A locally convex space which is not an ω-space

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The purpose of this paper is to exhibit an example of a locally convex topological vector space which is not an ω-space.

The class of locally convex spaces for which uniform holomorphy holds was first introduced by Nachbin [3]. It is extended by the class of ω-spaces whose definition was first published by Dineen [1]. Dineen has since then given a new definition, which covers more general situations [2]. In this paper we use the term "ω-spaces" for spaces which in Dineen's new terminology are called "$\mathbf{C}$-ω-spaces".

The notion of an ω-space is of a certain importance in infinite dimensional holomorphy, as Dineen shows in [1] and [2].

We have recently become aware that M. Schottenloher has also constructed a space which is not an ω-space. His example is a subspace of the space of holomorphic mappings from a certain Banach space into itself.

Definition. A locally convex space E is an *ω-space* if for every function $f\colon U \to \mathbf{C}$, where $U \subset E$ is open and connected, and f is Gâteaux holomorphic and continuous, there exists a sequence $\{p_i\}_{i=1}^{\infty}$ of continuous seminorms such that f is also continuous in the topology generated by the sequence $\{p_i\}_{i=1}^{\infty}$.

Let $E = \oplus_{r \in \mathbf{R}_+} \mathbf{C}$ algebraically, i.e.

$$E = \{(x_r)_{r \in \mathbf{R}_+};\ x_r \neq 0 \ \text{ only for a finite number of coordinates}\}.$$

If $A \subset \mathbf{R}_+$ we let $(x_r)_{r \in A}$ denote the point $(y_r)_{r \in \mathbf{R}_+}$ where $y_r = x_r$ if $r \in A$ and $y_r = 0$ otherwise.

Define a topology on E by the set of seminorms

$$\{p\} \cup \{p_{sk};\ s \in \mathbf{R}_+ \setminus \mathbf{Z}_+,\ k \in \mathbf{Z}_+\},$$

where

$$p(x) = \sum_{r \in \mathbf{R}_+} |x_r| \text{ if } x = (x_r)_{r \in \mathbf{R}_+},$$

and

$$p_{sk}(x) = \sum_{j=1}^{\infty} |x_j| \cdot k^{n_{sj}}.$$

Here $\{(n_{sj})_{j=1}^{\infty}; s \in \mathbf{R}_+ \setminus \mathbf{Z}_+\}$ is the set of all strictly increasing sequences of numbers belonging to $\mathbf{Z}_+$, where that set has been put in a one-to-one correspondence with $\mathbf{R}_+ \setminus \mathbf{Z}_+$.

PROPOSITION. *Define a function* $f: E \to \mathbf{C}$ *by*

$$f(x) = f((x_r)_{r \in \mathbf{R}_+}) = \sum_{s \in \mathbf{R}_+ \setminus \mathbf{Z}_+} x_s \cdot f_s(x),$$

where

$$f_s(x) = f_s((x_r)) = \sum_{j=1}^{\infty} x_j \cdot x_s^{n_{sj}}.$$

Then f is Gâteaux-holomorphic and continuous, but f is not continuous in any semi-metrizable locally convex topology on E, weaker than the given one.

Proof. a) *f is continuous.* The functions $x = (x_r)_{r \in \mathbf{R}_+} \mapsto x_{r_0}$ are continuous for every $r_0 \in \mathbf{R}_+$, for they are linear and $p(x) < \varepsilon$ implies $|x_{r_0}| < \varepsilon$.

For $s \in \mathbf{R}_+ \setminus \mathbf{Z}_+$ the seminorms p_{sk}, $k \in \mathbf{Z}_+$, are chosen so that f_s is continuous:

Let $x^0 = (x_r^0)$ be given and choose $m > \max\{r; x_r^0 \neq 0\}$. Then

$$f_s(x) = \sum_{j=1}^{\infty} x_j \cdot x_s^{n_{sj}} = \sum_{j=1}^{m} x_j \cdot x_s^{n_{sj}} + \sum_{j=m+1}^{\infty} x_j \cdot x_s^{n_{sj}} = f_{s1}(x) + f_{s2}(x),$$

for any $x \in E$, and f_{s1} is clearly continuous in the topology defined by the seminorm p, i.e. $|f_{s1}(x^0 + y) - f_{s1}(x^0)| < \varepsilon$ if $p(y)$ is small enough. Now

$$|f_{s2}(x^0 + y) - f_{s2}(x^0)| = |f_{s2}(x^0 + y)| = \left|\sum_{j=m+1}^{\infty} y_j (x_s^0 + y_s)^{n_{sj}}\right| < p_{sk}(y)$$

if $p(y) < 1$ and $k > |x_s^0| + 1$.

Thus f_{s1} and f_{s2} are both continuous in x^0 which implies that $f_s = f_{s1} + f_{s2}$ is continuous there. But x^0 was arbitrary, and f_s is then continuous everywhere.

Let $x^0 = (x_r^0)_{r \in \mathbf{R}_+}$ still denote an arbitrary element of E. Then we have

$$x^0 = (x_r^0)_{r \in \mathbf{R}_+} = (x_r^0)_{r \in A} + (x_j^0)_{j \in B},$$

where $A \subset \mathbf{R}_+ \setminus \mathbf{Z}_+$ and $B \subset \mathbf{Z}_+$ and both are finite.

$$f(x) = \sum_{r \in \mathbf{R}_+ \setminus \mathbf{Z}_+} x_r f_r(x) = \sum_{r \in A} x_r \cdot f_r(x) + \sum_{r \in M} x_r f_r(x) = f_1(x) + f_2(x)$$

for any $x \in E$, if $M = (\mathbf{R}_+ \setminus \mathbf{Z}_+) \setminus A$.

Now, f_1 is a finite sum of finite products of continuous functions and thus continuous. For $t \in M$ and $y \in U$, where

$$U = \{y; p(y) < [1 + \sum_{=1}^{\infty} |x_j^0|]^{-1}\} \cap \{y; p(y) < \tfrac{1}{2}\},$$

we have

$$|f_t(x^0 + y)| = |\sum_{j=1}^{\infty} (x_j^0 + y_j) y_t^{n_{tj}}| \le \sum_{j=1}^{\infty} |(x_j^0 + y_j) \cdot y_t| \cdot |y_t|^{n_{tj}-1} \le$$

$$\le \sum_{j=1}^{\infty} |y_t|^{n_{tj}-1} \le \sum_{j=0}^{\infty} |y_t|^j \le \sum_{j=0}^{\infty} \frac{1}{2^j} = 2.$$

This gives

$$|f_2(x^0 + y) - f_2(x^0)| = |f_2(x^0 + y)| = |\sum_{t \in M} y_t \cdot f_t(x^0 + y)| \le 2 \cdot \sum_{t \in M} |y_t| \le 2 \cdot p(y).$$

Thus, f_2 is continuous in x^0. Then $f = f_1 + f_2$ is continuous in x^0, but x^0 was arbitrary and therefore f is continuous in all of E.

b) *f is not continuous in any semi-metrizable locally convex topology weaker than the given one.*

Let P be a denumerable subset of the set of seminorms defining the topology on E. Then there is a number $t \in \mathbf{R}_+ \setminus \mathbf{Z}_+$ such that

$$\frac{n_{tj} - j}{n_{sj}} \to \infty \quad \text{when} \quad j \to \infty$$

for every s corresponding to a seminorm in P.

Every zero-neighbourhood in the topology defined by P contains a set of the form:

$$U = \{x; p_{sk}(x) < \varepsilon, p_{sk} \in P_1\} \cap \{x; p(x) < \varepsilon\},$$

where P_1 is a finite subset of P.

If $q(x) < 1$ and $x = (x_j)_{j \in \mathbf{Z}_+}$ then $x \in U$, provided

$$q(x) = \frac{1}{\varepsilon} \cdot \sum_{j=1}^{\infty} |x_j| \cdot h^{n_j}$$

where

$$h = \max \{j; p_{sj} \in P_1 \text{ for some } s \in \mathbf{R}_+ \setminus \mathbf{Z}_+\}$$

and

$$n_j = \max \{n_{sj}; p_{sk} \in P_1 \text{ for some } k \in \mathbf{Z}_+\}.$$

Let

$$x^m = \varepsilon \cdot \left(\frac{1}{2^{j+1}} \cdot \frac{1}{h^{n_j}}\right)_{j=1}^{m}, \quad m = 1, 2, \ldots$$

and $x^0 = (x_s^0)_{s \in \mathbf{R}_+}$, $x_s^0 = 0$ if $s \neq t$, $x_t^0 = 2$, where t is the number mentioned above (thus $(n_{tj} - j)/n_j \to \infty$ when $j \to \infty$). Then $q(x^m) < 1$, $m = 1, 2, \ldots$ and

$$f(x^0 + x^m) = x_t^0 \cdot f_t(x^0 + x^m) = 2 \cdot \sum_{j=1}^{m} \frac{\varepsilon}{2^{j+1}} \cdot \frac{2^{n_{tj}}}{h^{n_j}} = \varepsilon \cdot \sum_{j=1}^{m} \frac{2^{n_{tj}-j}}{h^{n_j}} \, .$$

If m is chosen large enough, the last term above is arbitrarily large, i.e. $f(x^0 + U)$ is not bounded.

References

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