An example of a strongly convex metric space

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1. Introduction

It is known (see [2, Sections 7 and 8]) that if (X, d) is a convex metric space[1]) and every point of X is locally passing, then every two points of X are joined by a geodesic. If every two points of X are joined by exactly one geodesic and every point is locally passing, then every point is passing and every geodesic is a straight line. Hence (X, d) is a strongly convex space. On the other hand the following theorem is a consequence of the results in [3].

Theorem 1. *If (X, d) is a strongly convex finitely-dimensional space and every point of X is locally passing, then every point of X is passing* (cf. [2, Problems and Theorems]).

Our aim is to give an example of a strongly convex 2-dimensional space (Y, φ) such that every point of Y is passing, but Y has a geodesic which is not a straight line (cf. [2, Problems and Theorems]).

2. Definitions and notations

Let (X, d) be a metric space with a metric d. If $x, y, z \in X$ then we say that *z lies between x and y* (writing xzy) provided that

$$d(x, z) + d(z, y) = d(x, y).$$

A *midpoint* of the pair x, y is a point m such that

$$d(x, m) = d(m, y) = \tfrac{1}{2} d(x, y).$$

[1]) In this paper by "space" we understand "finitely-compact space". For terminology and notation see Section 2 in this paper and [1], [2], [4].

Evidently every midpoint of the pair x, y lies between x and y. A space (X, d) is said to be *convex* if each pair of points of X has at least one midpoint, *strongly convex* if each pair of points has exactly one midpoint.

It is well known that in a complete (in particular finitely-compact) convex space (X, d) every two distinct points x, y are joined by a *segment* with *end-points* x, y, i.e. by a subset of X containing x and y and isometric to a real interval of length $d(x, y)$. For complete spaces, strong convexity is equivalent to the condition that each pair of distinct points x, y determines exactly one segment with end-points x, y.

A subspace (X_1, d_1) of a metric space (X, d) is said to be a *convex subspace* if X_1 contains all points of X which lie between two points of X_1.

A point z of a metric space (X, d) is said to be a *passing point in* X if for every point $x \in X$ there exists a point $y \in X \setminus \{z\}$ such that xzy. We say that z is a *locally passing point* if there exists a neighbourhood U of z such that z is a passing point in U.

A subset G of a metric space X is said to be a *geodesic (straight line)* if G is locally isometric (isometric) to the space of real numbers.

3. Sums of strongly convex spaces

Let (X_1, d_1) and (X_2, d_2) be two metric spaces such that $X_1 \cap X_2 \neq \emptyset$ and if $x, y \in X_1 \cap X_2$, then $d_1(x, y) = d_2(x, y)$. We define (see [1, Section 6] and [4, Section 10]) the function d_T as follows:

$$d_T(x, y) = \begin{cases} d_1(x, y) & \text{if } x, y \in X_1 \\ d_2(x, y) & \text{if } x, y \in X_2 \\ \min\limits_{z \in X_1 \cap X_2} [d_1(x, z) + d_2(z, y)] & \text{if } x \in X_1, y \in X_2 \end{cases} \tag{1}$$

This definition is correct, because the minimum exists, the spaces X_1, X_2 being finitely-compact. It is clear that the function d_T is a metric for the sum $X_1 \cup X_2$.

Let Y, Z be the subsets of a convex space (X, d). We consider the following three properties:

(α) If $x, y \in Y$, $z \in Z$ and all these points are distinct, then

$$2d(z, m) < d(x, z) + d(y, z)$$

for every midpoint m of the pair x, y.

(β) If $x, y \in Y$, $z \in Z$, then

$$2d(z, m) \leq d(x, z) + d(y, z)$$

for every midpoint m of the pair x, y.

(γ) If $x, y \in Y$, $p, q \in Z$ and m, m' are the midpoints of x, y and p, q respectively, then

$$2d(m, m') \leq d(x, p) + d(y, q).$$

We shall write: $(Y, Z) \in (\alpha)/(\beta), (\gamma)/$ *in the space* (X, d) or $(Y, Z) \in (\alpha)/(\beta), (\gamma)/$ if the latter does not lead to misunderstanding.

It is easy to see, that

3.1. (γ) *implies* (β).

3.2. *Let* (X_1, d_1) *be a convex subspace of the convex space* (X, d) *and let* Y, Z *be subsets of* X_1. *Then* $(Y, Z) \in (\alpha)/(\beta), (\gamma)/$ *in the space* (X_1, d_1) *if and only if* $(Y, Z) \in (\alpha)/(\beta), (\gamma)/$ *in the space* (X, d).

Since the metric-function is continuous, we have

3.3. $(\bar{Y}, \bar{Z}) \in (\beta)/(\gamma)/$ *if and only if* $(Y, Z) \in (\beta)/(\gamma)/$.

3.4. *If* (X, d) *is a convex space and* $(X, X) \in (\alpha)$, *then* (X, d) *is strongly convex.*

Proof. Suppose on the contrary that there exist two midpoints p, q of the pair x, y of the points of X. Let z be a midpoint of the pair p, q. Then from the definition of the property (α) we obtain

$$d(x, y) \leq d(x, z) + d(y, z) < \tfrac{1}{2}[d(x, p) + d(x, q)] + \tfrac{1}{2}[d(y, p) + d(y, q)] = d(x, y)$$

which is impossible.

3.5. *Let* (Y, d_i) *be a convex subspace of a convex space* (X_i, d_i) $(i = 1, 2)$ *such that* $Y = X_1 \cap X_2 \neq \emptyset$ *and* $d_1(x, y) = d_2(x, y)$ *for every two points* $x, y \in Y$. *Let* Z *be a subset of* Y. *If* $(Z, Y) \in (\gamma)$ *in* (X_1, d_1) *and* $(Y, X_2) \in (\beta)$ *in* (X_2, d_2), *then* $(Z, X_2) \in (\beta)$ *in* $(X_1 \cup X_2, d_T)$.

Proof. Since (Y, d_i) is a convex subspace of a convex space (X_i, d_i), we infer that (Y, d_i) and (X_i, d_i) are convex subspaces of $(X_1 \cup X_2, d_T)$ for $i = 1, 2$. It follows that $(X_1 \cup X_2, d_T)$ is a convex space. Let $x, y \in Z$ and $z \in X_2$. Then there exist two points $p, q \in Y$ such that xpz and yqz. Let m, m' be the midpoints of x, y and p, q, respectively. Applying 3.2 we infer that $(Z, Y) \in (\gamma)$ in $(X_1 \cup X_2, d_T)$ and $(Y, X_2) \in (\beta)$ in $(X_1 \cup X_2, d_T)$. This implies that

$$2d_T(m, z) \leq 2d_T(m, m') + 2d_T(m', z) \leq d_T(x, p) + d_T(y, q) + d_T(p, z) + d_T(q, z) = \\ = d_T(x, z) + d_T(y, z)$$

hence $(Z, X_2) \in (\beta)$ in the space $(X_1 \cup X_2, d_T)$.

Now we shall prove the following:

3.6. *Let* (Y, d_i) *be a convex subspace of a strongly convex space* (X_i, d_i) $(i = 1, 2)$ *such that* $Y = X_1 \cap X_2 \neq \emptyset$ *and* $d_1(x, y) = d_2(x, y)$ *for every two points* $x, y \in Y$.

If $(Y, X_1 \setminus Y) \in (\alpha)$ *in the space* (X_1, d_1) *and* $(Y, X_2) \in (\beta)$ *in the space* (X_2, d_2), *then the space* $(X_1 \cup X_2, d_T)$ *is strongly convex.*

Proof. Since (Y, d_i) is a convex subspace of a strongly convex space (X_i, d_i), we infer that (Y, d_i) and (X_i, d_i) are strongly convex subspaces of $(X_1 \cup X_2, d_T)$ for $i = 1, 2$. It follows that $(X_1 \cup X_2, d_T)$ is a convex space. Moreover, $(X_1 \cup X_2, d_T)$ is a finitely-compact space. Thus for every two distinct points $x, y \in X_1 \cup X_2$ there exists at least one segment with end-points x, y.

Now let us suppose that there exist two distinct segments L_1, L_2 with end-points x, y. Since (X_i, d_i) is a strongly convex subspace of $(X_1 \cup X_2, d_T)$ we may assume that $x \in X_1 \setminus Y$ and $y \in X_2 \setminus Y$. Then there exist two distinct points $a \in Y \cap (L_1 \setminus L_2)$ and $b \in Y \cap (L_2 \setminus L_1)$. Since (Y, d_i) is a strongly convex subspace of $(X_1 \cup X_2, d_T)$, we infer that there exists exactly one midpoint m of the pair a, b and $m \in Y$. Applying 3.2 we infer that $(Y, X_1 \setminus Y) \in (\alpha)$ in $(X_1 \cup X_2, d_T)$ and $(Y, X_2) \in (\beta)$ in $(X_1 \cup X_2, d_T)$. This implies that

$$2d_T(x, y) = d_T(x, a) + d_T(a, y) + d_T(x, b) + d_T(b, y) >$$
$$> 2d_T(x, m) + d_T(a, y) + d_T(b, y) \geq 2d_T(x, m) + 2d_T(m, y) \geq 2d_T(x, y)$$

which is impossible.

4. Cone over a strongly convex space

Let X be a compact space and $\mathbf{R}^+ = \{t \in \mathbf{R}; t \geq 0\}$. The space obtained from the cartesian products $X \times \mathbf{R}^+$ by identifying the set $X \times \{0\}$ to one point will be called a *cone* over X. The point corresponding to the set $X \times \{0\}$ in the identification space will be called a *vertex*.

Let (X, d) be a compact metric space with diameter < 2 and let X_1 be a cone over X with a vertex v. We define a function d_1 by the following equations:

$$d_1((x_1, t_1), (x_2, t_2)) = \min(t_1, t_2) d(x_1, x_2) + |t_1 - t_2| \tag{2}$$

$$d_1((x, t), v) = d_1(v, (x, t)) = t \tag{3}$$

$$d_1(v, v) = 0 \tag{4}$$

The proof of the following two propositions runs as in [4, § 11].

4.1. *The function* d_1 *is a metric for* X_1.

4.2. *Let* (X, d) *be a strongly convex space and* $p_1 = (x_1, t_1)$, $p_2 = (x_2, t_2)$, $0 < t_1 \leq t_2$ *be two distinct points of* X_1. *Then*

(i) *The space* (X_1, d_1) *is strongly convex.*

(ii) *The segment with end-points* p_1, p_2 *is the sum of the sets*

$$\{p \in X_1;\ p = (x, t_1), d(x_1, x) + d(x, x_2) = d(x_1, x_2)\}$$

and

$$\{p \in X_1;\ p = (x_2, t),\ \ t_1 \leq t \leq t_2\}.$$

(iii) *The set*

$$\{p \in X_1;\ p = (x_1, t),\ \ 0 < t \leq t_1\} \cup \{v\}$$

is the segment with end-points p_1, v.

Let $y, z \in X$ and let

$$Y = \{p \in X_1;\ p = (y, t),\ \ t > 0\} \cup \{v\}$$

and

$$Z = \{p \in X_1;\ p = (z, t),\ \ t > 0\} \cup \{v\}$$

be two subsets of the space (X_1, d_1).

We shall prove the following:

4. $(Y, Z) \in (\gamma)$ *in the space* (X_1, d_1).

Proof. By 3.3. it is sufficient to prove that $(Y \setminus \{v\}, Z \setminus \{v\}) \in (\gamma)$. Let $p_1, p_2 \in Y \setminus \{v\}, q_1, q_2 \in Z \setminus \{v\}$ and $p_i = (y, t_i), q_i = (z, s_i)$ for $i = 1, 2$. It follows from 4.2 (ii) that $m = (y, \frac{1}{2}(t_1 + t_2))$ and $m' = (z, \frac{1}{2}(s_1 + s_2))$ are the mid-points of p_1, p_2 and q_1, q_2, respectively. Applying (2) and the formula

$$2 \min (a, b) = a + b - |a - b|$$

we have

$$\begin{aligned} d_1(p_1, q_1) + d_1(p_2, q_2) &= [\min (t_1, s_1) + \min (t_2, s_2)]d(y, z) + |t_1 - s_1| + |t_2 - s_2| = \\ &= \tfrac{1}{2}[t_1 + t_2 + s_1 + s_2 - |t_1 - s_1| - |t_2 - s_2|]d(y, z) + |t_1 - s_1| + |t_2 - s_2| = \\ &= \tfrac{1}{2}[2 \min (t_1 + t_2, s_1 + s_2) + |t_1 + t_2 - s_1 - s_2| - |t_1 - s_1| - |t_2 - s_2|]d(y, z) + \\ &\quad + |t_1 - s_1| + |t_2 - s_2| = \\ &= \{2 \min [\tfrac{1}{2}(t_1 + t_2), \tfrac{1}{2}(s_1 + s_2)] + \tfrac{1}{2}|t_1 + t_2 - s_1 - s_2|\}d(y, z) + \\ &\quad + (|t_1 - s_1| + |t_2 - s_2|)[1 - \tfrac{1}{2}d(y, z)] \end{aligned}$$

Since X is a space with diameter < 2, we infer that $1 - \frac{1}{2}d(y, z) > 0$. Thus we have

$$\begin{aligned} &d_1(p_1, q_1) + d_1(p_2, q_2) \geq \\ &\geq \{2 \min [\tfrac{1}{2}(t_1 + t_2), \tfrac{1}{2}(s_1 + s_2)] + \tfrac{1}{2}|t_1 + t_2 - s_1 - s_2|\}d(y, z) + \\ &\quad + |t_1 + t_2 - s_1 - s_2|[1 - \tfrac{1}{2}d(y, z)] = \\ &= 2 \min [\tfrac{1}{2}(t_1 + t_2), \tfrac{1}{2}(s_1 + s_2)]d(y, z) + 2|\tfrac{1}{2}(t_1 + t_2) - \tfrac{1}{2}(s_1 + s_2)| = 2\, d_1(m, m'). \end{aligned}$$

5. The spaces $(Y_1, \varphi_1), (Y_2, \varphi_2)$

Let $X = \{(x, y) \in R^2; x, y \geq 0, x^2 + y^2 = 1\}$ and $a, b \in X$. Setting

$$d(a, b) = \arccos ab \text{ [1]} \tag{5}$$

we obtain a metric d for X. It is plain that (X, d) is a compact strongly convex space with diameter < 2. Let (X_1, d_1) be a cone over the space (X, d) with a vertex v. Setting

$$f((a, t)) = ta \quad \text{for} \quad a \in X \quad \text{and} \quad t \geq 0, \tag{6}$$

we get a function $f\colon X_1 \to \mathbf{R}^2$. It is easy to see that f is continuous and $1-1$. Moreover, we have

$$Y_1 = f(X_1) = \{(x, y) \in \mathbf{R}^2; x, y \geq 0\}. \tag{7}$$

Now let us put

$$\varphi_1(q_1, q_2) = d_1(f^{-1}(q_1), f^{-1}(q_2)) \tag{8}$$

for every two points $q_1, q_2 \in Y_1$. Evidently φ_1 constitutes a metric for Y_1. Applying the notation just introduced we shall prove the following

5.1. *The space* (Y_1, φ_1) *is strongly convex. Moreover,*

(i) *If* q_1, q_2 *are two distinct points of* Y_1 *such that* $0 < \|q_1\| \leq \|q_2\|$, *then the segment with end-points* q_1, q_2 *is the union of the sets*

$$A = \left\{q \in Y_1; \|q\| = \|q_1\|, d\left(\frac{q_1}{\|q_1\|}, \frac{q}{\|q\|}\right) + d\left(\frac{q}{\|q\|}, \frac{q_2}{\|q_2\|}\right) = d\left(\frac{q_1}{\|q_1\|}, \frac{q_2}{\|q_2\|}\right)\right\}$$

and $$B = \{q \in Y_1; q = sq_2, \|q_1\| \leq s\|q_2\| \leq \|q_2\|\}.$$

(ii) *If* $q_i \in Y_1 \setminus \{(0, 0)\}$ *and* $L_i = \{q \in Y_1; q = sq_i, s \geq 0\}$ $(i = 1, 2)$, *then* $(L_1, L_2) \in (\gamma)$ *in the space* (Y_1, φ_1).

Proof. Since (X, d) is a compact strongly convex space with diameter < 2, we infer by 4.2 (i) that (X_1, d_1) is a strongly convex space. It follows from (8) that (Y_1, φ_1) is a strongly convex space.

In order to prove (i) let us observe that (6) implies that

$$f^{-1}(q) = \begin{cases} (q/\|q\|, \|q\|) & \text{for } q \neq (0, 0) \\ v & \text{for } q = (0, 0). \end{cases} \tag{9}$$

Thus

$$f^{-1}(A) = \{p \in X_1; \ p = (q/\|q\|, \|q_1\|), \ q \in A\}$$

[1]) If $a = (a_1, a_2)$, $b = (b_1, b_2)$ are points of $\mathbf{R}^2$ and $t \in \mathbf{R}$, then $ab = a_1b_1 + a_2b_2$, $ta = (ta_1, ta_2)$ and $\|a\| = \sqrt{a_1^2 + a_2^2}$.

and

$$f^{-1}(B) = \{p \in X_1;\ p = (q_2/\|q_2\|, s\|q_2\|), \|q_1\| \leq s\|q_2\| \leq \|q_2\|\}.$$

It follows from 4.2 (ii) that the set $T = f^{-1}(A) \cup f^{-1}(B)$ is a segment with end-points $p_1 = f^{-1}(q_1), p_2 = f^{-1}(q_2)$. Then (8) implies that the set $f(T) = A \cup B$ is a segment with end-points q_1, q_2.

Passing to (ii), let us observe that (9) implies that

$$f^{-1}(L_i) = \{p \in X_1;\ p = (q_i/\|q_i\|, s\|q_i\|),\ s > 0\} \cup \{v\}.$$

According to 4.3, we have $(f^{-1}(L_1), f^{-1}(L_2)) \in (\gamma)$ in (X_1, d_1). Then (8) implies that $(L_1, L_2) \in (\gamma)$ in (Y_1, φ_1).

Remark. Let q_1, q_2 be the points of Y_1. According to the definition of the metric d_1 (see 4) and (5), (9), (8), we have

$$\varphi_1(q_1, q_2) = \min(\|q_1\|, \|q_2\|) \arccos\left(\frac{q_1}{\|q_1\|} \cdot \frac{q_2}{\|q_2\|}\right) + \big|\|q_1\| - \|q_2\|\big| \quad \text{for} \quad q_1 \neq (0, 0) \neq q_2 \tag{10}$$

$$\varphi_1(q_1, (0, 0)) = \|q_1\|. \tag{11}$$

Let $Y_2 = \{(x, y) \in \mathbf{R}^2;\ 0 \leq x, y \leq 0\}$ and we put in Y_2 the ordinary euclidean metric φ_2. It is easy to see, that $\varphi_1(x, y) = \varphi_2(x, y)$ for every two points $x, y \in Y = Y_1 \cap Y_2$. We put in $Y_1 \cup Y_2$ the metric φ_T, i.e.

$$\varphi_T(x, y) = \begin{cases} \varphi_1(x, y) & \text{for } x, y \in Y_1 \\ \varphi_2(x, y) & \text{for } x, y \in Y_2 \\ \min\limits_{z \in Y} [\varphi_1(x, z) + \varphi_2(z, y)] & \text{for } x \in Y_1, y \in Y_2 \end{cases}$$

We shall prove the following:

5.2. *The space* $(Y_1 \cup Y_2, \varphi_T)$ *is strongly convex.*

Proof. It is sufficient to show that the hypotheses in 3.6 hold. Indeed, by the definition of (Y_2, φ_2) and 5.1 the spaces (Y_1, φ_1) and (Y_2, φ_2) are strongly convex. Evidently, $(Y_2, Y_2) \in (\alpha)$ in the space (Y_2, φ_2), hence $(Y, Y_2 \setminus Y) \in (\alpha)$ in the space (Y_2, φ_2). According to 5.1 (ii) and 3.1 we have $(Y, Y_1) \in (\beta)$ in the space (Y_1, φ_1).

Let ϱ denote the ordinary euclidean metric. Applying (10) and (11) we obtain the following

5.3. $\varrho(p, q) \leq \varphi_T(p, q)$ *for every two points* $p, q \in Y_1 \cup Y_2$.

Now we shall prove the following:

5.4. *Let* $L = \{(x, y) \in \mathbf{R}^2;\ x = 0\}$. *For every point* $z \in Y_1 \cup Y_2$ *and every point* $m \in L$ *there exists a neighbourhood* U *of* m *in* L *such that* $(U, \{z\}) \in (\beta)$ *in the space* $(Y_1 \cup Y_2, \varphi_T)$.

Proof. Let $A = \{(x, y) \in \mathbf{R}^2;\ x = 0, y < 0\}$, $B = \{(x, y) \in \mathbf{R}^2;\ x = 0, y > 0\}$. Let us show that for every point $z \in Y_1 \cup Y_2$ we have $(A, \{z\}) \in (\beta)$ and $(B, \{z\}) \in (\beta)$ in $(Y_1 \cup Y_2, \varphi_T)$. We distinguish between two cases:

Case 1: $z \in Y_1$. By 5.1 (ii), 3.2 and 3.1 we obtain that $(B, \{z\}) \in (\beta)$ in $(Y_1 \cup Y_2, \varphi_T)$. It follows from 3.5 that $(A, \{z\}) \in (\beta)$ in $(Y_1 \cup Y_2, \varphi_T)$, because $(A, Y_1 \cap Y_2) \in (\gamma)$ in (Y_2, φ_2) and by 5.2 (ii), $(Y_1 \cap Y_2, \{z\}) \in (\beta)$ in (Y_1, φ_1).

Case 2: $z \in Y_2$. Evidently $(A, \{z\}) \in (\beta)$ in (Y_2, φ_2) hence from 3.2 we infer that $(A, \{z\}) \in (\beta)$ in $(Y_1 \cup Y_2, \varphi_T)$. By 5.1 (ii), $(Y_1 \cap Y_2, B) \in (\gamma)$ in (Y_1, φ_1) and applying 3.5 we obtain that $(B, \{z\}) \in (\beta)$ in $(Y_1 \cup Y_2, \varphi_T)$.

Hence for every point $z \in Y_1 \cup Y_2$ and every point $m \in L \setminus \{(0, 0)\}$ there exists a neighbourhood U of m in L such that $(U, \{z\}) \in (\beta)$ in $(Y_1 \cup Y_2, \varphi_T)$.

Now let $p, q \in L, m = (0, 0) = m(p, q), z \in Y_1 \cup Y_2$ and all these points are distinct. Applying 5.3 and (11) we have

$$\varphi_T(p, z) + \varphi_T(q, z) - 2\varphi_T(z, m) \geq \varrho(p, z) + \varrho(q, z) - 2\varrho(z, m) > 0.$$

Since the metric-function ϱ is continuous, there exists a neighbourhood V of $m = (0, 0)$ in $Y_1 \cup Y_2$ such that the inequality

$$\varphi_T(x, z) + \varphi_T(y, z) > 2\varphi_T(z, m(x, y))$$

holds for every two distinct points $x, y \in V$. Let $U = V \cap L$. Then $(U, \{z\}) \in (\beta)$ in the space $(Y_1 \cup Y_2, \varphi_T)$.

6. Construction of the space (Y, φ)

Let $Y_3 = \{(x, y) \in \mathbf{R}^2;\ x \leq 0\}$ and let φ_3 be the ordinary euclidean metric in Y_3. Then for every two points $x, y \in L = Y_3 \cap (Y_1 \cup Y_2)$ we have $\varphi_3(x, y) = \varphi_T(x, y)$. Let $Y = Y_1 \cup Y_2 \cup Y_3$ and we define the metric φ as follows:

$$\varphi(x, y) = \begin{cases} \varphi_T(x, y) & \text{for } x, y \in Y_1 \cup Y_2 \\ \varphi_3(x, y) & \text{for } x, y \in Y_3 \\ \min\limits_{z \in L} [\varphi_T(x, z) + \varphi_3(z, y)] & \text{for } x \in Y_1 \cup Y_2, y \in Y_3 \end{cases} \tag{12}$$

It is not difficult to verify that the space (Y, φ) is topologically a plane E^2. We shall prove that

6.1. *The space* (Y, φ) *is strongly convex.*

Proof. Evidently the space (Y_3, φ_3) is strongly convex, hence by (12) and 5.2 it is sufficient to show that for every two points $p \in (Y_1 \cup Y_2) \setminus L$, $q \in Y_3 \setminus L$ there exists exactly one segment with end-points p, q.

It is easy to see that (Y, φ) is a complete convex space. Thus there exists at least one segment with end-points p, q. Suppose on the contrary that p, q are joined by at least two segments in Y with end-points p, q. Let F be the set of all points x of the set $Y_3 \setminus L$ such that there exist at least two segments in Y with end-points x, p. We shall show that the set F is closed in Y_3. Let $x_n \in F$ and $x = \lim_{n\to\infty} x_n$. Then there exist $y_n, z_n \in L$ such that $y_n \neq z_n$, $x_n y_n p$ and $x_n z_n p$ $(n = 1, 2, \ldots)$. Since $(Y_3, L) \in (\alpha)$ in (Y, φ), we infer by 5.4 that $\inf_n \varphi(y_n, z_n) > 0$. Without loss of generality suppose that $y = \lim_{n\to\infty} y_n$, $z = \lim_{n\to\infty} z_n$. Then $y \neq z$ and xyp, xzp, because $x_n y_n p$ implies that $\lim_{n\to\infty} y_n$ is between $\lim_{n\to\infty} x_n$ and p. It follows that $x \in F$, hence F is closed in Y_3.

Since F is a closed subset of the complete space (Y_3, φ_3) and $(Y_1 \cup Y_2, \varphi_T)$ is strongly convex, there exists $q \in F$ such that

$$r = \varphi(p, q) = \inf_{x \in F} \varphi(p, x) \tag{13}$$

and

$$r > \inf_{x \in L} \varphi(p, x). \tag{14}$$

Let us observe that for every point $x \in Y_3 \setminus (F \cup L)$ there exists exactly one point $f(x) \in L$ such that $xf(x)p$. It is not difficult to see that the function $f: Y_3 \setminus (F \cup L) \to L$ defined in this way is continuous.

Since $q \in F$, there exist $a, b \in L$ such that $a \neq b$ and qap, qbp. Let x_n, y_n be points such that $x_n \neq q \neq y_n$, $qx_n a$, $qy_n b$ $(n = 1, 2, \ldots)$ and $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = q$. Since $\varphi(x_n, q) < r$ and $\varphi(y_n, q) < r$, $x_n, y_n \in Y_3 \setminus (F \cup L)$. It follows that $f(x_n) = a$ and $f(y_n) = b$. Since the set $D_n = \{z \in Y;\ x_n z y_n\}$ is connected, there exists $z_n \in D_n$ such that $f(z_n) = m$, where m is a midpoint of the pair a, b. Since $\lim_{n\to\infty} z_n = q$, we infer that qmp. It follows that the set $C = \{x \in L;\ qxp\}$ is convex and $\dim C = 1$. Choose $a_n, b_n \in C$ such that $0 < \varphi(a_n, b_n) < 1/n$ and let m_n be a midpoint of the pair a_n, b_n for $n = 1, 2, \ldots$ It follows from 5.4 that an index N exists such that

$$\varphi(a_n, p) + \varphi(b_n, p) \geq 2\varphi(m_n, p) \quad \text{for} \quad n > N.$$

Since $(Y_3, L) \in (\alpha)$ in (Y, φ), we obtain

$$\begin{aligned} 2\varphi(p, q) &= \varphi(p, a_n) + \varphi(a_n, q) + \varphi(p, b_n) + \varphi(b_n, q) > \\ &> 2\varphi(m_n, q) + \varphi(p, a_n) + \varphi(p, b_n) \geq \\ &\geq 2\varphi(m_n, q) + 2\varphi(p, m_n) \geq 2\varphi(p, q) \end{aligned}$$

which is impossible.

6.2. *Every point of the space* (Y, φ) *is passing in* Y.

Proof. Supposing the contrary, we have two distinct points $p, q \in Y$ such that pqx implies $x = q$. Since (Y, φ) is strongly convex, for every $x \in Y$ and $0 \leq t \leq 1$, there exists exactly one point $h(x, t) \in Y$ such that

$$\varphi(x, h(x, t)) = t\varphi(p, x)$$

and

$$\varphi(p, h(x, t)) = (1 - t)\varphi(p, x).$$

It is not difficult to verify that the function $h : (Y \setminus \{q\}) \times [0,1] \to Y$ is continuous and $h(x, t) \neq q$ for every $x \in Y \setminus \{q\}$ and $t \in [0,1]$. But $h(x, 0) = x$ and $h(x, 1) = p$ for every point $x \in Y \setminus \{q\}$ hence $Y \setminus \{q\}$ is contractible into itself, which is impossible, because (Y, φ) is topologically a plane E^2.

6.3. (Y, φ) *has a geodesic which is not a straight line.*

Proof. Consider the sets

$$A = \{(x, y) \in \mathbf{R}^2;\ x, y \geq 0,\ \|(x, y)\| = a\},$$
$$B = \{(x, y) \in \mathbf{R}^2;\ x = 0,\ y \geq a\},$$
$$C = \{(x, y) \in \mathbf{R}^2;\ x \geq a,\ y = 0\},$$

where $a > 0$. It follows from (7), (5) and 5.1 (i) that the sets $A \cup B$ and $A \cup C$ are isometric to the real closed half-line. By the same argument we infer that the set A is a segment in the space (Y, φ). This implies that the set $G = A \cup B \cup C$ is a geodesic in Y.

On the other hand let $q_1 = (2a, 0)$, $q_2 = (0, 2a)$. It follows from 5.1 (i) that the set

$$D = \{(x, y) \in \mathbf{R}^2;\ x, y \geq 0,\ \|(x, y)\| = 2a\}$$

is a segment with end-points q_1, q_2. Since $D \cap G = \{q_1, q_2\}$, we infer from 6.1 that G is not isometric to the real line. Thus G is the desired geodesic.

Applying 6.1, 6.2 and 6.3 we obtain the following result

THEOREM 2. *There exists a strongly convex 2-dimensional space* (Y, φ) *such that every point of* Y *is passing, but* Y *has a geodesic which is not a straight line.*

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