On colimits of injectives in Grothendieck categories

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1. Colimits of injectives

A *Grothendieck category* is an Abelian category $\mathscr{A}$ which has a set of generators and admits colimits that are exact functors when taken over directed sets. Any Grotendieck category has arbitrary products and sufficiently many injective objects (see [1]).

An object D of $\mathscr{A}$ is said to be a *strict cogenerator* if every object of $\mathscr{A}$ admits an embedding in a suitable coproduct of copies of D.

In [6] (see also [2] and [5]) J. E. Roos has proved that *the following conditions are equivalent for a Grothendieck category* $\mathscr{A}$

(i) *There exists an injective object* I *of* $\mathscr{A}$ *such that every injective in* $\mathscr{A}$ *is a coproduct of direct summands of* I.

(ii) *Every coproduct of injectives in* $\mathscr{A}$ *is still injective.*

(iii) $\mathscr{A}$ *admits a strict cogenerator.*

The aim of this paper is to prove that in a Grothendieck category $\mathscr{A}$ each of the above equivalent conditions *(i)—(iii) implies that a directed colimit of injectives* is injective. This answers a question of J. E. Roos [6, p. 202].

Throughout this paper colim is a colimit functor, $\coprod$ and $\prod$ denote coproduct and product respectively.

We start with the following

LEMMA. *Let* $\{M_\alpha, h_{\beta\alpha}\}_{\alpha \le \beta < \gamma}$ *be a well-ordered directed system in a Grothendieck category* $\mathscr{A}$ *such that* $M_0 = 0$, $h_{\beta\alpha}$ *are monomorphisms and* $M_\eta = \operatorname{colim}_{\xi<\eta} M_\xi$ *whenever* η *is a limit ordinal number. If* $h_{\xi+1,\xi}\colon M_\xi \to M_{\xi+1}$ *splits for any* $\xi < \gamma$ *then*

$$\operatorname{colim}_{\alpha<\gamma} M_\alpha \cong \coprod_{\xi<\gamma} M_{\xi+1}/M_\xi .$$

Proof. Let $B_\alpha = \coprod_{\xi<\alpha} M_{\xi+1}/M_\xi$ and let $s_{\beta\alpha}: B_\alpha \to B_\beta$, $\alpha \le \beta < \gamma$, be a natural monomorphism. For any $\alpha < \gamma$ we fix such splitting maps d_α and t_α of $h_{\alpha+1,\alpha}$ and $v_\alpha: M_{\alpha+1} \to M_{\alpha+1}/M_\alpha$ respectively, that the identity map on $M_{\alpha+1}$ is equal to $h_{\alpha+1,\alpha} d_\alpha + t_\alpha v_\alpha$.

Let $f_\beta: B_\beta \to M_\beta$ be a map defined by commutative diagrams

$$\begin{array}{ccc} B_\beta & \xrightarrow{f_\beta} & M_\beta \\ \uparrow & & \uparrow{\scriptstyle h_{\beta,\xi+1}} \\ M_{\xi+1}/M_\xi & \xrightarrow{t_\xi} & M_{\xi+1}, \quad \xi < \beta. \end{array}$$

It is easy to verify that $\{f_\beta\}: \{B_\beta, s_{\alpha\beta}\} \to \{M_\beta, h_{\alpha\beta}\}$ is a map of directed systems and that the following diagram commutes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B_\alpha & \underset{s_{\alpha+1,\alpha}}{\overset{j_\alpha}{\leftrightarrows}} & B_{\alpha+1} & \overset{p_\alpha}{\leftrightarrows} & M_{\alpha+1}/M_\alpha & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f_\alpha} & & \downarrow{\scriptstyle f_{\alpha+1}} & & \downarrow{\scriptstyle id} & & \\ 0 & \longrightarrow & M_\alpha & \underset{h_{\alpha+1}}{\overset{d_\alpha}{\leftrightarrows}} & M_{\alpha+1} & \overset{t_\alpha}{\leftrightarrows} & M_{\alpha+1}/M_\alpha & \longrightarrow & 0 \end{array}$$

where j_α is a natural projection and p_α is a natural injection. It follows that $f_{\alpha+1}$ is an isomorphism whenever so is f_α. Clearly f_0 is an isomorphism. If η is a limit ordinal and f_ξ are isomorphisms for $\xi < \eta$ then so is

$$f_\eta = \operatorname{colim}_{\xi<\eta} f_\xi: B_\eta \cong \operatorname{colim}_{\xi<\eta} B_\xi \to M_\eta.$$

Consequently, $\operatorname{colim}_{\alpha<\gamma} f_\alpha$ is an isomorphism and the lemma is proved.

THEOREM 1. *If a Grothendieck category $\mathscr{A}$ admits a strict cogenerator then any directed colimit of injectives in $\mathscr{A}$ is injective.*

Proof. Consider an exact sequence

$$0 \longrightarrow K \longrightarrow \coprod_{i\in I} M_i \xrightarrow{h} \operatorname{colim}_{i\in I} M_i \longrightarrow 0 \qquad (*)$$

where M_i, $i \in I$, is a direct system of injective objects in $\mathscr{A}$ and h is the natural colimit morphism. We shall prove by transfinite induction on the cardinality $|I|$ of I that any sequence of the form (*) splits. Clearly it is true if I is finite. Now suppose that I is arbitrary and that our statement holds for any directed set of cardinality less than $|I|$. It follows from [4, Lemma 1.4] that I is a union of an ascending transfinite sequence of directed subsets I_ξ such that $|I_\xi| < |I|$ for any ξ and $I_\eta = \bigcup_{\xi<\eta} I_\xi$ whenever η is a limit ordinal number. Then for any ξ we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_\xi & \longrightarrow & \coprod_{i\in I_\xi} M_i & \longrightarrow & \operatorname{colim}_{i\in I_\xi} M_i & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & K_{\xi+1} & \longrightarrow & \coprod_{i\in I_{\xi+1}} M_i & \longrightarrow & \operatorname{colim}_{i\in I_{\xi+1}} M_i & \longrightarrow & 0 \end{array} \qquad (**)$$

with splitting rows by the inductive assumption. It follows that $K_\xi \hookrightarrow K_{\xi+1}$ splits for any ξ and therefore $K_{\xi+1}/K_\xi$ is injective. Since the exact sequence

$$0 \longrightarrow \operatorname{colim} K_\xi \longrightarrow \operatorname{colim} \coprod_{i\in I_\xi} M_i \longrightarrow \operatorname{colim} \operatorname{colim}_{i\in I_\xi} M_i \longrightarrow 0$$

is isomorphic to (*) then by Lemma $K \simeq \coprod K_{\xi+1}/K_\xi$. Hence K is injective and our conclusion follows.

COROLLARY. *Let $\mathcal{A}$ be a Grothendieck category with a strict cogenerator. Then*

$$\text{inj dim colim } M_i \le \sup \text{inj dim } M_i$$

for any directed system $M_i, i \in I$, in $\mathcal{A}$.

Proof. It follows from our assumption that $\mathcal{A}$ has an injective cogenerator E. For each object M of $\mathcal{A}$ we set $Q(M) = \prod E_f$ where $E_f = E$ and f runs through all morphisms from M to E. It is clear that the natural morphism $M \to Q(M)$ is a monomorphism and any morphism $h: M \to N$ induces a natural morphism $Q(h): Q(M) \to Q(N)$ such that Q becomes a covariant endofunctor of $\mathcal{A}$.

We shall prove the corollary by induction on $d = \sup \text{inj dim } M_i$, since there is nothing to prove provided $d = \infty$. In virtue of Theorem 1 it is sufficient to pass from $d - 1$ to d. For this purpose observe that from a directed system of exact sequences

$$0 \longrightarrow M_i \longrightarrow Q(M_i) \longrightarrow K_i \longrightarrow 0$$

we derive the exact sequence

$$0 \longrightarrow \operatorname{colim} M_i \longrightarrow \operatorname{colim} Q(M_i) \longrightarrow \operatorname{colim} K_i \longrightarrow 0$$

where the middle object is injective by Theorem 1 and the last one has injective dimension less then d by the inductive assumption. Hence the corollary follows.

2. Pure-injectives and pure-projectives

Let $\mathcal{A}$ be a Grothendieck category. Recall that an object M of $\mathcal{A}$ is *finitely generated* if for each directed family M_i, $i \in I$, of subobjects of M with $M = \bigcup_{i\in I} M_i$ there is an $j \in I$ with $M_j = M$. M is *finitely presented* if it is finitely generated and every epimorphism $N \to M$, where N is finitely generated, has a finitely generated kernel. $\mathcal{A}$ is said to be a *locally finitely presented* category if it has a family of finitely presented generators. A short exact sequence in $\mathcal{A}$ is *pure* if every finitely presented object is relatively projective for it.

By [8], pure sequences in a locally finitely presented category form a proper class which is closed under directed colimits. Moreover, in this case there exist enough pure-projective objects.

The following result is an extension of the Theorem 4.2 in [3] from module categories to locally finitely presented ones.

THEOREM 2. *Let $\mathcal{A}$ be a locally finitely presented category. The following statements are equivalent:*

(a) *All objects of $\mathcal{A}$ are pure-projective.*

(b) *Every pure-projective object of $\mathcal{A}$ is pure-injective.*

Moreover, if $\mathcal{A}$ has enough pure-injectives then (a) *is equivalent to the following statement*

(c) *Every pure-injective object in $\mathcal{A}$ is pure-projective.*

Proof. The implications (a) $\Rightarrow$ (b) and (a) $\Rightarrow$ (c) are trivial, and (c) $\Rightarrow$ (a) may be proved as the one in [3, Theorem 4.2].

To prove (b) $\Rightarrow$ (a) observe that every sequence of the form (*) is pure. It is trivial if the set I is finite. The general case follows by transfinite induction on the cardinality $|I|$ using the fact that (*) is a directed colimit of sequences of the form (**). Assume (b). By [8, Lemma 4] every object is a directed colimit of finitely presented (so pure-injective) objects. Then it is sufficient to show that every sequence of the form (*), with M_i pure-injective, splits. But this follows from the above remarks and the arguments from the proof of Theorem 1. This completes the proof.

After not hard modifications of arguments from [3, Sec. 5] and [7, Sec. 1] one obtains

THEOREM 3. *Let $\mathcal{A}$ be a locally finitely presented category. If*

$$0 \longrightarrow K \longrightarrow P_n \longrightarrow \dots \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

is a pure exact sequence in $\mathcal{A}$ and $P_0, \dots, P_n$ are pure-projective objects then K is an $\aleph_n$-directed union of pure-projective subobjects (which are $\aleph_{n-1}G$-pure subobjects of K in the sense of [3, Sec. 5]*).*

Note Added in Proof. Theorem 1 and Corollary have been proved by Jan-Erik Roos in the paper »On the structure of abelian categories with generators and exact direct limits» (to appear).

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