xi^0)$ and that $n'' = 2$. Set $W = \{x'' \in \mathbf{R}^2\,;\ x_{n'+2} \geq 0,\ x_{n'+1} \geq -cx_{n'+2}\}$, $c > 0$. Then $W^* = \{\xi'' \in \mathbf{R}^2\,;\ \xi_{n'+1} \geq 0,\ \xi_{n'+2} \geq c\xi_{n'+1}\}$. Let R and A be given. If $\eta_{n'+2}$ is real and

$$\zeta_1\zeta_{n'+1} + i\eta_{n'+2} + B(\zeta^0) = 0.$$

then

$$|\zeta_1|^2\eta_{n'+1} = -\xi_1(\eta_{n'+2} + \operatorname{Im} B(\zeta^0)) - \eta_1 \operatorname{Re} B(\zeta^0).$$

Assume that $|B(\zeta^0)| \leq C$ for all ζ^0 with $|\zeta^0| \leq R$ and take $\eta_{n'+2} < -(2A + C)$. Thus, if we let θ_1 be large we see that $(\eta_{n'+1}, \eta_{n'+2}) \in -W_A^*$ for all ζ' with $|\zeta' - (\theta_1, 0)| \leq R$. This proves that the necessary conditions of Theorem 2.6 are not true with respect to W. Because of symmetry we then see that V_0 must be contained in $\operatorname{int} V$ or $-\operatorname{int} V$.

THEOREM 7.3. *If P is a polynomial with* $\deg P = 2$, *that satisfies the necessary conditions of Theorem* 2.6, *then the operator $P(D)$ has a fundamental solution $E \in B^{\mathrm{loc}}_{\infty, \tilde{P}}$ with support in Γ.*

Proof. If P is hyperbolic then the theorem follows immediately.

If P is of the form (i), (ii), (iii) or (iv) with $a = 0$ in Theorem 7.1, then $P(D)$ is an evolution operator with respect to the half space $x_{n'+1} \geq 0$ in $\mathbf{R}^{n'+1}$. From [6] we obtain that there is a fundamental solution $E_1 \in B^{\mathrm{loc}}_{\infty, \tilde{P}}(\mathbf{R}^{n'+1})$, with support in the half space $x_{n'+1} \geq 0$. Set $E = E_1 \otimes \delta_{n'+2} \otimes \ldots \otimes \delta_n$. Then E has the required properties.

It remains to consider the case $P(\xi) = \xi_1\xi_{n'+1} + \xi_{n'+2} + B(\xi^0)$. However, since B is real, it follows from Remark 6.23 that $P(D)$ has a fundamental solution with the required properties.

8. The two dimensional case

Let $P(D) = \sum_0^m a_j(D_1)D_2^j$, $a_m \not\equiv 0$, be an evolution operato r with respect to the half space $x_2 \geq 0$. Furthermore, let $\Delta(\zeta_1)$ be the discrim nant of P considered as a polynomial of ζ_2 and assume that $a_m(\zeta_1)\Delta(\zeta_1) \neq 0$ when $|\zeta_1| \geq r$ or $\operatorname{Im} \zeta_1 = \theta_0$.

Let $\tau_1(\zeta_1), \ldots, \tau_m(\zeta_1)$ denote the solutions of the equation $P(\zeta_1, \tau) = 0$, which are analytic in a neighbourhood of the curves in Fig. 1. If we choose suitable θ_j^+ and θ^- then it follows from the Puiseux expansion that $\operatorname{Im} \tau_j(\zeta_1) \geq 1$ for large $\zeta_1 \in \gamma_j$, where γ_j is chosen as one of the four possible curves in Fig. 1. (See Lemma 4.3 in [6].) For small $\zeta_1 \in \gamma_j$ we can obtain this by a complex translation in the ζ_2 variable. Thus we can assume that $\operatorname{Im} \tau_j(\zeta_1) \geq 1$ for all $\zeta_1 \in \gamma_j$, $j = 1, 2, \ldots, m$.

It is now possible to define a fundamental solution E of $P(D)$. Let $\varrho_j(\zeta_1) = 1/\partial P(\zeta_1, \zeta_2)/\partial \zeta_2$ for $\zeta_2 = \tau_j(\zeta_1)$ and set

$$\check{E}(u) = (2\pi)^{-2} \sum_1^m \int_{\gamma_j \times \mathbf{R}} \hat{u}(\zeta_1, \xi_2)\varrho_j(\zeta_1)/(\xi_2 - \tau_j(\zeta_1))\,d\zeta_1 d\xi_2, \quad u \in C_0^\infty(\mathbf{R}^2). \tag{8.1}$$

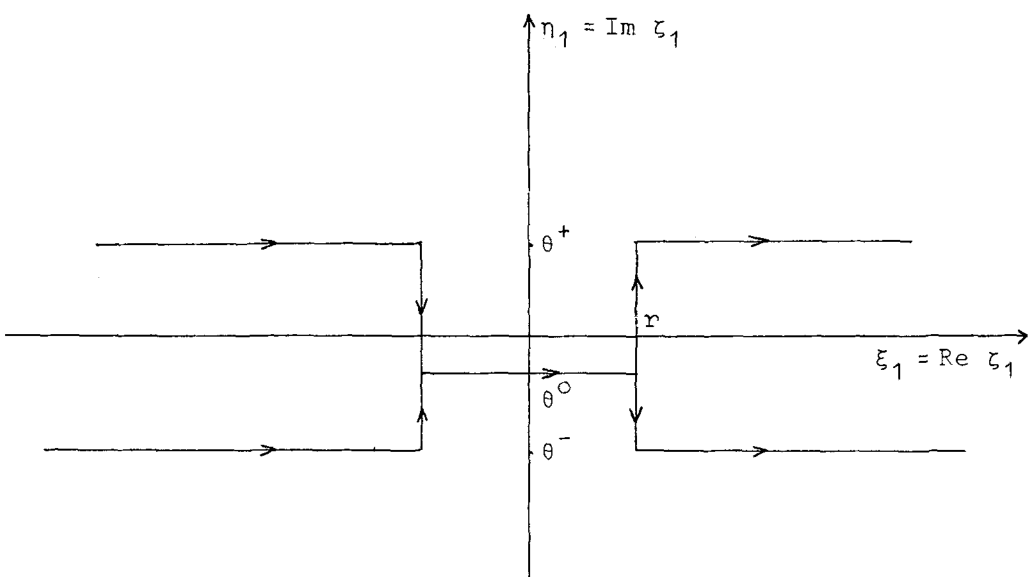

Fig. 1

The integrals of (8.1) are convergent for if $\zeta_1 \in \gamma_j$ then

$$|\hat{u}(\zeta_1, \xi_2)\varrho_j(\zeta_1)/(\xi_2 - \tau_j(\zeta_1))| \leq C_N(1 + |\zeta_1| + |\xi_2|)^{-N}$$

where N is arbitrary. The support of E lies in the half plane $\{x ; x_2 \geq 0\}$. To see this we change the integration contour:

$$\check{E}(u) = (2\pi)^{-2}\sum_1^m \int_{\gamma_j \times \mathbf{R}} \hat{u}(\zeta_1, \xi_2 - i\eta_2)\varrho_j(\zeta_1)/(\xi_2 - i\eta_2 - \tau_j(\zeta_1))d\zeta_1 d\xi_2.$$

If $u(x) = 0$ for $x_2 \leq 0$ we let $\eta_2 \to \infty$. Then we obtain that $\check{E}(u) = 0$ for all $u \in C_0^\infty(\mathbf{R}^2)$ with $u(x) = 0$ for $x_2 \leq 0$. Thus $\operatorname{supp} E \subset \{x ; x_2 \geq 0\}$. To show that E is a fundamental solution, we observe that

$$\begin{aligned}
&\check{E}(P(D)u) = \\
&= (2\pi)^{-2}\sum_1^m \int_{\gamma_j \times \mathbf{R}} P(\zeta_1, \xi_2)\hat{u}(\zeta_1, \xi_2)\varrho_j(\zeta_1)/(\xi_2 - \tau_j(\zeta_1))d\zeta_1 d\xi_2 = \\
&= (2\pi)^{-2}\sum_1^m \int_{\gamma_j \times \mathbf{R}} \hat{u}(\zeta_1, \xi_2)\varrho_j(\zeta_1)a_m(\zeta_1)\prod_{k \neq j}(\xi_2 - \tau_k(\zeta_1))d\zeta_1 d\xi_2 = \\
&= (2\pi)^{-2}\sum_1^m \int_{\gamma \times \mathbf{R}} \hat{u}(\zeta_1, \xi_2)\varrho_j(\zeta_1)a_m(\zeta_1)\prod_{k \neq j}(\xi_2 - \tau_k(\zeta_1))d\zeta_1 d\xi_2 = \\
&= (2\pi)^{-2}\int_{\gamma \times \mathbf{R}} \hat{u}(\zeta_1, \xi_2)d\zeta_1 d\xi_2 = (2\pi)^{-2}\int_{\mathbf{R}^2} \hat{u}(\xi)d\xi = u(0),
\end{aligned}$$

where γ is the path in Fig. 2.

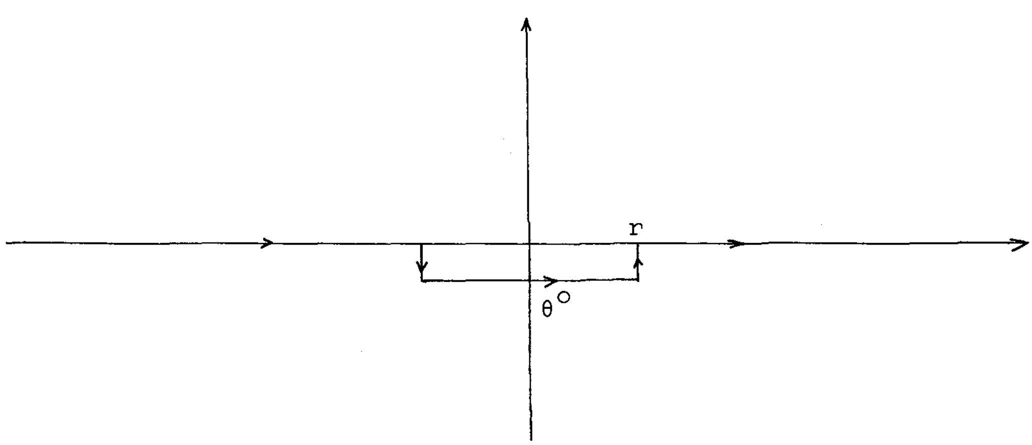

Fig. 2

We also want to show that $E \in B^{\mathrm{loc}}_{\infty, \tilde{P}}$ and then we need the following lemma, which is related to the proof that E is a fundamental solution.

LEMMA 8.1. *Let $Q(\zeta_1, \xi_2)$ be a polynomial in ξ_2 with coefficients analytic in a neighbourhood of the set between the curves γ_j defined above and assume that the coefficients are bounded by some power of $(2 + |\zeta_1|)$. Furthermore, let $Q = RP + Q_0$ where $\deg_{\xi_2} Q_0 < \deg_{\xi_2} P$. Then for every $u \in C_0^\infty(\mathbf{R}^2)$ we have that*

$$\sum_1^m \int_{\gamma_j \times \mathbf{R}} Q(\zeta_1, \xi_2)\hat{u}(\zeta_1, \xi_2)\varrho_j(\zeta_1)/(\xi_2 - \tau_j(\zeta_1))d\zeta_1 d\xi_2 =$$

$$= \sum_1^m \int_{\gamma_j \times \mathbf{R}} Q_0(\zeta_1, \tau_j(\zeta_1))\hat{u}(\zeta_1, \xi_2)\varrho_j(\zeta_1)/(\xi_2 - \tau_j(\zeta_1))d\zeta_1 d\xi_2 +$$

$$+ \int_{\gamma \times \mathbf{R}} R(\zeta_1, \xi_2)\hat{u}(\zeta_1, \xi_2)d\zeta_1 d\xi_2.$$

Proof. $Q(\zeta_1, \xi_2)\varrho_j(\zeta_1)/(\xi_2 - \tau_j(\zeta_1)) = R_j(\zeta_1, \xi_2) + Q(\zeta_1, \tau_j(\zeta_1))\varrho_j(\zeta_1)/(\xi_2 - \tau_j(\zeta_1))$, where R_j is a polynomial in ξ_2. We obtain that

$$\sum_1^m \int_{\gamma_j \times \mathbf{R}} Q(\zeta_1, \xi_2)\hat{u}(\zeta_1, \xi_2)\varrho_j(\zeta_1)/(\xi_2 - \tau_j(\zeta_1))d\zeta_1 d\xi_2 =$$

$$= \sum_1^m \int_{\gamma \times \mathbf{R}} R_j(\zeta_1, \xi_2)\hat{u}(\zeta_1, \xi_2)d\zeta_1 d\xi_2 +$$

$$+ \sum_1^m \int_{\gamma_j \times \mathbf{R}} Q_0(\zeta_1, \tau_j(\zeta_1))\hat{u}(\zeta_1, \xi_2)\varrho_j(\zeta_1)/(\xi_2 - \tau_j(\zeta_1))d\zeta_1 d\xi_2.$$

However, $R = \sum_1^m R_j$ for if $a_m(\zeta_1)\Delta(\zeta_1) \neq 0$ we have

$$\sum_1^m Q(\zeta_1, \xi_2)\varrho_j(\zeta_1)/(\xi_2 - \tau_j(\zeta_1)) = Q(\zeta_1, \xi_2)/P(\zeta_1, \xi_2) =$$

$$= R(\zeta_1, \xi_2) + \sum_1^m Q(\zeta_1, \tau_j(\zeta_1))\varrho_j(\zeta_1)/(\xi_2 - \tau_j(\zeta_1)).$$

To show that $E \in B^{\mathrm{loc}}_{\infty, \tilde{P}}$ it is sufficient to prove that $P^{(\alpha)}(D)E \in B^{\mathrm{loc}}_{\infty, 1}$ for all α:

LEMMA 8.2. *Let E be a distribution and P a polynomial. Then $E \in B^{\mathrm{loc}}_{\infty, \tilde{P}}$ if and only if $P^{(\alpha)}(D)E \in B^{\mathrm{loc}}_{\infty, 1}$ for every α.*

Proof. Let $\varphi \in C_0^\infty$ and assume that $P^{(\alpha)}(D)E \in B^{\mathrm{loc}}_{\infty, 1}$ for all α. Then

$$P^{(\alpha)}(\xi)(\widehat{\varphi E})(\xi) = \mathcal{F}(P^{(\alpha)}(D)\varphi E)(\xi) = \sum_{\beta} (\mathcal{F}((P^{(\alpha+\beta)}(D)E)D^{\beta}\varphi/\beta!)(\xi))$$

is bounded. This shows that

$$\tilde{P}(\xi)(\widehat{\varphi E})(\xi)$$

is bounded, i.e. $\varphi E \in B_{\infty, \tilde{P}}$. The converse follows from Theorem 2.3.4 in Hörmander [5].

Thus we want to estimate

$$\check{E}(P^{(\alpha)}(D)u) = \tag{8.2}$$

$$= (2\pi)^{-2} \sum_{1}^{m} \int_{\gamma_j \times \mathbf{R}} P^{(\alpha)}(\zeta_1, \xi_2)\hat{u}(\zeta_1, \xi_2)\varrho_j(\zeta_1)/(\xi_2 - \tau_j(\zeta_1))d\zeta_1 d\xi_2 =$$

$$= (2\pi)^{-2} \int_{\gamma \times \mathbf{R}} a_m^{(\alpha)}(\zeta_1)\hat{u}(\zeta_1, \xi_2)/a_m(\zeta_1)d\zeta_1 d\xi_2 +$$

$$+ (2\pi)^{-2} \sum_{1}^{m} \int_{\gamma_j \times \mathbf{R}} P^{(\alpha)}(\zeta_1, \tau_j(\zeta_1))\hat{u}(\zeta_1, \xi_2)\varrho_j(\zeta_1)/(\xi_2 - \tau_j(\zeta_1))d\zeta_1 d\xi_2.$$

The first integral on the right side of (8.2) occurs if $\alpha = (\alpha_1, 0)$ and then it can be estimated by $\text{const} \cdot \| \cosh(a|x|) \cdot u\|_{1, 1}$. When $|\zeta_1|$ is bounded, the remaining integrals can be estimated in the same way. To be able to estimate these integrals for large $|\zeta_1|$, we must group the zeros into classes. From Lemma 4.3 in [6] we know that for large $|\zeta_1|$ we can write $\tau_j(\zeta_1) = c_j\zeta_1^{k_j} +$ lower order terms, where k_j is a non-negative integer. We will now say that τ_i and τ_j are equivalent if $k_i = k_j$ and $c_i = c_j$. Thus, there is a constant $d > 0$ such that if τ_i and τ_j are not equivalent then

$$|\tau_i(\zeta_1) - \tau_j(\zeta_1)| \geq 4d(|\zeta_1|^{k_i} + |\zeta_1|^{k_j} + 1) \quad \text{when} \quad \zeta_1 \in \gamma_i.$$

If θ^+ and θ^- (see Fig. 1) are chosen in a suitable way, then we obtain from Lemma 4.3 in [6] that

(i) (c_j real.)
There is a constant $c > 0$ such that
$\operatorname{Im} \tau_j(\zeta_1) \geq 3c(|\zeta_1|^{k_j-1} + 1)$, $\zeta_1 \in \gamma_j$ and
$|\tau_i(\zeta_1) - \tau_j(\zeta_1)| \leq c\, |\zeta_1|^{k_j-1}$, $\zeta_1 \in \gamma_j$, τ_i and τ_j equivalent.

(ii) ($\operatorname{Im} c_j > 0$.)
If the constant d above is chosen small enough then
$\operatorname{Im} \tau_j(\zeta_1) \geq 3d\, |\zeta_1|^{k_j}$, $\zeta_1 \in \gamma_j$ and
$|\tau_i(\zeta_1) - \tau_j(\zeta_1)| = o(|\zeta_1|^{k_j})$, $\zeta_1 \in \gamma_j$, τ_i and τ_j equivalent.

Set $r_j = 2c\,|\zeta_1|^{k_j-1}$ if c_j is real and $r_j = 2d\,|\zeta_1|^{k_j}$ if c_j is not real. From (i) and (ii) we see that if $\zeta_1 \in \gamma_j$ is fixed, then we can find a circle $\sigma_j = \sigma_j(\zeta_1)$ in the ζ_2 plane with radius r_j such that $\operatorname{Im}\theta \geq \text{const.} > 0$ for all $\theta \in \sigma_j$ and σ_j has winding number 0 with respect to the zeros that are not equivalent to τ_j and such that a circle with radius $r_j/2$ concentric to σ_j has winding number 1 with respect to the zeros that are equivalent to τ_j. From the mean value theorem we now obtain that there exists a constant $\varepsilon > 0$ such that the distance in $\mathbf{C}^2$ from $(\zeta_1, \theta) \in \gamma_j \times \sigma_j(\zeta_1)$ to the zeros of P is $\geq \varepsilon > 0$. We can now rewrite the remaining part of (8.2) in the following way

$$(2\pi)^{-2}\sum_1^m \int\limits_{\substack{\gamma_j\times\mathbf{R}\\ |\zeta_1|\geq C}} P^{(\alpha)}(\zeta_1, \tau_j(\zeta_1))\hat{u}(\zeta_1, \xi_2)\varrho_j(\zeta_1)/(\xi_2 - \tau_j(\zeta_1))d\zeta_1 d\xi_2 =$$

$$= (2\pi)^{-2}\sum \int\limits_{\substack{\gamma_j\times\mathbf{R}\\ |\zeta_1|\geq C}} (\hat{u}(\zeta_1, \xi_2) \int\limits_{\sigma_j(\zeta_1)} P^{(\alpha)}(\zeta_1, \theta)/P(\zeta_1, \theta)(\xi_2 - \theta)d\theta)d\zeta_1 d\xi_2$$

where the last sum is taken over the equivalence classes. However,

$$\left| \int\limits_{\sigma_j} P^{(\alpha)}(\zeta_1, \theta)/P(\zeta_1, \theta)(\zeta_2 - \theta)d\theta \right| \leq C.$$

We have now proved the following estimate

$$|\check{E}(P^{(\alpha)}(D)u)| \leq C\sum_1^m \int\limits_{\gamma_j\times\mathbf{R}} |\hat{u}(\zeta_1, \xi_2)|d\zeta_1 d\xi_2 \leq C\,\|u\cdot\cosh(a|x|)\|_{1,1}$$

for all $u \in C_0^\infty(\mathbf{R}^2)$ and arbitrary α. This shows that $P^{(\alpha)}(D)E \in B^{\mathrm{loc}}_{\infty,1}$ for all α, which by Lemma 8.2 implies that $E \in B^{\mathrm{loc}}_{\infty,\tilde{P}}$. Thus we have proved:

THEOREM 8.3. *Let $P(D)$ be an evolution operator in $\mathbf{R}^2$ with respect to the half-plane $H = \{x\,;\,x_2 \geq 0\}$. Then the distribution E defined by* (8.1) *is a fundamental solution to $P(D)$ with* $\operatorname{supp} E \subset H$ *and* $E \in B^{\mathrm{loc}}_{\infty,\tilde{P}}$.

Remark 8.4. The estimates

$$|\check{E}(P^{(\alpha)}(D)u)| \leq C\,\|u\cosh(a|x|)\|_{1,1}$$

show that $F = E/\cosh(a|x|)$ lies in $B_{\infty,\tilde{P}} \subset \mathcal{S}$.

Appendix

Since the proof of Lemma 6 in Trèves [10] is not quite clear, we will here give a partially different proof of Theorems 1 and 1′ in [10]. The new ideas of this proof are due to L. Hörmander. Another correction has been proposed by F. Trèves.

THEOREM A1. (*Trèves.*) *Let* $P(\nu, D)$ *be a differential operator with coefficients that are* C^∞ *functions of* $\nu \in U$, *where* U *is convex. Furthermore, let* $P(\nu, D)$ *be of analytic type* (*see below*) *in* U *and* $\omega \subset \mathbf{R}^n$ *a neighbourhood of zero and let* $k \in \mathcal{K}(\mathbf{R}^n)$. *If there is a distribution* $E(\nu)$ *which is a* C^∞ *function of* ν *with values in* $B^{\mathrm{loc}}_{\infty, k}(\omega)$, *such that* $P(\nu, D)E(\nu) = \delta$ *for all* $\nu \in U$, *then* $P(\nu, D)$ *has constant strength when* $\nu \in U$.

According to Definition 1 in [10] we say that $P(\nu, D)$ is of *analytic type* if every linear combination of coefficients of $P(\nu, D)$ which has a zero of infinite order in U is identically zero in U.

Proof. It follows from section 1 in [10] that we can assume that U is a neighbourhood of zero in $\mathbf{R}$ and that it is sufficient to prove that P_k is weaker than P_0 for all k. Here

$$P_k = [(d/dt)^k P(t, D)]_{t=0}.$$

Let $\mathcal{A}$ be a commutative ring with unit element 1 and let $\mathcal{B}$ be a unitary $\mathcal{A}$-module. Consider two sequences $X_j \in \mathcal{A}$, $Y_j \in \mathcal{B}$ such that $X_0 = 1$ and $\sum_0^m \binom{m}{j} X_j Y_{m-j} = 0$, $m \geq 1$. By recursion we obtain that Y_m is a polynomial of $X_1, \ldots, X_m$. We have the recursion formula

$$Y_m = F_m(X_1, \ldots, X_m) Y_0 = -\sum_1^m \binom{m}{j} X_j F_{m-j}(X_1, \ldots, X_{m-j}) Y_0, \tag{A1}$$

where $m \geq 1$ and $F_0 = 1$. If $F_m(X_1, \ldots, X_m) = \Sigma c_\alpha X_1^{\alpha_1} \ldots X_m^{\alpha_m}$ we immediately see that $\Sigma j\alpha_j = m$.

Now, consider the differential operator $P(t, D)$, $t \in U$. We have $\deg_\xi P(t, \xi) \leq$ const., $t \in U$, and

$$P(t, D)E(t) = \delta, \quad t \in U, \tag{A2}$$

where $E(t)$ is an infinitely differentiable function of $t \in U$ with values in $B^{\mathrm{loc}}_{\infty, k}(\omega)$. It follows from (A2) that $P_0 E_0 = \delta$ and

$$\sum_0^m \binom{m}{j} P_j E_{m-j} = 0, \quad m \geq 1, \tag{A3}$$

where $P_j = [(d/dt)^j P(t, D)]_{t=0}$ and $E_j = [(d/dt)^j E(t)]_{t=0}$. If we multiply (A3) by P_0^m and set $X_0 = 1$, $X_j = P_0^{j-1}P_j$ and $Y_j = P_0^{j+1}E_j$ then we obtain from (A1) the Faà di Bruno formula:

$$P_0^{m+1}E_m = F_m(P_1, P_0P_2, \ldots, P_0^{m-1}P_m)\delta, \quad m \geq 1. \tag{A4}$$

(Cf. Trèves [10] page 477.) By assumption we have that $P_0^{m+1}E_m \in B^{\text{loc}}_{\infty, k/\tilde{P}_0^{m+1}}$ so that

$$|F_m(P_1(\xi), \ldots, P_0^{m-1}(\xi)P_m(\xi))| \leq C\tilde{P}_0^{m+1}(\xi)/k(\xi) \leq C_m\tilde{P}_0^m(\xi)(1 + |\xi|)^M \tag{A5}$$

for all $m \geq 1$ and all $\xi \in \mathbf{R}^n$. Since P_j, $j = 1, 2, \ldots$, have bounded degrees there is a smallest $\gamma \geq 0$ such that $P_j(\xi)/\tilde{P}_0(\xi)(1 + |\xi|)^{\gamma j}$ is bounded for every j. Set $d(\xi) = 1/(P_0(\xi)(1 + |\xi|)^\gamma)$ and multiply (A5) by $(d(\xi))^m$. By means of the homogeneity conditions of F_m we obtain that

$$|F_m(P_1(\xi)d(\xi), \ldots, P_0^{m-1}(\xi)P_m(\xi)(d(\xi))^m)| \leq C_m(1 + |\xi|)^{M-\gamma m}.$$

Now, assume that $\gamma > 0$ and let m_0 be so large that $\gamma m_0 > \mathrm{M}$. Then

$$F_m(P_1(\xi)d(\xi), \ldots, P_0^{m-1}(\xi)P_m(\xi)(d(\xi))^m) \to 0 \quad \text{as} \quad |\xi| \to \infty.$$

Furthermore, there is a sequence ξ_j such that $|\xi_j| \to \infty$ and

$$(P_1(\xi_j)d(\xi_j), \ldots, P_0^{m-1}(\xi_j)P_m(\xi_j)(d(\xi_j))^m) \to (q_1, \ldots, q_m) \neq (0, \ldots, 0) \quad \text{as} \quad j \to \infty.$$

(Cf. Lemma A2 in [6].) Since the degree of P_m is bounded we have that $q_j = 0$ for $j > m_0$ if m_0 is large enough. Thus,

$$F_m(q_1, \ldots, q_{m_0}, \; 0, \ldots, 0) = 0 \quad \text{for} \quad m \geq m_0. \tag{A6}$$

Set $Q(t) = 1 + \sum_1^{m_0} q_j t^j/j!$. Then $Q(t)(1/Q(t)) = 1$ near zero. In the same way as we obtained (A4) we now get that

$$[(d/dt)^m(1/Q(t))]_{t=0} = F_m(q_1, \ldots, q_{m_0}, \; 0, \ldots, 0), \quad m \geq m_0,$$

so that (A6) implies that $[(d/dt)^m(1/Q(t))]_{t=0} = 0$ for all $m \geq m_0$. Thus $1/Q(t)$ is a polynomial, which implies that $q_1 = \ldots = q_{m_0} = 0$. This is a contradiction so that γ must be zero which proves that P_j is weaker than P_0 for all j.

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Arne Enqvist
Department of Mathematics
University of Lund
Sweden