Hilbert algebras as topological algebras

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0. Introduction

Hilbert algebras (for references and definitions see § 1) have been found to be useful in various ways in analysis. Referring to them in his book [4], Dixmier states »Elles constituent, on le verra, un puissant moyen d'étude des algebres de von Neumann.» For other instances see the work of Godement [6] and Rieffel [15].

In this paper we present a rather systematic study of Hilbert algebras from the point of view of the theory of topological algebras. This leads first (but not exclusively) to an investigation of ideals in Hilbert algebras as well as to quotient algebras formed by a Hilbert algebra modulo a closed ideal.

In several situations below properties of a Hilbert algebra A are obtained by first embedding A in its fulfillment A_b (all the bounded elements in its completion, see § 1 for notation), then working in A_b and finally dropping back to A. In this way it is shown that if K is a closed ideal in A then $K = K^*$ so that K is a Hilbert algebra. Also this procedure is used to show that A/K is a Hilbert algebra in the quotient algebra norm. Yet again we follow this route to see that any topologically simple Hilbert algebra with a minimal one-sided ideal is equivalent to a dense *-subalgebra of an H^*-algebra and to see that a homomorphism of a Banach algebra onto a Hilbert algebra must be continuous. The success of this program is made possible by the theory of full Hilbert algebras $(A = A_b)$ as developed first by Godement [6] and then furthered by Rieffel [15].

We consider some special classes of Hilbert algebras in § 2 as well as full Hilbert algebras. Every full Hilbert algebra A is orthocomplemented ($A = J \oplus J^{\perp}$ for all closed right (left) ideals J). Every orthocomplemented Hilbert algebra is a dual Hilbert algebra. The notions of dual Hilbert algebra and annihilator Hil-

[1] This research was supported in part by NSF GRANT GP 20226. Much of this work was reported in an invited address to the North British Functional Analysis Seminar, Edinburgh, Scotland, on March 9, 1970.

bert algebra turn out to be the same. For K a closed ideal in A it is shown in § 2 that, if A is full or orthocomplemented or dual, then so are K and A/K.

The work of Rieffel [15] was also drawn upon to help develop in § 4 a theory for Hilbert algebras with dense socle (see Corollary 4.7 and, for full Hilbert algebras, Corollary 4.8). For a full Hilbert algebra A with dense socle its completion H is the Hilbert space direct sum of the minimal closed ideals of A each of which is equivalent to an H^*-algebra. (It is stressed that the »simple components» of the decomposition of A are complete (Banach algebras) while A is, in general, incomplete.) As a by-product of this investigation we obtain a characterization of H^*-algebras. A Hilbert algebra A is equivalent to an H^*-algebra if and only if A is full and there exists $c > 0$ such that $\|p\| \geqq c$ for all non-zero projections p in A. Again note that A turns out to be complete from a set-up which is *a priori* incomplete.

In many examples of Hilbert algebras such as $C(G)$, where G is a compact group, made into an algebra by convolution multiplication, multiplication is completely continuous [10, p. 700]. Every c.c. Hilbert algebra is the (Hilbert space) direct sum of its minimal closed ideals each being a full finite-dimensional matrix algebra. See Theorem 6.4.

In § 3 and § 4 we consider some more analytical aspects of the theory of Hilbert algebras. A starting point is the discovery in Theorem 3.1. that the involution in a full Hilbert algebra is symmetric. This avenue is explored for Hilbert algebras not full. The relation of a Hilbert algebra to the natural C^*-algebra in which it is embedded is examined in detail there.

1. Notation

For convenience and to set forth notation we start with a definition for Hilbert algebras. Our notation is that of [15] with some minor changes. For references to the original papers on Hilbert algebras we refer the reader to [4] as well as to [15]; we call attention to [6].

A *Hilbert algebra* is an algebra A over the complex field with an involution $x \to x^*$ which is a pre-Hilbert space with inner product (x, y) where

(a) $(x, y) = (y^*, x^*)$ for all $x, y \in A$;
(b) $(xy, z) = (y, x^*z)$ for all $x, y, z \in A$;
(c) for each $a \in A$ the linear operator L_a defined by $L_a(x) = ax$, $x \in A$, is continuous;
(d) the set of elements of the form xy, where $x, y \in A$, is dense in A.

Let R_a be the operation on A of right multiplication by $a \in A$. We denote the completion of A by $H = H(A)$. The operators L_a and R_a extend to bounded linear operators on H which are denoted by $\bar{L}_a$ and $\bar{R}_a$ respectively.

It is known [6, p. 51] that, in the presence of (a), (b) and (c), the condition (d) is equivalent to the requirement that x lies in the closure of xA (Ax) for each $x \in A$. This fact is used below. From this, or by the more elementary argument of [3, p. 331], the mapping $x \to \bar{L}_x$ of A into the algebra $B(H)$ of all bounded linear operators on H is a *-isomorphism. Therefore, by [14, Theorem 4.1.19], A is semisimple.

For $a \in A$, $\|\bar{L}_a\| = \|\bar{L}_{a*}\|$ as $\bar{L}_{a*}$ is the adjoint of $\bar{L}_a$. Likewise $\|\bar{R}_a\| = \|\bar{R}_{a*}\|$. It turns out that all four norms are equal. For

$$\|\bar{R}_a(x)\| = \|(xa)^*\| = \|L_{a*}(x^*)\| \leqq \|L_{a*}\|\|x\|.$$

Thus $\|\bar{R}_a\| \leqq \|\bar{L}_{a*}\|$ and similarly $\|\bar{L}_{a*}\| \leqq \|\bar{R}_a\|$.

For the theory of bounded elements for Hilbert algebras see [6] and [15]. An element a in $H = H(A)$ is *left* (*right*) *bounded* if the mapping $L_a(R_a)$ defined on A with values in H by $L_a(x) = \bar{R}_x(a)$ $(R_a(x) = \bar{L}_x(a))$ is bounded. Fortunately, a is left bounded if and only if a is right bounded; we then call *a bounded*. We denote the set of bounded elements of $H(A)$ by A_b. As shown in [6], A_b is itself a Hilbert algebra in the inner product of H with a multiplication and involution extending that of A.

The Hilbert algebra A is called *full* if $A = A_b$. We shall sometimes refer to A_b as the *fulfillment* of A. This is the language used by Rieffel in [15]; in [6] and elsewhere other terminology is adopted.

For a subset S in A we set $S^\perp = \{x \in A\colon (x, S) = (0)\}$, $\mathbf{L}(S) = \{x \in A\colon xS = (0)\}$ and $\mathbf{R}(S) = \{x \in A\colon Sx = (0)\}$. Arguments of Kaplansky [10, Theorem 12] show that $\mathbf{L}(K) = K^{\perp *}$, $\mathbf{R}\,\mathbf{L}(K) = K^{\perp\perp}$, $\mathbf{R}(J) = J^{\perp *}$ and $\mathbf{LR}(J) = J^{\perp\perp}$ for every closed right ideal K (left ideal J). We shall consider Hilbert algebras which are annihilator algebras and dual algebras in the sense of [14, Chapter 2].

As in [15, Definition 7.1] by the *C^*-algebra* $C^*(A)$ of A is meant the operator norm closure of $\{\bar{L}_x\colon x \in A\}$ in the algebra $B(H)$ of all bounded linear operators on $H = H(A)$. It is convenient to think of A as being algebraically embedded in $C^*(A)$ via the algebraic *-isomorphism $x \to \bar{L}_x$.

2. Some basic theory for Hilbert algebras

We are mainly concerned here with closed ideals K in a Hilbert algebra A and with the quotient algebras A/K. These are studied also in special classes of Hilbert algebras — full, dual and orthocomplemented (see Definition 2.3 for the latter term).

As is well known [8] even for a semi-simple Banach algebra B, B can be an annihilator algebra without being dual (the reverse implication is trivial). For certain classes of topological algebras the two notions agree. This is the case for B^*-algebras [2] and, as we now see, for Hilbert algebras.

THEOREM 2.1. *The following statements about a Hilbert algebra A are equivalent.*
(1) *A is an annihilator algebra.*
(2) *Every right or left ideal K in A for which $K^{\perp} = (0)$ is dense in A.*
(3) *$J + J^{\perp}$ is dense in A for each right or left ideal J in A.*
(4) *A is a dual algebra.*

Proof. It was shown in [22, Lemma 3.2] that (1) and (2) are equivalent. Clearly (2) implies (3). Next assume (3). Consider a closed right ideal K in A. As noted in § 1, $\mathbf{R}\,\mathbf{L}(K) = K^{\perp\perp}$. Let K^c be the closure of K in the Hilbert space completion H of A and $K^{c\#}$ be its orthogonal complement in H. We must show that $K^{\perp\perp} \subset K$. Let $x \in K^{\perp\perp}$ and write $x = \lim (u_n + v_n)$ where each $u_n \in K$ and $v_n \in K^{\perp}$. Whereas also $u_n \in K^c$ and $v_n \in K^{c\#}$ we see that $\{u_n\}$ and $\{v_n\}$ are Cauchy sequences in the Hilbert space H. Then there exist $a \in K^c$, $b \in K^{c\#}$ where $u_n \to a$ and $v_n \to b$ and $x = a + b$. As $x \in K^{\perp\perp}$ we have $0 = (x, v_n) \to (x, b)$. Thus $(b, b) = (x, b) - (a, b) = 0$. Therefore $x = a \in K^c \cap A = K$. In the same way $\mathbf{LR}I = I$ for a closed left ideal I. Inasmuch as A is semisimple, (4) implies (1).

As shown by Kaplansky [10, Theorem 2] a closed ideal I in a semisimple dual algebra is a dual algebra. This can fail if I is not closed. Again we have a contrast with the situation for Hilbert algebras.

COROLLARY 2.2. *A $*$-ideal W (not necessarily closed) in a dual Hilbert algebra A is a dual Hilbert algebra.*

Proof. As shown by Dixmier [4, p. 72], W is a Hilbert algebra (in the involution $*$). So also is its closure $\bar{W}$. But by the result of Kaplansky cited just above $\bar{W}$ is a dual algebra. Therefore, without loss of generality, we may assume that W is dense in A.

We must distinguish between orthogonality in W and in A. As usual for a set $S \subset A$, $S^{\perp} = \{x \in A \colon (x, S) = (0)\}$ whereas for a set T in W we let $T^{\#} = \{x \in W \colon (x, T) = (0)\}$. Let J be a right ideal in W. By Theorem 2.1 our task is to show that J is dense in W if $J^{\#} = (0)$. Now $\bar{J}$ is a right ideal in A and it suffices, by Theorem 2.1, to show that $J^{\perp} = (0)$.

Clearly $J^{\perp}W \subset J^{\perp} \cap W = J^{\#} = (0)$. As W is dense in A, we get $J^{\perp}A = (0)$ so that $J^{\perp} = (0)$ (see § 1).

Definition 2.3. We say that the Hilbert algebra A is an *ortho*complemented Hilbert algebra if $A = J \oplus J^{\perp}$ for every closed right or left ideal J in A.

Example 2.4. By Theorem 2.1, an orthocomplemented Hilbert algebra is a dual algebra. A Hilbert algebra can be dual without being orthocomplemented. Consider the algebra $C(G)$ of all complex valued continuous functions on the compact topological group G. Here the multiplication operation is convolution

$$fg(s) = \int_G f(st^{-1})g(t)d\mu$$

where μ is normalized Haar measure on G and

$$(f, g) = \int_G f(t)\overline{g(t)}d\mu.$$

The involution is given by $f^*(t) = \overline{f(t^{-1})}$.

$C(G)$ is an *-ideal in the H^*-algebra $L_2(G)$ (see [10, p. 700]) and hence dual in the pre-Hilbert space topology by Corollary 2.2 (other arguments could be used to see this).

Take the special case $G = T$, the multiplicative group of all complex numbers of absolute value one. Each $f \in C(T)$ has a Fourier expansion $f \sim \sum a_n \exp(\text{int})$. Let $I = \{f \in C(T): a_n = 0, n < 0\}$. Then I is a closed ideal in $C(T)$ in the pre-Hilbert space topology, $I^\perp = \{f \in C(T): a_n = 0, n \geq 0\}, I \oplus I^\perp$ is dense in $C(T)$ and $I \oplus I^\perp \neq C(T)$.

THEOREM 2.5. *A full Hilbert algebra A is an orthocomplemented Hilbert algebra and is dual.*

Proof. Let J be a closed right ideal in A and J^c its closure in $H = H(A)$. We show that J^c is a right-invariant subspace of H in the sense of [15, p. 272], that is, $\bar{R}_a(\xi) \in J^c$ for each $a \in A$ and $\xi \in J^c$. Let $\xi = \lim w_n, w_n \in J$. Then $\bar{R}_a(\xi) = \lim \bar{R}_a(w_n) = \lim w_n a \in J^c$. Let P be the orthogonal projection of H onto J^c. As shown in the proof of [15, Proposition 2.7], $P(A) = A \cap (J^c) = J$. For a set S in H, let $S^\# = \{x \in H: (x, S) = (0)\}$. Then $H = J^c \oplus (J^c)^\#$. Given $x \in A$ we can write $x = u + v, u \in J^c, v \in (J^c)^\#$. As $u = P(x)$, we have $u \in J$ and $v \in (J^c)^\# \cap A = J^\perp$. That A is dual now follows from Theorem 2.1.

Example 2.6. At this point it is appropriate to point out that not every orthocomplemented Hilbert algebra is full. Consider the algebras c_p, $1 \leqq p \leqq 2$, studied by McCarthy [11] and others. This is the class of operators on a Hilbert space for which the c_p norm $|T|_p = [\text{trace}\,(T^*T)^{p/2}]^{1/p}$ is finite. Let the Hilbert space be infinite-dimensional. The algebra c_1 is the trace class algebra (τc) in the notation of Schatten's book [16] while c_2 is the H^*-algebra (σc), the Schmidt-class of operators. For $1 \leqq p < 2, c_p$ is contained in c_2. If $c_p, 1 \leqq p < 2$, is given the pre-Hilbert space topology inherited from c_2, then it becomes an orthocomplemented Hilbert algebra which is not full. The author is indebted to Professor James F. Smith for pointing this out to him. See [18] where this author, in a manuscript just completed, has shown more general results. For another example obtained from different considerations see Example 4.4 below.

THEOREM 2.7. *Suppose that* K *is a closed ideal in a Hilbert algebra* A. *Then* $K = K^*$ *and* K *is a Hilbert algebra. If* A *is full or orthocomplemented or dual then so is* K.

Proof. If we show that $K = K^*$, we have K a Hilbert algebra by [4, p. 72].

We suppose at first that A is a full Hilbert algebra. Let $x \in K$. We claim that $x^* \in K$. Since A is full this follows from Theorem 2.5 if we show $x^* \in K^{\perp\perp}$. Let $w \in A, z \in K^{\perp}$. Clearly $xz = 0 = (xz, w) = (z, x^*w)$. Therefore $x^*A \subset K^{\perp\perp}$ so that $x^* \in K^{\perp\perp}$. Next we show that K is full.

Consider the completion $H(K)$ of K; clearly $H(K) \subset H(A)$. Let $a \in K_b$ be a bounded element for K. Then L_a defined by $L_a(x) = \bar{R}_x(a)$ is a linear operator on A and is bounded as a mapping of K into $K(H)$ with operator bound M. There is a sequence $\{v_n\}$ in K with $a = \lim v_n$. For $y \in K^{\perp}$ we have $L_a(y) = \lim v_n y = 0$ since each $v_n y \in K \cap K^{\perp}$. For $z \in A$ we can write, by Theorem 2.5, $z = x + y$ with $x \in K, y \in K^{\perp}$. Then, as $\|z\|^2 = \|x\|^2 + \|y\|^2$, we get

$$\|L_a(z)\| = \|L_a(x)\| \leqq M\|x\| \leqq M\|z\|.$$

Consequently $a \in A_b = A$. Hence $a = \lim v_n$ lies in K.

Now we turn to the case when A is an arbitrary Hilbert algebra. Let K^c denote the closure of K in the full Hilbert algebra A_b. The involution on A_b extends that of A and moreover K^c is a closed ideal in A_b. Then, by the above, if $x \in K$, we get $x^* \in K^c \cap A = K$. Therefore K is a Hilbert algebra.

Since $K = K^*$ the statement on dual algebras is immediate by Corollary 2.2. Suppose that A is orthocomplemented and let J be a closed right ideal in K. As K is dual, J is a right ideal in A by [10, Theorem 2] and so $A = J \oplus J^{\perp}$. From this it is readily seen that $K = J \oplus J^{\perp} \cap K$.

By contrast with Corollary 2.2, a *-ideal in a full (orthocomplemented) Hilbert algebra need not be full (orthocomplemented). In the positive direction we have the following result.

PROPOSITION 2.8. *A *-ideal* W *in an orthocomplemented Hilbert algebra* A *with identity* 1 *is an orthocomplemented Hilbert algebra.*

Proof. Let J be a right ideal of W, J closed in W. Now J is not necessarily a right ideal in A but its closure $\bar{J}$ in A is a right ideal in $\bar{W}$ and therefore, by [10, Theorem 2] and Theorem 2.1, is a right ideal in A. We can then write $1 = u + v$ where $u \in \bar{J}$ and $v \in J^{\perp}$. For each $z \in W$ we have $z = uz + vz$ where $uz \in \bar{J}W \subset J$ and $vz \in J^{\perp} \cap W$.

We make a detailed study of the quotient algebra A/K where A is a Hilbert algebra and K is a closed ideal in A. First of all, by Theorem 2.7, A/K is a *-algebra where the involution is given by

$$(x + K)^* = x^* + K. \tag{2.1}$$

We consider the quotient algebra norm

$$\|x + K\| = \inf_{y \in K} \|x + y\|. \tag{2.2}$$

Elementary arguments show that, for each $x, y \in A$,

$$\|x + y + K\|^2 + \|x - y + K\|^2 = 2(\|x + K\|^2 + \|y + K\|^2).$$

Therefore, in the norm of (2.2), A/K becomes a pre-Hilbert space where the inner product $(x + K, y + K)$ is given by the quantity

$$[\|x + y + K\|^2 - \|x - y + K\|^2 + i\|x + iy + K\|^2 - i\|x - iy + K\|^2]/4 \tag{2.3}$$

THEOREM 2.9. *Let K be a closed ideal in a Hilbert algebra A. Then A/K is a Hilbert algebra in terms of the involution* (2.1) *and the inner product* (2.3). *If A is full or orthocomplemented or dual then so is A/K.*

Proof. Since $\|x^* + K\| = \|x + K\|$ for each $x \in A$ we see immediately that

$$(x + K, y + K) = (y^* + K, x^* + K)$$

for each $x, y \in A$. Also

$$\|(x + K)(y + K)\| = \|xy + K\| \leq \inf_{z \in K} \|x(y + z)\| \leq \|L_x\| \|y + K\|$$

so that left multiplication by $x + K$ is a bounded linear operator on A/K. The natural homomorphism of A onto A/K is continuous. This makes the set of products $\{(x + K)(y + K)\colon x, y \in A\}$ a continuous image of a dense set in A and therefore dense in A/K. It remains for us to show that

$$(xy + K, z + K) = (y + K, x^*z + K) \tag{2.4}$$

for each $x, y, z \in A$.

We demonstrate (2.4) first in the special case where $A = K \oplus K^\perp$. It is readily shown that the mapping $Q(x) = x + K$ of $K^\perp$ onto A/K is an algebraic *-isomorphism and as seen above Q preserves inner products. By Theorem 2.7, we get (2.4) and the fact that A/K and $K^\perp$ are equivalent as Hilbert algebras. Using Theorem 2.5 and 2.7 we now see our statements in the full or orthocomplemented cases.

Now we consider an arbitrary Hilbert algebra A. Let K^c denote the closure of K in the fulfillment A_b of A. By the above, A_b/K^c is a Hilbert algebra in terms of the quotient space norm. The mapping σ defined by

$$\sigma(f + K) = f + K^c$$

for $f \in A$ is an algebraic *-isomorphism of A/K into A_b/K^c since $K^c \cap A = K$. By the definition of the quotient algebra norms, $\|\sigma(f)\|^2 = \|f\|^2$. Now we have already seen that A/K is a pre-Hilbert space in the quotient algebra norm. Therefore

$$(f + K, g + K) = (\sigma f, \sigma g) \tag{2.5}$$

for all $f, g \in A$. Formula (2.4) for A follows directly from (2.5) and the validity of (2.4) for A_b/K^e.

Finally suppose that A is dual. Let π be the natural homomorphism of A onto A/K and let W be a closed right ideal in A/K, $W \neq A/K$. Then $\pi^{-1}(W)$ is a closed right ideal in A, $\pi^{-1}(W) \neq A$, $\pi^{-1}(W) \supset K$. Since, for some $x \in A$, $x \neq 0$ and $x\pi^{-1}(W) = (0)$ we see that $\mathbf{L}(W) \neq (0)$ in A/K. By Theorem 2.1, A/K is a dual Hilbert algebra.

This argument shows that if B is a topological algebra which is a continuous homomorphic image of a dual Hilbert algebra A, then B is a semisimple annihilator algebra.

3. Hilbert algebras in the Rieffel norm

For an element x in the Hilbert algebra A we set

$$\|x\|_r = \|x\| + \|\bar{L}_x\|. \tag{3.1}$$

We refer to (3.1) as the *Rieffel norm* for x. As noted in [15, p. 270] A is a normed algebra in this norm and, if A is full, A is complete in this norm. Let A_r denote the completion of A in the Rieffel norm and call A *replete* if $A = A_r$. Clearly A_r is a *-subalgebra of the fulfillment A_b of A. In general $A_r \neq A_b$ (see Example 4.4 below). The Rieffel norm is especially useful to us in discussing spectral properties for A and the connection between A and its C^*-algebra $C^*(A)$.

A straight-forward computation yields

$$\|xy\|_r \leqq \|\bar{L}_x\|\|y\|_r \tag{3.2}$$

for all $x, y \in A$.

Lemma 3.1. *If A is replete, A is a *-ideal in $C^*(A)$ and the involution on A is symmetric.*

Proof. Let $y \in A$ and $W \in C^*(A)$ where $W = \lim \bar{L}_{z_n}$, $z_n \in A$, $n = 1, 2, \ldots$. Using (3.2) we see that

$$\|z_n y - z_m y\|_r \leqq \|\bar{L}_{z_n} - \bar{L}_{z_m}\|\|y\|_r \to 0$$

as $m, n \to \infty$. Therefore there exists $v \in A$ where $\|v - z_n y\|_r \to 0$. Then also $\|\bar{L}_v - \bar{L}_{z_n y}\| \to 0$. On the other hand

$$\|W\bar{L}_y - \bar{L}_{z_n y}\| \leqq \|W - \bar{L}_{z_n}\|\|\bar{L}_y\| \to 0.$$

We see then that $W\bar{L}_y = \bar{L}_v$ and A is a left ideal in $C^*(A)$, indeed a two-sided ideal as A is a *-subalgebra of $C^*(A)$. The symmetry of the involution on $C^*(A)$ forces the involution to be symmetric on its ideal A.

It is obvious that not all Hilbert algebras have the symmetry property. We next seek a larger class of Hilbert algebras (see Corollary 3.5) than the replete ones where we can be sure of having a hermitian involution ($\mathrm{sp}(h)$ real for h self-adjoint, where $\mathrm{sp}(h)$ is the spectrum of h).

We shall use the notation $\nu_r(x) = \lim \|x^n\|_r^{1/n}$.

LEMMA 3.2. *If x is normal in A then $\nu_r(x) = \|\bar{L}_x\|$ and $\nu_r(xx^*) = (\nu_r(x))^2$.*

Proof. Since $\bar{L}_x$ is a normal operator on H, we get

$$\|\bar{L}_x\| = \|\bar{L}_{x^n}\|^{1/n} \leqq \|x^n\|_r^{1/n}$$

for each positive integer n. Then $\|\bar{L}_x\| \leqq \nu_r(x)$.

We have also

$$\|x^n\|_r = \|\bar{L}_{x^{n-1}}(x)\| + \|\bar{L}_{x^n}\| \leqq \|\bar{L}_x\|^{n-1}\|x\|_r .$$

Taking n^{th} roots and letting $n \to \infty$ we see that $\nu_r(x) \leqq \|\bar{L}_x\|$. We also have $\nu_r(xx^*) = \|\bar{L}_{xx^*}\| = \|\bar{L}_x\|^2 = (\nu_r(x))^2$.

By a Q-algebra [9] is meant a topological algebra whose quasiregular elements form an open set. Obviously every Banach algebra is a Q-algebra. We consider a normed Q-algebra B and use the notation $\nu(x) = \lim \|x^n\|^{1/n}$ and $\varrho(x)$ for the spectral radius of $x \in B$. We use the fact [20, Lemma 2.1] that the normed Q-algebras are just those normed algebras B for which $\nu(x) = \varrho(x)$ for all $x \in B$.

THEOREM 3.3. *Let B be a complex normed Q-algebra with an involution $x \to x^*$. Then $\nu(xx^*) = (\nu(x))^2$ for all normal $x \in B$ if and only if the involution is hermitian.*

Proof. Assume the condition on normal elements and let h be self-adjoint. If h has a complex number $a + bi$, a, b real, $b \neq 0$, in its spectrum, there is a self-adjoint element $w = ch + dh^2$, c, d real, such that $i \in \mathrm{sp}\,(w)$. We now modify an argument of Arens [1] (see also [14, p. 190]) for our purposes. Consider the normal element

$$z = (w + ni)^m w$$

where n, m are positive integers. Then

$$-\,[i(1 + n)]^{2m} \in \mathrm{sp}\,(z^2).$$

Therefore, using the remarks on Q-algebras given above, and the rules of [14, p. 10] we get

$$\begin{aligned}(1 + n)^{2m} &\leqq \varrho(z^2) = [\nu(z)]^2 = \nu(zz^*) = \nu[(w^2 + n^2)^m w^2] \\ &\leqq \sum_{k=0}^{m} \binom{m}{k} \nu(w)^{2(k+1)} n^{2(m-k)} = [\nu(w^2) + n^2]^m \nu(w^2).\end{aligned}$$

If we fix n, take m^{th} roots and let $m \to \infty$, we get

$$(1 + n)^2 \leqq \nu(w^2) + n^2$$

for all $n = 1, 2, \ldots$ Clearly this is impossible for n sufficiently large. Therefore $\operatorname{sp}(h)$ is real.

Suppose the condition on the spectra and let $x \in B$ be normal. We consider B as embedded in its Banach algebra completion B^c. The involution on B need not extend to an involution on B^c but in any case there is a maximal commutative subalgebra E of B^c containing x and x^*. E is a commutative Banach algebra with space of modular maximal ideals $\mathcal{M}$. Consider $z \in E$. By [14, Theorem 1.6.14] we have $\operatorname{sp}(z|E) = \operatorname{sp}(z|B^c) \subset \operatorname{sp}(z|B)$. It then follows from our hypothesis that, for each $M \in \mathcal{M}$, $(x + x^*)(M)$, $i(x - x^*)(M)$ are real numbers. Therefore $x^*(M) = \overline{xM}$ for each $M \in \mathcal{M}$. Consequently $\nu(xx^*) = (\nu(x))^2$.

COROLLARY 3.4. *Let B be a complex Banach algebra with an involution $x \to x^*$. The involution is symmetric if and only if $\nu(xx^*) = (\nu(x))^2$ for all normal elements x.*

Proof. This is immediate from Theorem 3.3 and the work of Shirali and Ford [17].

In [13] Pták made an interesting and detailed study of the function $p(x) = \varrho(x^*x)^{1/2}$ on a Banach $*$-algebra with hermitian involution (and identity). Theorem 3.3 shows that a normed $*$-Q-algebra has hermitian involution if and only if $p(x) = \varrho(x)$ for all normal x.

COROLLARY 3.5. *The involution is hermitian in a Hilbert algebra which is a Q-algebra in its Rieffel norm.*

Proof. This is immediate from Theorem 3.3 and Lemma 3.2.

Instances of Hilbert algebras which are incomplete Q-algebras in the Rieffel norm occur quite naturally. Consider the Hilbert algebra $C(G)$ of Example 2.4. The norm $|f| = \sup |f(t)|$ is a Banach algebra norm related to the Hilbert algebra norm by the inequalities $\|f\| \leqq |f|$ and $|fg| \leqq \|f\|\|g\|$ for all $f, g \in C(G)$. If $C(G)$ were complete in the Rieffel norm we would have, for some $K > 0$, $|f| \leqq K\|f\|$ for all $f \in C(G)$ and $C(G)$ would be complete in the norm $\|f\|$. But the completion of $C(G)$ in that norm is $L_2(G)$. To see that $C(G)$ is a Q-algebra in the Rieffel norm take any $f \in C(G)$ with $\|f\|_r < 1$. Then $|f^2| < 1$ so that f^2 and hence also f is quasi-regular. By [9, Lemma 2] $C(G)$ is a Q-algebra in the norm $\|g\|_r$.

By a *minimal idempotent* we mean an idempotent generator for a minimal one-sided ideal. For a minimal idempotent e in a Hilbert algebra A we have $eAe = \{ce\colon c \text{ complex}\}$. This follows from the fact that A is a normed algebra in the Rieffel norm so that the Gelfand-Mazur theorem applies to the normed division algebra eAe.

We return to the study of the relation of a replete Hilbert algebra to its C^*-algebra. When convenient we consider A as embedded in $C^*(A)$ (see § 1).

LEMMA 3.6. *For a replete Hilbert algebra A, A and $C^*(A)$ have the same minimal one-sided ideals.*

Proof. For convenience set $B = C^*(A)$. Let p be a minimal idempotent in A. Then $\{c\bar{L}_p: c \text{ complex}\} = \{\bar{L}_p\bar{L}_x\bar{L}_p: x \in A\}$. As this set is closed in B it coincides with $\bar{L}_pB\bar{L}_p$. Thus p is a minimal idempotent in B. Take a minimal idempotent W in B. Then $WAW = \{cW: c \text{ complex}\}$ as A is dense in B. Inasmuch as $WAW \subset A$ by Lemma 3.1, W is a minimal idempotent of A.

Let p be a minimal idempotent. Using Lemma 3.1 we get $pB \subset A$. Then obviously $pB = pA$.

A fact relating a closed right ideal I in A to its closure I^c in $C^*(A)$ is the following. If j is an idempotent in A and $j \notin I$ then $j \notin I^c$. For otherwise $\|\bar{L}_j - \bar{L}_{x_n}\| \to 0$ for some sequence $\{x_n\}$ in I. Then $\bar{L}_{x_n}(j) \to \bar{L}_j(j)$ or $x_nj \to j$ This is impossible as each $x_nj \in I$.

THEOREM 3.7. *Let A be a replete Hilbert algebra where $C^*(A)$ is dual. Then A is dual if and only if A has dense socle.*

Proof. Suppose that A has dense socle. Let I be a closed right ideal in $A, I \neq A$. There exists a minimal idempotent p not in I. By the above remark, $p \notin I^c$. As $C^*(A)$ is dual, there exists a minimal idempotent q of $C^*(A)$ with $qI^c = (0)$. By Lemma 3.6, $q \in A$ so that $\mathbf{L}(I) \neq (0)$. Apply Theorem 2.1.

The converse follows with the aid of [14, Corollary 2.8.16]. For more on the question of when $C^*(A)$ is dual see Theorem 4.3 below.

4. Projections in Hilbert algebras

We first use projections (self-adjoint idempotents) to continue our study of the relation of A to $C^*(A)$. However our main interest here is to discuss the theory of Hilbert algebras with dense socle.

Following Rieffel [15, p. 272] we use $E = E(A)$ to denote the set of all non-zero projections in A_b. Note that we consider the projections in A_b rather than A. By [15, Theorem 2.3], A_b can be rich in projections whereas A has no non-zero projections. An example of this is the Hilbert algebra of all complex continuous functions $f(t)$ on $[0, 1]$ with $f(0) = 0$ with the inner product

$$(f, g) = \int_0^1 f(t)\overline{g(t)}dt$$

and the usual pointwise operations.

Definition 4.1. We say that A is *projection bounded from above (below)* if there exists $c > 0$ such that $\|p\| \leqq c$ ($\|p\| \geqq c$) for all $p \in E$.

By standard Hilbert space theory, E is a partially ordered space if we define $p_1 \leqq p_2$ to mean $p_1 = p_1 p_2$ and $\|p_1\| \leqq \|p_2\|$ if $p_1 \leqq p_2$. Consequently A is projection bounded from above if A has an identity.

THEOREM 4.2. *The following statements about a Hilbert algebra A are equivalent.*

(1) *A is projection bounded from above.*

(2) *There exists $K > 0$ such that $\|x\| \leqq K\|\bar{L}_x\|$ for all $x \in A$.*

(3) *There exists $K > 0$ such that $\|h\| \leqq K\|\bar{L}_h\|$ for all self-adjoint elements h in A.*

(4) *A_r is equivalent, in its Rieffel norm, to a B^*-algebra.*

(5) *$A_r = C^*(A_r)$, where A_r is viewed as a Hilbert algebra.*

(6) *A_b is equivalent, in its Rieffel norm, to a B^*-algebra.*

(7) *$A_b = C^*(A_b)$.*

Proof. Assume (1). By [15, Proposition 2.12] A_b has an approximate identity $\{p_\gamma\}$. Here the set of norms $\{\|p_\gamma\|\}$ is bounded above by some $K < \infty$. Then, as $xp_\gamma \to x$ in A, $\|\bar{L}_x(p_\gamma)\| \leqq K\|\bar{L}_x\|$ and $\|\bar{L}_x(p_\gamma)\| \to \|x\|$. This yields (2).

Assume (3). Let a be any self-adjoint element in A_r. It is clear that there exist self-adjoint elements h_n in A such that $\|a - h_n\|_r \to 0$. But then $\|a - h_n\| \to 0$ and $\|\bar{L}_a - \bar{L}_{h_n}\| \to 0$. Therefore (3) gives $\|a\| \leqq K\|\bar{L}_a\|$. From this and Lemma 3.1 and 3.2 we see that sp (a) is real and $2\nu_r(a) \geqq \min(1, K^{-1})\|a\|_r$. Then (4) follows from [23, Corollary 1].

Assume (4). The embedding mapping of A_r (in its Rieffel norm) into $C^*(A_r)$ is bicontinuous by [14, Theorem 4.8.5]. Therefore $A_r = C^*(A_r)$.

Note that (5) implies (4) by the uniqueness of norm property for B^*-algebras. From (4) and (5) there is some $M > 0$ such that $\|x\| + \|\bar{L}_x\| \leqq M\|\bar{L}_x\|$ for all $x \in A_r$. This gives (2).

Now suppose (2) and $a \in A_b$. By [15, Proposition 1.17] there exists a sequence $\{a_n\}$ in A such that $\|a_n - a\| \to 0$ and $\|\bar{L}_{a_n}\| \leqq \|\bar{L}_a\|$ for $n = 1, 2, \ldots$ Then (2) allows us to conclude that $\|a\| \leqq K\|\bar{L}_a\|$ and so the inequality of (2) persists on A_b. The above argument that (2) implies (5), now applied to A_b, gives (6). From (6) we deduce (7) by [14, Theorem 4.8.5] again.

Next suppose that (7) holds. Arguments above give (2) valid on A_b for some $K > 0$. For $p \in E$ we then have $\|p\| \leqq K\|\bar{L}_p\| \leqq K$. This completes the proof.

For the notion of modular annihilator algebra see [21].

THEOREM 4.3. *Let A be a replete Hilbert algebra which is projection bounded from above. Then $C^*(A)$ is dual if and only if A is a modular annihilator algebra. In that case A is orthocomplemented and has the same closed right and left ideals as $C^*(A)$.*

Proof. Suppose that $C^*(A)$ is dual. Then the socle S is dense in $C^*(A)$ by [14, Corollary 2.8.16]. By Theorem 4.2, $A = C^*(A)$ and the identity mapping of $C^*(A)$ onto A is continuous. Therefore S is dense in A. It follows from Theorem 3.7 that A is dual. Since $I = \mathbf{R}\,\mathbf{L}(I)$ for closed right ideals in A (or $C^*(A)$) the continuity of multiplication shows that a closed right ideal in either topology is closed in the other topology. Therefore A is clearly a modular annihilator algebra. For a closed right ideal K we have, as $C^*(A)$ is a dual B^*-algebra, the rule $C^*(A) = K \oplus \mathbf{L}(K)^*$ via [2, Theorem 3]. However (see § 1) $\mathbf{L}(K)^* = K^\perp$. Thus A is orthocomplemented.

Now suppose that A is a modular annihilator algebra. This notion is purely algebraic so that $C^*(A)$ is a modular annihilator B^*-algebra. But then $C^*(A)$ must be a dual algebra by [21, Theorem 4.1].

Example 4.4. Theorem 4.3 shows how to find orthocomplemented Hilbert algebras which are not full yet are replete. Let A be the set of all sequences $c = \{c_n\}$, $n = 1, 2, \ldots,$ of complex numbers such that

$$\sum_{k=1}^{\infty} |c_k|^2/k^2 < \infty \quad \text{and} \quad \lim c_k = 0$$

made into an algebra by defining the algebraic operations componentwise. If $c, d \in A$ we set $c^* = \{\bar{c}_n\}$ and

$$(c, d) = \sum_{k=1}^{\infty} (c_k \bar{d}_k)/k^2$$

A is a commutative Hilbert algebra. Simple computations show that $A = A_r$ but that A_b is the larger algebra of all sequences $\{a_n\}$ such that

$$\sum_{k=1}^{\infty} |a_k|^2/k^2 < \infty \quad \text{and} \quad \sup |a_k| < \infty.$$

Thus A is replete but not full. As any projection in A_b must be a sequence $\{a_n\}$ where each $a_n = 0$ or $a_n = 1$, A is projection bounded from above. If M is modular maximal ideal in A it is the null space of a multiplicative linear functional G since A is a Banach algebra in the Rieffel norm. Standard arguments show that there is an integer n_0 such that $G(c) = c_{n_0}$ for each $c \in A$. Therefore A is a modular annihilator algebra. Theorem 4.3 shows that A is orthocomplemented.

By a *trivial renorming* of a Hilbert algebra A is meant the introduction of a new inner product $(x, y)_1$ where, for some $c > 0$, $(x, y)_1 = c(x, y)$ for all $x, y \in A$. If A is equivalent, as a normed linear space, to a normed algebra then, as is easily seen, there is a trivial renorming in terms of which it is a normed algebra.

It is convenient to have at hand a slightly expanded version of [15, Theorem 1.16].

LEMMA 4.5. *The following statements about the Hilbert algebra A are equivalent.*
(a) *xy is a continuous function of the two variables x and y simultaneously.*
(b) *The mapping $x \to \bar{L}_x$ is a bounded linear mapping of A into $B(H)$.*
(c) *There exists $K > 0$ such that $\|xy\| \leqq K\|x\|\|y\|$ for all $x, y \in A$.*
(d) *The Rieffel norm is equivalent to the given norm on A.*
(e) *$A_b = H$.*
(f) *A_b is trivially renormable to be an H^*-algebra.*

This is an immediate consequence of [15, § 1].

The involution in the Hilbert algebra A is proper ($xx^* = 0$ implies $x = 0$) by [3, p. 331]. By a lemma of Rickart [14, Lemma 4.10.1], every minimal one-sided ideal in A is generated by a projection (we call such a projection a *minimal projection* of A). For a minimal projection p, $pAp = \{cp: c \text{ complex}\}$ (see § 3). Completeness in the Rieffel norm is not required. Thus the arguments of Rieffel [15, Lemma 5.4] show that

$$\|\bar{L}_{ap}\| = \|ap\|/\|p\| \quad \|\bar{R}_{pa}\| = \|pa\|/\|p\| \tag{4.1}$$

for each $a \in A$.

THEOREM 4.6. *Let A be a topologically simple Hilbert algebra with a minimal idempotent. Then A_b is trivially renormable to be an H^*-algebra.*

Proof. As just noted, A has a minimal projection p. Let W be the (two-sided) ideal of A which is algebraically generated by p. Then, in the language of [15], W is contained in the minimal bi-invariant subspace I of $H(A)$ containing p. By [15, Theorem 5.14], we get $I \subset A_b$. But as W is dense in A we see that $I = A_b$. Another appeal to [15, Theorem 5.14] gives the desired result.

As a consequence of Theorem 4.6 we see that any full topologically simple Hilbert algebra with a minimal idempotent is automatically complete and in fact equivalent to an H^*-algebra. For the question of when a Hilbert algebra is equivalent to an H^*-algebra see Theorem 4.9.

For the notion of direct topological sum see [14, p. 46].

COROLLARY 4.7. *Let A be a Hilbert algebra with dense socle. Then A is the direct topological sum of its minimal closed ideals each of which is trivially renormable to be a dense $*$-subalgebra of an H^*-algebra.*

Proof. By [21, Lemma 3.11], A is the direct topological sum of its minimal closed ideals. Arguments of [7, p. 65] show readily that each minimal closed ideal N is topologically simple. Let S be the socle of A. The socle of N is SN ([21, Lemma 3.10]) which is dense in N. Moreover N is a Hilbert algebra by Theorem 2.7. We apply Theorem 4.6 to complete the proof.

We can give a more detailed analysis for full Hilbert algebras.

COROLLARY 4.8. *Let A be a full Hilbert algebra. Then A has dense socle if and only if the completion H of A is the Hilbert space direct sum of the minimal closed ideals of A each of which is trivially renormable to be a topologically simple H^*-algebra.*

Proof. Suppose that the socle S is dense. Then A is the direct topological sum of its minimal closed ideals by Corollary 4.7. If $N_1 \neq N_2$ are two distinct minimal closed ideals then, $N_1N_2 = N_2N_1 = (0)$ and each $N_j^* = N_j$, we get $(N_1, N_2) = (0)$. Moreover each minimal closed ideal is complete and of the required form by Theorem 4.6. We then get the Hilbert space decomposition of H.

The converse follows readily from the facts that the socle of each minimal closed ideal N is dense in N and is included in the socle of the algebra A.

THEOREM 4.9. *A full Hilbert algebra A is equivalent to an H^*-algebra if and only if A is projection bounded from below.*

Proof. The forward implication is trivial. Assume that A is full and that $\|p\| \geq c > 0$ for all non-zero projections in A. We show next that a non-zero right ideal I must contain a minimal projection. First of all, by [15, Theorem 2.3], I contains a non-zero projection p_1. If p_1 is not a minimal projection then there is a right ideal J, $(0) \neq J \subset p_1A$, $J \neq p_1A$. Now J contains a non-zero projection q. Since $p_1q = q$, $(p_1 - q, q) = 0$. Also $p_1 - q$ is a projection $\neq 0$ and $\|p_1\|^2 = \|q\|^2 + \|p_1 - q\|^2$. Let p_2 be one of q, $p_1 - q$ with $\|p_2\| \leq \|p_1\|/2^{1/2}$. If p_2 is not a minimal projection we can find a non-zero projection p_3 in I with $\|p_3\| \leq \|p_2\|/2^{1/2}$. If we continue this process for n stages without reaching a minimal projection, $\|p_n\| \leq \|p_1\|/2^{(n-1)/2}$. Hence this process cannot continue indefinitely.

It follows from this that $L(S) = (0)$ where S is the socle of A. Inasmuch as A is an annihilator algebra by Theorem 2.5, we see that S is dense in A. Now Corollary 4.8 applies to show A is the direct topological sum of its minimal closed ideals each equivalent to an H^*-algebra. A minimal closed ideal N is full by Theorem 2.7. Theorem 4.6 and [15, Theorem 5.14] show that $\|xy\| \leq c^{-1}\|x\|\|y\|$ for all $x, y \in N$.

Let H be the completion of A, $\{N_\alpha\}$ be the set of all minimal closed ideals in A and let P_α be the orthogonal projection of H onto its closed subspace N_α. Arguing as in the proof of Corollary 4.8 we see that each $a \in A$ is the Hilbert space sum $a = \sum P_\alpha(a)$. If also $b \in A$,

$$ab = \sum P_\alpha(a) \sum P_\alpha(b) = \sum P_\alpha(a)P_\alpha(b) = \sum P_\alpha(ab)$$

Therefore $P_\alpha(ab) = P_\alpha(a)P_\alpha(b)$. Then, for $x, y \in A$

$$\|xy\|^2 = \sum \|P_\alpha(x)P_\alpha(y)\|^2 \leqq c^{-2}\sum \|P_\alpha(x)\|^2\|P_\alpha(y)\|^2 \leqq c^{-2}\sum \|P_\alpha(x)\|^2 \sum \|P_\alpha(y)\|^2 = c^{-2}\|x\|^2\|y\|^2.$$

An application of Lemma 4.5 shows that A is equivalent to an H^*-algebra.

We now relate the socle S of A to the intersection D_r (D_l) of the closed modular maximal right (left) ideals.

PROPOSITION 4.10. *For a dual Hilbert algebra* A *we have* $D_r = D_l = S^{\perp}$.

Proof. Let I be any minimal right ideal. As noted above $I = eA$ where e is a projection. A simple computation shows that $(1 - e)A \subset I^{\perp}$. Since $A = eA \oplus (1 - e)A$ it follows that $I^{\perp} = (1 - e)A$. If M is a closed modular maximal right ideal, $M^{\perp} \neq (0)$ by Theorem 2.1. But $M^{\perp}$ must be a minimal right ideal so as $A = M \oplus M^{\perp}$ we see that $M = (M^{\perp})^{\perp}$. Consequently every closed modular maximal right ideal has the form $(1 - e)A$ for a minimal projection A. We see that $x \in S^{\perp}$ if and only if x lies in every closed modular maximal right ideal. Likewise $S^{\perp} = D_l$.

Consequently, for the dual Hilbert algebra A, S is dense if and only if $D_r = D_l = (0)$.

It seems to us unlikely that Proposition 4.10 is valid for all Hilbert algebras but we have no example at hand. In any case, $D_r = D_l$.

PROPOSITION 4.11. *For any Hilbert algebra* A *we have* $D_r = D_l$.

Proof. By [12, Lemma 9.1], D_r is a left as well as a right ideal. From Theorem 2.7 we have $D_r = D_r^*$. But obviously $D_r^* = D_l$.

5. Homomorphisms and Hilbert algebras

We tie up the theory of Hilbert algebras with that of Banach algebras through a study of homomorphisms.

PROPOSITION 5.1. *Let* T *be a homomorphism of a Banach algebra* B *onto a Hilbert algebra* A. *Then* T *is continuous whether the given or the Rieffel norm is used for* A.

Proof. We can consider A as a *-subalgebra of the Hilbert algebra A_b. Inasmuch as A_b is complete in the Rieffel norm [15, p. 270] it is an A^*-algebra [14, p. 181]. By [14, Theorem 4.1.20] there exists $K > 0$ such that

$$\|T(x)\| \leqq \|T(x)\|_r \leqq K\|x\|, \quad x \in B.$$

Easy examples show that a homomorphism T of a Hilbert algebra onto a Banach algebra can be discontinuous. If T is continuous it has some strong properties.

THEOREM 5.2. *Let T be a continuous homomorphism of a Hilbert algebra A onto a Banach algebra B. Then T is an open mapping and B is equivalent to an H^*-algebra.*

Proof. Let π denote the natural homomorphism of A onto $A/T^{-1}(0)$ which is a Hilbert algebra in the quotient algebra norm by Theorem 2.9. We have a naturally defined isomorphism T_1 of $A/T^{-1}(0)$ onto B. Since T_1^{-1} is continuous by Proposition 5.1, and π is an open mapping we see readily that T is an open mapping.

Consider $A \subset A_b \subset H = H(A)$. T extends to a bounded linear mapping T' of H onto B. We show that T' restricted to A_b is a homomorphism. Let $f, g \in A_b$, $f = \lim x_n$, $g = \lim y_n$ where each x_n, y_n lies in A. Then

$$T'(fy_k) = \lim_n T(x_n y_k) = T'(f)T(y_k).$$

But clearly $fy_k \rightarrow fg$. Then $T'(fg) = T'(f)T'(g)$. Now we restrict the domain of definition of T' to A_b. By Theorem 2.9, A_b/K is a full Hilbert algebra in the quotient algebra norm where K is the Kernel of T'. We have a naturally defined continuous isomorphism T_1' of A_b/K onto B which is bi-continuous as it is open. Let p be a non-zero projection on A_b/K. Then $T_1'(p)$ is a non-zero idempotent in B. Therefore

$$1 \leqq \|T_1'(p)\| \leqq \|T_1'\|\|p\|.$$

Then A_b/K is projection bounded from below. Theorem 4.9 shows that A_b/K is equivalent to an H^*-algebra and, therefore, so is B.

Consider a homomorphism T of a Hilbert algebra A into a Hilbert algebra A_1. We may think of A (A_1) as embedded in $C^*(A)$ $(C^*(A_1))$ and ask about extending T. A homomorphism σ of $C^*(A)$ into $C^*(A_1)$ is said to *extend* T if $\bar{L}_{T(x)} = \sigma(\bar{L}_x)$ for all $x \in A$. Under certain conditions we can obtain such an extension.

THEOREM 5.3. *Suppose there is an *-homomorphism T of a replete Hilbert algebra A onto a Hilbert algebra A_1. Then there is a continuous *-homomorphism of $C^*(A)$ onto $C^*(A_1)$ which extends T.*

Proof. We let $|x|_r$, $|\bar{L}_x|$ and $\gamma(x)$ be the Rieffel norm on A_1, the operator norm on $H(A_1)$ and $\lim |x^n|_r^{1/n}$ respectively. We can use Proposition 5.1 to see that, for some $M > 0$, we have $|T(x)|_r \leq M\|x\|_r$ for all $x \in A$. By replacing x by x^n we are led to the rule $\gamma(T(x)) \leq \nu_r(x)$, $x \in A$. We use Lemma 3.2 to obtain

$$|\bar{L}_{T(x)}|^2 = |\bar{L}_{T(x^*x)}| = \gamma(T(x^*x)) \leq \nu_r(x^*x) = \|\bar{L}_x\|^2 \tag{5.1}$$

for all $x \in A$. The desired homomorphism σ when restricted to A is given by

$$\sigma(\bar{L}_x) = \bar{L}_{T(x)}$$

and is, by (5.1), a continuous *-isomorphism there in the C^*-algebra norms. It extends to a *-homomorphism σ of $C^*(A)$ into $C^*(A_1)$. The mapping σ determines a *-isomorphism τ of the C*-algebra $C^*(A)/\sigma^{-1}(0)$ into $C^*(A_1)$. By [14, Theorem 4.8.5], τ is an isometry. But its range contains A_1 which is dense in $C^*(A_1)$. Therefore the range of σ is all of $C^*(A_1)$.

6. Hilbert algebras with completely continuous multiplication

In many of the standard examples of Hilbert algebras such as Example 2.4, multiplication is completely continuous. Here a Hilbert algebra is considered as a normed linear space, in general not complete. We use as the definition of complete continuity that given for normed linear spaces, for example, in [24, p. 274]. Accordingly we say that the multiplication in a Hilbert algebra A is *completely continuous* if, given a bounded sequence $\{x_n\}$ and $y \in A$, there exists a subsequence $\{x_{n_k}\}$ and $w \in A$ such that $yx_{n_k} \to w$.

We verify this property for the Hilbert algebra $A = C(G)$ of Example 2.4. For $x \in A$, let $\|x\|$ denote the Hilbert algebra norm and $|x|$ denote the supremum norm of x. Clearly $\|x\| \leq |x|$ for all $x \in A$. Also, with convolution multiplication xy we have $|xy| \leq \|x\|\|y\|$ for all $x, y \in A$. Moreover [10] multiplication is completely continuous if the complete norm $|x|$ is used. Now let $\{x_n\}$ be a sequence where each $\|x_n\| \leq 1$. Let $a, b \in A$. Then $|bx_n| \leq \|b\|$ for each n. Then the set $\{abx_n\}$ is totally bounded in the $|x|$ norm.

Let $y \in A$. There are sequences $\{a_n\}$ and $\{b_n\}$ in A where $\|y - a_n b_n\| \to 0$. There is a subsequence $\{x_{1,m}\}$ of $\{x_n\}$ with $\{a_1 b_1 x_{1,m}\}$ a Cauchy sequence in the $|x|$ norm. Likewise there is a subsequence $\{x_{2,r}\}$ of $\{x_{1,m}\}$ with $\{a_2 b_2 x_{2,r}\}$ a Cauchy sequence in the $|x|$ norm. Continuing in this way we see that the sequence $\{x_{r,r}\}$ has the property that $\{a_n b_n x_{r,r}\}$ is Cauchy in the $|x|$ norm for each n. Take $\varepsilon > 0$ and choose m so large that $\|y - a_m b_m\| < \varepsilon/4$. Now choose N where $|a_m b_m (x_{r,r} - x_{s,s})| < \varepsilon/2$ for $r, s > N$. For such r, s we see that $|y(x_{r,r} - x_{s,s})| \leq 2\|y - a_m b_m\| < |a_m b_m (x_{r,r} - x_{s,s})| < \varepsilon$. Therefore there exists $w \in A$ with $|yx_{r,r} - w| \to 0$. Then also $\|yx_{r,r} - w\| \to 0$.

LEMMA 6.1. *If multiplication in the Hilbert algebra A is completely continuous then A is a *-ideal in A_b.*

Proof. Let $a \in A_b$ and $x_n \to a$ where each $x_n \in A$. Take any $y \in A$. There exists a subsequence $\{x_{n_k}\}$ and an element $z \in A$ with $yx_{n_k} \to z$. As $yx_{n_k} \to ya$ in A_b we see that $ya = z \in A$, that is A is a right ideal in A_b.

This simple result suggests a definition.

Definition 6.2. We call a Hilbert algebra A *almost full* if A is an ideal in A_b. Such algebras are dual by Corollary 2.2.

Consider next a dense *-subalgebra W of a full Hilbert algebra F. By [4, p. 72] W is itself a Hilbert algebra. A tedious exercise on the definition of W_b shows that $W_b = F$. This allows us to recast Definition 6.2 to say that A is almost full if (and only if) A is a *-ideal in some full Hilbert algebra B. For then the closure $\bar{A}$ of A in B is full by Theorem 2.7 and $\bar{A} = A_b$.

THEOREM 6.3. *Let A be an almost full Hilbert algebra.*

(1) *For a closed ideal K in A, K and A/K are almost full.*

(2) *Any one-sided ideal $\neq (0)$ in A contains a non-zero projection.*

Proof. From Theorem 2.7, $K = K^*$ is a Hilbert algebra. Let $x \in K$. There exists a sequence $\{y_n\}$ in K with $xy_n \to x$. Now let $z \in A_b$. We show $xz \in K$. For $xz = \lim x(y_n z)$ and each $y_n z \in A$ as A is almost full. Then $x(y_n z) \in K$. This argument shows that K is a *-ideal in the full Hilbert algebra A_b. Next let K^c denote the closure of K in A_b. As noted in the proof of Theorem 2.9, there is an algebraic *-isomorphism σ of A/K into A_b/K^c which preserves inner products. Thus A/K is identifiable as a Hilbert algebra with its image in A_b/K^c. Let $x + K^c$, $x \in A$, be a typical element in that image and let $w + K^c$, $w \in A_b$, be any element of A_b/K^c. Then $(x + K^c)(w + K^c) = xw + K^c$ lies in $\sigma(A/K)$ as $xw \in A$. Thus $\sigma(A/K)$ is almost full being a *-ideal of the full Hilbert algebra A_b/K^c (see Theorem 2.9).

We turn to (2). Let F be a full Hilbert algebra. Arguments of Rieffel [15, Theorem 2.3] show that if $x \neq 0$ in F there exists $y \in F$ with yx a non-zero projection in F. Consider a non-zero left ideal J in A. We can take $w \neq 0$ in J and $v \in A_b$ such that vw is a non-zero projection in A_b. As $vwv \in A$ we see that $vw = (vwv)w$ lies in J.

THEOREM 6.4. *If the multiplication in the Hilbert algebra A is completely continuous then $H = H(A)$ is the Hilbert space direct sum of the minimal closed ideals of A each of which is, for some positive integer n, the algebra of all $n \times n$ matrices over the complex field.*

Proof. It is to be understood that the integer n can vary from one minimal closed ideal to another.

For each idempotent e, eA and Ae are finite-dimensional by the Riesz-Schauder theory. Let $J \neq (0)$ be a left ideal in A. Lemma 6.1 and Theorem 6.3 show that there is a non-zero projection p_1 in J. If p_1 is not a minimal projection there is a left ideal I, $(0) \neq I \subset Ap_1, I \neq Ap_1$. In that case there is a non-zero projection p_2 in I where the dimension of Ap_2 is smaller than that of Ap_1. If

p_2 is not a minimal projection we can continue this process. This process must terminate at some stage with a minimal projection in J.

Let S denote the socle of A. If $S^{\perp} \neq (0)$ then $S^{\perp}$ would contain a minimal projection which is clearly impossible. As A is dual, by Theorem 2.1 we see that S is dense in A. Corollary 4.7 gives A as the direct topological sum of its minimal closed ideals. Let N be such an ideal and e be a minimal projection in N. Clearly AeA is finite-dimensional so that $AeA = N$. Moreover N is, say by Theorem 4.6, an H^*-algebra and so is the algebra of all $n \times n$ matrices over the complex field for some N. As each such N is complete we argue as in the proof of Corollary 4.8 to obtain the H as the Hilbert space direct sum of these ideals N.

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Received August 21, 1972

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