On Absolutely Convergent Fourier Series

C. W. ONNEWEER

1. Introduction

Let $T$ denote the circle group and let $Z$ denote the group of integers. We shall consider functions $f$ which are integrable on $T$ and we shall denote their Fourier coefficients by $\hat{f}(n)$, where $n \in Z$. For $\beta > 0$ the set of all $f \in L_1(T)$ such that $\sum_{n=\infty}^{\infty} |\hat{f}(n)|^\beta < \infty$ will be denoted by $A(\beta)$. Among the classical results in the theory of absolutely convergent Fourier series are the following theorems [6, Vol. 1, Chapter VI, 3].

**Theorem 1** (Bernstein). If $f \in \text{Lip } \alpha$ for some $\alpha > \frac{1}{2}$, then $f \in A(1)$.

**Theorem 2** (Zygmund). If $f$ is of bounded variation on $T$ ($f \in \text{BV}$) and if $f \in \text{Lip } \alpha$ for some $\alpha > 0$, then $f \in A(1)$.

Attempts to generalize these theorems have led to the following.

**Theorem 1A** (Szs). If $f \in \text{Lip } \alpha$ for some $\alpha$ with $0 < \alpha \leq 1$, then $f \in A(\beta)$ for all $\beta$ such that $\beta > 2/(2\alpha + 1)$.

**Theorem 1B** (Hardy). If $f \in \text{Lip } \alpha$ for some $\alpha$ with $0 < \alpha \leq 1$, then $\sum_{n=1}^{\infty} |n|^{-\beta} |\hat{f}(n)| < \infty$ for all $\beta$ such that $\beta > (1 - 2\alpha)/2$.

**Definition 1.** Let $f$ be a function defined on $T$ and for $r \geq 1$, let

$$V_r[f] = \sup \left( \sum_{k=0}^{n-1} |f(x_{k+1}) - f(x_k)|^r \right)^{1/r},$$

where the supremum is taken over all finite partitions $0 \leq x_0 < x_1 \cdots < x_n < 2\pi$ of $T$. The function $f$ is of $r$-bounded variation ($f \in r\text{-BV}$) if $V_r[f] < \infty$. 
THEOREM 2A (Hirschman [2]). If \( f \in \text{r-BV} \) for some \( r \) with \( 1 \leq r < 2 \), and if \( f \in \text{Lip } \alpha \) for some \( \alpha > 0 \), then \( f \in A(1) \).

It is well-known that each of the foregoing theorems is the best possible in a certain sense [6, Vol. 1]. In [4, Exercise I.6.6] Katznelson gave a new and very simple example of a function in \( \text{Lip } \frac{1}{2} \) that does not belong to \( A(1) \). In the remainder of this section we shall give a simple extension of Katznelson's example which can be used to show that all the previous theorems are sharp. We first give the definition of the so-called Rudin-Shapiro polynomials \( P_n(x) \) and \( Q_n(x) \). Let \( P_0(x) = Q_0(x) = 1 \), and for \( m \geq 0 \), let

\[
P_{m+1}(x) = P_m(x) + e^{2\pi i m}Q_m(x) \quad \text{and} \quad Q_{m+1}(x) = P_m(x) - e^{2\pi i m}Q_m(x).
\]

Next, let \( f_{m+1}(x) = P_{m+1}(x) - P_m(x) \) and for each \( \alpha \) with \( 0 < \alpha < 1 \), let

\[
g_{\alpha}(x) = \sum_{k=1}^{\infty} 2^{-k(\alpha + \frac{1}{2})} f_k(x).
\]

It follows immediately from the definition of \( g_{\alpha} \) that \( \hat{g}_{\alpha}(n) = 0 \) if \( n \leq 0 \) and that \( \hat{g}_{\alpha}(n) = \epsilon(n)2^{-k(\alpha + \frac{1}{2})} \) if \( 2^{k-1} \leq n \leq 2^k \) for some \( k \geq 1 \) and with \( \epsilon(n) = \pm 1 \).

A proof similar to the one given by Katznelson for the case \( \alpha = \frac{1}{2} \) yields the following.

THEOREM 3. For each \( \alpha \) with \( 0 < \alpha < 1 \) we have

(i) \( g_{\alpha} \in \text{Lip } \alpha \) and \( g_{\alpha} \notin \text{Lip } \gamma \) for any \( \gamma > \alpha \),
(ii) \( g_{\alpha} \notin \text{Lip } \gamma \),
(iii) \( g_{\alpha} \notin \text{Lip } \gamma \),
(iv) \( \sum_{n=1}^{\infty} n^{\alpha-1/2} |\hat{g}_{\alpha}(n)| = \infty \).

2. Convolution functions

Throughout this section we shall denote the conjugate of a number \( p > 1 \) by \( q \), that is, \( 1/p + 1/q = 1 \). For \( f, g \in L_1(T) \) the convolution \( f \ast g \) is defined by

\[
(f \ast g)(x) = \int_T f(x - t)g(t)dt.
\]

Then \( (f \ast g)^\wedge(n) = \hat{f}(n)^\wedge g(n) \) for all \( n \in \mathbb{Z} \). The following theorem is due to M. Riesz (6, Vol. 1, page 251).

THEOREM 4. A continuous function \( f \) has an absolutely convergent Fourier series if and only if there exist functions \( g, h \in L_2(T) \) such that \( f = g \ast h \).
The next theorem gives a partial extension of this result.

**Theorem 4A.** If \( g, h \in L_p(T) \) for some \( p \) with \( 1 < p \leq 2 \), then
\[
g \ast h \in A(p/(2p - 2)).
\]

**Proof.** It follows from Young's inequality and the Hausdorff-Young inequality that
\[
\sum_{n=-\infty}^{\infty} |\hat{g}(n)|^{q_2} |\hat{h}(n)|^{q_2} \leq \frac{1}{2} \sum_{n=-\infty}^{\infty} |g(n)|^q + \frac{1}{2} \sum_{n=-\infty}^{\infty} |h(n)|^q \leq \frac{1}{2} \|g\|_p^q + \frac{1}{2} \|h\|_p^q < \infty.
\]
that is, \( g \ast h \in A(q/2) = A(p/(2p - 2)). \)

We next show that Theorem 4A is sharp.

**Theorem 5.** For every \( p \) with \( 1 < p \leq 2 \) there exist functions \( g, h \in L_p(T) \) such that \( g \ast h \notin A(\beta) \) for any \( \beta < p/(2p - 2) \).

**Proof.** We define the functions \( g \) and \( h \) by
\[
\hat{g}(n) = \hat{h}(n) = \begin{cases} n^{1/q} \log n^{-1} & \text{if } n > 1, \\ 0 & \text{if } n \leq 1. \end{cases}
\]
Clearly, \( \hat{g}(n) \searrow 0 \) as \( n \to \infty \) and
\[
\sum_{n=2}^{\infty} (\hat{g}(n))^{p-2} = \sum_{n=2}^{\infty} n^{p-2-p/q} (\log n)^{-p} = \sum_{n=2}^{\infty} n^{-1} (\log n)^{-p} < \infty,
\]
because \( p > 1 \). A theorem due to Hardy and Littlewood [6, Vol. 2, page 129] implies that \( g \), and hence also \( h \), belongs to \( L_p(T) \). Furthermore, if \( \beta < p/(2p - 2) \) then
\[
\sum_{n=-\infty}^{\infty} |(g \ast h)^(n)|^\beta = \sum_{n=2}^{\infty} (n^{1/q} \log n)^{-2\beta} = \infty,
\]
because \( 2\beta/q < 1 \). Therefore, \( g \ast h \notin A(\beta) \).

**Theorem 6.** If \( g \in L_p(T) \) with \( 1 < p \leq 2 \) and if \( h \in \text{Lip } x \) with \( 0 < x \leq 1 \), then \( g \ast h \in A(\beta) \) for all \( \beta \) such that \( 2p/(2xp + 3p - 2) < \beta \).

**Proof.** First choose \( \beta \) such that \( 2p/(2xp + 3p - 2) < \beta < q \). Then Young's inequality implies that
\[
\sum_{n=-\infty}^{\infty} |\hat{g}(n)\hat{h}(n)|^\beta \leq \frac{\beta}{q} \sum_{n=-\infty}^{\infty} |\hat{g}(n)|^q + \frac{q-\beta}{q} \sum_{n=-\infty}^{\infty} |\hat{h}(n)|^{q_2(q-\beta)} = A + B.
\]
Since \( \beta > 2p/(2x + 3p - 2) \), we have \( \beta q/(q - \beta) > 2/(2x + 1) \). Hence, Theorem 1A implies that \( B \) is finite. Also, the Hausdorff-Young inequality implies that \( A \) is finite. Therefore, \( g \ast h \in A(\beta) \).

Choosing \( \beta = 1 \) in Theorem 6 we obtain the following corollary. It shows how we can ameliorate functions in \( \text{Lip} \alpha \) with \( 0 < \alpha \leq \frac{1}{2} \), which are not necessarily in \( A(1) \), into functions in \( A(1) \) by means of the convolution operator.

**Corollary 1.** Let \( g \in L_p(T), \ 1 < p \leq 2, \) and \( h \in \text{Lip} \alpha, \ 0 < \alpha \leq \frac{1}{2}. \) If \( (2x + 1)p > 2 \), then \( g \ast h \in A(1) \).

We now show to what extent Theorem 6 and Corollary 1 are the best possible.

**Theorem 7.** Let \( p \) and \( \alpha \) satisfy the conditions \( 1 < p \leq 2 \) and \( 0 < \alpha < 1/p \). Then, (i) for all \( \alpha \), with \( 0 < \alpha < 1/\alpha \) there exist functions \( g \) and \( h \) with \( g \in L_p(T) \) and \( h \in \text{Lip} \alpha \) and such that \( g \ast h \notin A(2p/(2x + 3p - 2)) \), (ii) for all \( p \), with \( 1 < p < p \) there exist functions \( g \) and \( h \) with \( g \in L_p(T) \) and \( h \in \text{Lip} \alpha \) and such that \( g \ast h \notin A(2p/(2x + 3p - 2)) \).

**Proof.** (i) If \( \gamma \) is defined by \( \gamma = x + 1/q \), then \( 2/(2\gamma + 1) = 2p/(2xp + 3p - 2) \). Let \( h = g_\alpha \), then, according to Theorem 3(i), \( h \in \text{Lip} \alpha \). Let \( g \) be defined by

\[
\hat{g}(n) = \begin{cases} 2^{-k(\gamma - \alpha)} & \text{if } 2^{k-1} \leq n < 2^k \text{ for some } k \geq 1, \\ 0 & \text{if } n \leq 0. \end{cases}
\]

Then \( \hat{g}(n) \searrow 0 \) as \( n \to \infty \) and

\[
\sum_{k=0}^{\infty} \hat{g}(n)p^{p-2} \leq \sum_{k=0}^{\infty} 2^{k-1}2^{-k(\gamma - \alpha)p}2^{(k-1)(p-2)} < \infty,
\]

because \( 1 - (\gamma - \alpha)p + p - 2 = (\alpha_1 - \alpha)p < 0 \). Thus, \( g \in L_p(T) \). Furthermore, if \( n \in \mathbb{Z} \) and if \( 2^{k-1} \leq n < 2^k \) for some \( k \geq 1 \), then

\[
\hat{g}(n)\hat{h}(n) = 2^{-k(\gamma - \alpha)} \epsilon(n)2^{-k(\alpha + \frac{1}{2})} = \epsilon(2^{-k(\gamma + \frac{1}{2})} = \hat{g}(n),
\]

that is, \( g \ast h = g_\gamma \). Since, according to Theorem 3(iii), \( g_\gamma \notin A(2p/(2xp + 3p - 2)) \), we have established (i).

(ii) The proof of (ii) is similar to the proof of (i). In this case the functions \( g \) and \( h \) are chosen as follows. Let \( h = g_\alpha \) and let \( \hat{g}(n) = 0 \) if \( n \leq 0 \) and let \( \hat{g}(n) = 2^{-k(\gamma - \alpha)} \) if \( 2^{k-1} \leq n < 2^k \) for some \( k \geq 0 \) and with \( \gamma = x + 1/q \). Then it is clear that the functions \( g \) and \( h \) satisfy the conditions mentioned in (ii).

**Remark 1.** The following case of Theorem 7 is of special interest. For each \( p \) such that \( 1 < p < 2 \) and each \( \alpha \) such that \( 0 < \alpha < (2 - p)/2p \) there exist functions \( g \) and \( h \) with \( g \in L_p(T), \ h \in \text{Lip} \alpha \) and \( g \ast h \notin A(1) \). This improves
a result of M. and S. Izumi [3, Theorem 3] who proved for each \( p \) with \( 1 < p < 2 \) and each \( s \) with \( s > 2 \) the existence of functions \( g \) in \( L_p(T) \) and \( h \) in \( L_s(T) \) such that \( g \ast h \notin A(1) \).

3. Multipliers of type \((l_p(Z^n), l_p(Z^n))\)

In this section we shall define a collection of functions on the \( n \)-dimensional torus \( T^n \). We shall use these functions to show that certain results of Hahn [1] about \( p \)-multipliers on \( T^n \) are the best possible. Furthermore, for \( n = 1 \) these new functions will be the same as the functions \( g_s \) which were defined in Section 1. Throughout this section we shall use the notation \( x = (x_0, x_1, \ldots, x_{n-1}) \) for \( x \) in \( T^n \) and \( m = (m_0, m_1, \ldots, m_{n-1}) \) for \( m \) in \( Z^n \).

Definition 2 [2]. A bounded and measurable function \( f \) defined on \( T^n \) is a \( p \)-multiplier, \( 1 \leq p \leq \infty \), if for every function \( F \) in \( l_p(Z^n) \), the function \( T(f)F \) is again in \( l_p(Z^n) \), where \( T(f) \) is defined by

\[
T(f)F(m) = \sum_{k \in Z^n} F(m - k)\hat{f}(k).
\]

The set of \( p \)-multipliers will be denoted by \( M_p \).

Definition 3 [1, page 327]. Let \( \alpha \) be a positive real number and let \( \alpha_* \) be the largest integer less than \( \alpha \). For \( 1 \leq p \leq \infty \), \( \text{Lip}(\alpha, p) \) is the class of all functions \( f \) defined on \( T^n \) such that for \( |k| < \alpha_* \) we have \( (\partial/\partial x)^{\alpha} f \in L_p(T^n) \) and for \( |k| = \alpha_* \) we have

\[
\left\| \Lambda_h \left( \frac{\partial}{\partial x} \right)^k f \right\|_p = O(|h|^{\alpha - \alpha_*}) \quad \text{if} \quad \alpha - \alpha_* < 1,
\]

\[
\left\| \Lambda_h \left( \frac{\partial}{\partial x} \right)^k f \right\|_p = O(|h|) \quad \text{if} \quad \alpha - \alpha_* = 1,
\]

where for each \( h \) and \( x \) in \( T^n \) we set \( \Lambda_h f(x) = f(x + h) - f(x) \).

Obviously, if \( p_1 \geq p_2 \), then \( \text{Lip}(\alpha, p_1) \subset \text{Lip}(\alpha, p_2) \); so, in particular, \( \text{Lip}(\alpha, \infty) \subset \text{Lip}(\alpha, p) \) for all \( p \geq 1 \) and for all \( \alpha > 0 \). Hahn proved the following [1, Theorems 12' and 20].

Theorem 8.

(a) If \( 1 < p \leq 2 \) and \( \alpha > n/p \), then \( \text{Lip}(\alpha, p) \subset M_r \) for \( 1 \leq r < \infty \).

(b) If \( p > 2 \) and \( \alpha > n/p \) then \( \text{Lip}(\alpha, p) \subset M_r \) for \( 2p/(p + 2) \leq r \leq 2p/(p - 2) \).

(c) If \( n/p < \alpha \leq n/2 \), then \( \text{Lip}(\alpha, p) \subset M_r \) for \( 2n/(n + 2x) < r < 2n/(n - 2x) \).
We shall prove that these results are sharp in the sense that for \( p \geq 2 \) we cannot replace \( \alpha > n/p \) by \( \alpha \geq n/p \) in Theorem 8(a) and (b), whereas the conclusion of Theorem 8(c) does not hold for \( r = 2n/(n + 2\alpha) \) or \( r = 2n/(n - 2\alpha) \). We do not know whether the conclusion of Theorem 8(a) holds if \( 1 < p < 2 \) and \( \alpha = n/p \).

**Theorem 9.**

(a) If \( p \geq 2 \) and if \( \alpha = n/p \), then \( \text{Lip}(x, \infty) \subseteq M_{2p/(p+2)} \); in particular, \( \text{Lip}(n/2, \infty) \subseteq M_1 \).

(b) If \( 0 < \alpha \leq n/2 \), then \( \text{Lip}(x, \infty) \subseteq M_{2n/(n + 2\alpha)} \).

In order to prove Theorem 9 we first define functions \( h_\alpha(x) \) for each \( \alpha > 0 \). For convenience we shall write \( \tilde{n} \) for \( 2^n \) and \( \omega_n \) for \( \exp(2\pi n/\tilde{n}) \). For \( i = 0, 1, 2, \ldots \) and \( l = 0, 1, 2, \ldots, \tilde{n} - 1 \) we define the trigonometric polynomials \( P_i(x) \) inductively. Let \( P_{00}(x) = \cdots = P_{0\tilde{n}-1}(x) = 1 \) for all \( x \in T^n \). Next, assume that the polynomials \( P_k(x) \) have been defined for some \( k \geq 0 \) and all \( l \) with \( 0 \leq l < \tilde{n} \). Each \( j \) with \( 0 \leq j < \tilde{n} \) has a unique representation of the form

\[
\hat{j} = j_0 + 2j_1 + \ldots + 2^{n-1}j_{n-1},
\]

with \( j_i \in \{0, 1\} \). Let \( \hat{j} = (j_0, \ldots, j_{n-1}) \in Z^n \) and let \( j \cdot x = j_0x_0 + \ldots + j_{n-1}x_{n-1} \). Next, for \( l \) with \( 0 \leq l < \tilde{n} \) we define \( P_{k+1}(x) \) by

\[
P_{k+1}(x) = \sum_{j=0}^{\tilde{n}-1} \omega_j \bar{c}_j \cdot x_{2k} P_j(x).
\]

Since

\[
\omega_j \bar{c}_j = \begin{cases} \tilde{n} & \text{if } l = 0, \\
0 & \text{if } l = 1, 2, \ldots, \tilde{n} - 1,
\end{cases}
\]

we have for arbitrary complex numbers \( c_0, \ldots, c_{\tilde{n}-1} \)

\[
\sum_{l=0}^{\tilde{n}-1} \sum_{j=0}^{\tilde{n}-1} |c_j|^2 = \tilde{n} \sum_{j=0}^{\tilde{n}-1} |c_j|^2.
\]

Therefore,

\[
\sum_{l=0}^{\tilde{n}-1} \sum_{j=0}^{\tilde{n}-1} |P_{k-1}(x)|^2 = \sum_{l=0}^{\tilde{n}-1} \sum_{j=0}^{\tilde{n}-1} \omega_j \bar{c}_j \cdot x_{2k-1} P_{k-1}(x) = \tilde{n} \sum_{j=0}^{\tilde{n}-1} |P_{k-1}(x)|^2 = \tilde{n}^{k+1}.
\]

Hence, for each \( k \geq 0 \) we have

\[
\|P_{k0}(x)\|_{\infty} \leq \tilde{n}^{(k+1)/2}.
\]

Also, \( |\hat{P}_{k0}(m)| = 1 \) if \( m = (m_0, \ldots, m_{n-1}) \) with \( 0 \leq m_i < 2^k \) for \( i = 0, 1, \ldots, n-1 \), and \( \hat{P}_{k0}(m) = 0 \) otherwise. For \( k \geq 1 \) let \( f_k(x) = \hat{P}_{k0}(x) - P_{k-10}(x) \), and for \( \alpha > 0 \) let
We can show that \( h_\alpha \in \text{Lip}(\alpha, \infty) \). The proof requires a long and tedious computation which we shall omit. We only observe that we need an \( n \)-dimensional version of Bernstein's inequality: if \( f \) is a trigonometric polynomial on \( T^n \) of degree \( k \), that is, 
\[
f(x) = \sum_{j \in \mathbb{Z}^n} c_j e^{i \cdot j \cdot x},
\]
with \( \max_j (|j_0| + |j_1| + \ldots + |j_{n-1}|) = k \), then for each of the first order partial derivatives of \( f \) we have 
\[
\left\| \frac{\partial f}{\partial x_1} \right\|_\infty \leq k \|f\|_\infty.
\]

**Proof of Theorem 9.** (a) Consider the function \( F \) which is defined on \( \mathbb{Z}^n \) by 
\[
F(0) = F(0, \ldots, 0) = 1 \quad \text{and} \quad F(m) = 0 \quad \text{for} \quad m \neq 0 \quad \text{and} \quad m \in \mathbb{Z}^n.
\]
Clearly, \( F \in L_p(\mathbb{Z}^n) \). We shall prove that 
\[
h_{n/p} \notin M_{2p/(p+2)}.
\]
For each \( m \in \mathbb{Z}^n \) we have 
\[
T(h_{n/p})F(m) = \sum_{k \in \mathbb{Z}^n} F(m - k)\hat{h}_{n/p}(k) = \hat{h}_{n/p}(m).
\]
Also,
\[
\sum_{k \in \mathbb{Z}^n} |\hat{h}_{n/p}(k)|^{2p/(p+2)} = \sum_{k=1}^{\infty} \left( 2^{kn} - 2^{k-1} \right) 2^{-n(k+1)/(p+2)} \cdot 2^{k(p+2)/(p+2)} \\
\geq \frac{1}{2} \sum_{k=1}^{\infty} 2^{kn} 2^{-kn} = \infty,
\]
that is, \( T(h_{n/p})F \notin L_{2p/(p+2)} \). Therefore, \( h_{n/p} \notin M_{2p/(p+2)} \).
Since \( M_{2p/(p+2)} = M_{2p/(p-2)} \) we also have \( h_{n/p} \notin M_{2p/(p-2)} \).

(b) For each \( \alpha \) such that \( 0 < \alpha \leq n/2 \) we have 
\[
\sum_{k \in \mathbb{Z}^n} |\hat{h}_\alpha(k)|^{2n/(n+2\alpha)} \geq \frac{1}{2} \sum_{k=1}^{\infty} 2^{kn} 2^{-n(2\alpha+n)/(2n)} = \infty.
\]
Therefore, an argument as in (a) shows that \( h_\alpha \notin M_{2n/(n+2\alpha)} \), and hence also, \( h_\alpha \notin M_{2n/(n-2\alpha)} \).

**Remark 2.** For each \( n \) the function \( h_{n/2} \) provides an example of a function in \( \text{Lip}(n/2, \infty) \) which does not have an absolutely convergent Fourier series. Hence the \( n \)-dimensional version of Theorem 1 is sharp. This and related results were established by Wainger [5].
Remark 3. The functions $g_\alpha$ as defined in Section 1 also provide new examples that show that several of the results of Hirschman on $p$-multipliers cannot be improved as was already pointed out by Hirschman in [2].

The author would like to thank Professor L.-S. Hahn for a number of helpful conversations on the subject of this paper.

References


Received January 10, 1973