Jordan decomposition for a class of singular differential operators

A. H. M. Levelt

0. Introduction

In 1955 H. L. Turrittin published a theorem on canonical forms of certain differential operators (cf. [1], Theorem I).

We shall not describe that theorem in all detail in this introduction. However, in order to understand the main result of the present paper it is useful to know that Turrittin considers differential operators of the type

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \to \left\{ \tau^g \frac{d}{d\tau} + A(\tau) \right\} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

where $g \in N$, $x_i \in \mathbf{C}[[\tau]]$, the ring of formal power series in one variable τ, and A is a square matrix of n rows and columns and elements in $\mathbf{C}[[\tau]]$. Turrittin's statement is roughly as follows: By a convenient "coordinate transformation"

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = P(t) \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix},$$

where y_i and the elements of the matrix $P(t)$ belong to $\mathbf{C}[[t]]$, $t = \tau^{1/p}$ (p positive integer) the differential operator can be expressed as

$$\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \to \left\{ t^h \frac{d}{dt} + B(t) \right\} \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix},$$

where $B(t)$ is a matrix with elements in $\mathbf{C}[[t]]$ having an explicitly prescribed canonical form closely resembling the Jordan canonical form for ordinary linear transformations.

As is more or less known Turrittin's proof can essentially be simplified by using cyclic vectors. Such a proof can equally lead to an estimate of p: for instance $p=n!$ will suffice. This point was missing in Turrittin's theorem (c.f. [2], p. 7).

However, there is still another gap, namely a statement on uniqueness. Filling this gap is the main purpose of the present paper. Here again the well-known decomposition of a linear transformation into the sum of a semisimple transformation and a nilpotent one can be imitated. This leads to the main results the Theorems I and II of section 1.

In section 6 we give some applications. Our main theorems serve well to understand the various invariants known in the literature ([3], Theorem I, 1,9 et [4], Introduction).

We shall also deduce a better estimate for p, namely

$$\text{l. c. m } \{m \mid 1 \leqq m \leqq n\}.$$

In section 1 we develop the appropriate invariant notions concerning defferential operators, and we define and study semisimple differential operators.

Section 2 contains a well-known splitting lemma. For completeness'sake we shall give a full prof.

In section 3 the existence of a decomposition is shown over an extension of the ground field, and uniqueness is proved in a special case in section 4.

Section 5 contains the proof of the main theorems.

1. Differential operators, semisimplicity

Throughout this paper we shall use the following notations:

k: field of characteristic zero.
$\mathfrak{D}=k[[t]]$: ring of formal power series in one determinate and coefficients in k.
$K=k((t))$: field of fractions of $\mathfrak{D}$.
n: positive integer.
V: vector space of dimension n over K.
Λ: $\mathfrak{D}$-lattice in V, i.e. (free) $\mathfrak{D}$-submodule of V of rank n.
ϑ: the derivation $t(d/dt)$ on K.
D: differential operator in V, i.e. k-linear endomorphism of V satisfying

$$D(av) = \left(t\frac{d}{dt}\,a\right)v + aDv$$

all $a \in K$, $v \in V$.

When $\lambda \in k$, $\lambda \neq 0$, we shall also call λD differential operator.

When D: $V \to V$ and D': $V' \to V'$ are differential operators, (V, D) and (V', D') are said to be isomorphic, if an K-isomorphism φ: $V \to V'$ exists such that $D' \circ \varphi = \varphi \circ D$.

Let $K\langle \Delta \rangle$ the K-vector space having as a basis the formal powers of Δ

$$\Delta^0 = 1, \ \Delta^1 = \Delta, \ \Delta^2, \ \Delta^3, \ \dots .$$

We define a multiplication in $K\langle \Delta \rangle$ by putting first

$$\Delta(a_0 + a_1\Delta + \dots + a_m\Delta^m) = \vartheta a_0 + (a_0 + \vartheta a_1)\Delta + \dots + (a_{m-1} + \vartheta a_m)\Delta^m + a_m\Delta^{m+1},$$

next by induction

$$\Delta^l(a_0 + a_1\Delta + \dots + a_m\Delta^m) = \Delta^{l-1}(\Delta(a_0 + \dots + a_m\Delta^m)),$$

and finally

$$\left(\textstyle\sum_i b_i\Delta^i\right)\left(\textstyle\sum_j a_j\Delta^j\right) = \textstyle\sum_i b_i\Delta^i\left(\textstyle\sum_j a_j\Delta^j\right).$$

It can be verified that in this way $K\langle \Delta \rangle$ becomes an associative, non commutative k-algebra, and that $K\langle \Delta \rangle$ contains K as a subring.

Let D: $V \to V$ be a differential operator. We can make V into a left $K\langle \Delta \rangle$-module by putting

$$(a_0 + a_1\Delta + \dots + a_m\Delta^m)x = a_0x + a_1Dx + \dots + a_mD^mx \quad (a_i \in K, \ x \in V).$$

Conversely, let V be a left $K\langle \Delta \rangle$-module. Then the multiplication by elements of $K \subset K\langle \Delta \rangle$ defines a structure of K-vector space on V. Suppose $\dim_K V < \infty$, and define D: $V \to V$ by

$$D\colon x \to \Delta x.$$

Then D is a differential operator on V.

In the sequel we often make the transition from couples (V, D) to $K\langle \Delta \rangle$-modules and backward without further explication.

We notice that (V, D) and $(V'D')$ are isomorphic if and only if V and V' are isomorphic as $K\langle \Delta \rangle$-modules.

We recall some basic facts which shall frequently be used:

a) *Bases and coordinates*

Let $(e_1, \dots, e_n)$ be a basis of the K-vector space V. Then elements $a_{ij} \in K$ $(1 \leq i, j \leq n)$ are uniquely determined by

$$De_i = \textstyle\sum_{j=1}^n a_{ji}e_j .$$

The matrix $A = (a_{ji})_{1 \leq i, \, j \leq n}$ shall also be denoted by Mat $(D, (e_1, \dots, e_n))$ or Mat $(D, (e))$.

Now let $(\xi_1, \ldots, \xi_n)$ be the coordinates of $x \in V$ with respect to $(e_1, \ldots, e_n)$, and let $(\eta_1, \ldots, \eta_n)$ be the coordinates of Dx.

Then we have

$$\eta_i = t\frac{d}{dt}\xi_i + \sum_j a_{ij}\xi_j \quad (1 \leqq i \leqq n).$$

So using matrix notation we can say that D is the differential operator having the coordinate representation

$$\begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix} \to t\frac{d}{dt}\begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix} + A\begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix}$$

with respect to the basis $(e_1, \ldots, e_n)$.

Let $(f_1, \ldots, f_n)$ be a second K-basis of V and suppose

$$f_i = \sum_{j=1}^n t_{ji}e_j \quad (1 \leqq i \leqq n)$$

where $t_{ji} \in K$ and the matrix $T = (t_{ji})$ is invertible. Let B be the matrix of D with respect to $(f_1, \ldots, f_n)$. Then we have the obvious relation

$$B = T^{-1}AT + T^{-1}t\frac{d}{dt}T.$$

b) *Field extensions*

We shall often have to make finite extensions $K \subset L$. It is well known that the t-adic valuation on K can be extended to a valuation on L in a unique way (cf. [7], Chap. II, §2, Prop. 3). We denote by $\mathfrak{O}_L$ the integral closure of $\mathfrak{O}$ in L. $\mathfrak{O}_L$ is a discrete valuation ring; in fact it is the valuation ring corresponding to the valuation just defined on L.

In general we only need extension fields of the type $L = k'((s))$, where $k \subset k'$ is a finite extension and s satisfies $s^m = t$ for some positive integer m. (It can be shown that every finite field extension of K is contained in such a field.) When $L \subset M$ is another finite extension of the above type, the same holds for $K \subset M$.

We shall denote by V_L the vector space $L \otimes_K V$ (extension of scalars), and by D_L the map of V_L into itself defined by

$$D_L(a \otimes v) = (\vartheta a) \otimes v + a \otimes Dv,$$

all $a \in L$, $v \in V$. Here we mean by $\vartheta \colon L \to L$ the unique extension of the derivation $\vartheta = t(d/dt)$ of $k((t))$. Notice that D_L is k-linear and that $D_L(bx) = (\vartheta b)x + bD_L x$, all $b \in L$, $x \in V_L$. We shall still call D_L differential operator; D_L is *the extension of D to* V_L (Notice that $D_{L|V} = D$. We identify V with the K-subspace $\{1 \otimes v | v \in V\}$ of V_L).

In the special case of $L=k'((s))$ as above we have

$$\vartheta = \frac{1}{m} s\, ds$$

which shows that the use of the term "differential operator" for D_L is consistent with the earlier definition. In this case D_L is even k'-linear.

c) *Cyclic vectors*

It is well-known ([3], Lemma II, 1,3 p. 42) that V admits a cyclic vector e, i.e. an element $e \in V$ such that $e, De, D^2 e, \ldots, D^{n-1} e$ are linearly independent over K.

This fact can also be expressed by saying that the $K\langle \Delta \rangle$-module V is cyclic, i.e. $V = K\langle D \rangle e$.

d) *Simple differential operators*

We shall call $D\colon V \to V$ a *simple differential operator*, if V is a simple $K\langle \Delta \rangle$-module, i.e. $V \neq 0$ and V contains no proper $K\langle \Delta \rangle$-submodules. This is equivalent to saying, that $V \neq 0$ and that V contains no proper K-subspace invariant under D.

Obviously D is simple when $\dim_K V = 1$. We shall now classify this type of simple differential operators. For this we define:

$$\mathfrak{O}^{\cdot} = \{a \mid a \in \mathfrak{O},\ \exists u \in K,\ u \neq 0,\ a = u^{-1} \vartheta u\}.$$

One verifies without difficulty that $\mathfrak{O}^{\cdot}$ is the subring of $\mathfrak{O}$ consisting of the elements

$$\alpha_0 + \alpha_1 t + \alpha_2 t^2 + \ldots \quad (\alpha_i \in k,\ \alpha_0 \in \mathbf{Z}).$$

In the sequel $\mathfrak{O}^{\cdot}$ is considered as a subgroup of the additive group of $\mathfrak{O}$ (resp. K). When $a \in K$, we denote by $[a]$ the class of a modulo $\mathfrak{O}^{\cdot}$.

Now let e be a generator of our one dimensional vector space V. Then $De = ae$ for some $a \in K$, and we call $[a]$ *the type of* D, notation: $[D]$. This is a correct definition. For if e' is another generator of V as a K-vector space, we have $e' = ue\ (u \in K,\ u \neq 0)$, whence

$$De' = D(ue) = (\vartheta u) e + u De = (\vartheta u) e + uae = (a + u^{-1} \vartheta u) e',$$

and

$$[a + u^{-1} \vartheta u] = [a].$$

When D (resp. D') is a differential operator on V (resp. V'), $\dim_K V = \dim_K V' = 1$, then (V, D) and $V', D')$ are isomorphic if and only if $[D] = [D']$. This is an easy consequence of the definitions; the proof is left to the reader as well as that of the following simple fact: Every element of $K/\mathfrak{O}^{\cdot}$ appears as type of a differential operator on a one dimensional vector space.

In this way we have obtained a complete classification of isomorphism classes of couples (V, D), $\dim_K V=1$ and $D\colon V\to V$ differential operator, by their types.

Finally a remark on types and field extensions. Let $K\subset L$ a finite extension. We define:

$$\mathfrak{D}_L^{\cdot} = \{a \mid a\in \mathfrak{D}_L,\ \exists u\in L,\ u\neq 0,\ a = u^{-1}\vartheta u\}.$$

Since $\mathfrak{D}^{\cdot}\subset\mathfrak{D}_L^{\cdot}$ there is an induced homomorphism

$$\Psi\colon K/\mathfrak{D}^{\cdot} \to L/\mathfrak{D}_L^{\cdot}$$

defined by

$\Psi([a])=$class of a modulo $\mathfrak{D}_L^{\cdot}$, when $a\in K$ (and hence $\in L$). In general Ψ is not surjective and not injective. For instance take $L=k((s))$, $s^2=t$. Then $\mathfrak{D}_L^{\cdot}$ is the subring of $\mathfrak{D}_L$ consisting of the elements

$$\beta_0+\beta_1 s+\beta_2 s^2+\dots \quad (\beta_i\in k,\ 2\beta_0\in \mathbf{Z}).$$

Now the class mod $\mathfrak{D}^{\cdot}$ of $\frac{1}{2}\in K$ is different from 0, whereas $\Psi([\frac{1}{2}])=0$. On the other hand the class of $1/s\in L$ mod $\mathfrak{D}_L^{\cdot}$ is not in the image of Ψ.

In the special case of $L=k'((s))$, $k\subset k'$ finite, $s^m=t$, it is not difficult to see that

$$\operatorname{Ker}\Psi = \left\{[0], \left[\frac{1}{m}\right], \dots, \left[\frac{m-1}{m}\right]\right\}.$$

Let again V be a one dimensional vector space over K and $D\colon V\to V$ a differential operator. Then it may easily be verified that

$$\Psi([D]) = [D_L],$$

where $[D_L]$ the type of D_L is defined as an element of $L/\mathfrak{D}_L^{\cdot}$ in an evident way. Using the above example it is not difficult to construct non isomorphic differential operators on V which become isomorphic over an extension field L of K.

e) *Semisimple and diagonalizable operators*

Let $D\colon V\to V$ be a differential operator. We shall call D *semisimple* if V is a semisimple $K\langle\Delta\rangle$-module, i.e. (cf. [5], §3, Déf. 3) if the following equivalent conditions hold:

(i) V is a sum of simple submodules.
(ii) V is a direct sum of simple submodules.
(iii) Every submodule of V is a direct factor.

Condition (iii) is obviously equivalent to:

(iv) To every K-subspace W of V, invariant under D, there exists an invariant complement, i.e. a K-subspace W' of V, invariant under D, such that $V=$ $=W+W'$ is a direct sum.

Examples of semisimple differential operators are diagonalizable operators. (V, D) is called *diagonalizable* over K, when V is a direct sum of $K\langle\Delta\rangle$-submodules which are one-dimensional over K (and hence simple). This is equivalent to saying that V admits a K-basis $(e_1, \ldots, e_n)$ such that $\mathrm{Mat}(D, (e))$ is a diagonal matrix.

Proposition. *For any differential operator D': $V \to V$ and finite field extension $K \subset L$ the following conditions are equivalent:*

(i) *D is semisimple.*
(ii) *D_L is semisimple.*

Proof. (ii)$\Rightarrow$(i). Let $W \subset V$ be an invariant subspace, and choose a K-basis $(e_1, \ldots, e_n)$ of V such that $e_1, \ldots, e_r$ generate W. Then

$$\mathrm{Mat}(D, (e)) = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix},$$

where A and B have r rows, B and C $n-r$ columns and all the matrix elements are in K.

By extension of scalars we have $W_L \subset V_L$ invariant under D_L, and by (ii) there exists a L-subspace Z of V_L invariant under D_L and complementary to W_L. We can find $u_{ij} \in L$ such that:

$$f_i = e_i + \textstyle\sum_{j=1}^{r} u_{ji} e_j \quad (r+1 \leqq i \leqq n)$$

form a basis of Z over L. Hence

$$\mathrm{Mat}(D_L, (e_1, \ldots, e_r, f_{r+1}, \ldots, f_n)) = T^{-1} \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} T + T^{-1} \vartheta T$$

where

$$T = \begin{pmatrix} I & U \\ 0 & I \end{pmatrix}, \quad U = (u_{ji}).$$

On the other hand we know that the matrix of D_L with respect to that basis looks like

$$\begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix}.$$

Consequently

$$A = P, \quad C = Q, \quad AU + B + \vartheta U = UC.$$

Taking traces with respect to $K \subset L$ in the last equality, we find

$$AV + B + \vartheta V = VC,$$

where

$$(v_{ji}) = V = \frac{1}{[L:K]} Tr_{L/K} U$$

is a matrix with elements in K. (Notice that ϑ commutes with Tr; see below) A simple computation shows at one

$$\mathrm{Mat}\,(D, (e_1', \ldots, e_n')) = \begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix},$$

when

$$e_i' = e_i \quad (1 \leqq i \leqq r)$$

$$e_i' = e_i + \sum\nolimits_{j=1}^{r} v_{ji} e_j \quad (r+1 \leqq i \leqq n).$$

This proves that $V = W + W'$, $W' = \sum_{i=r+1}^{n} K e_i'$, is a direct sum of invariant subspaces.

(i)$\Rightarrow$(ii) We first reduce the statement to the special case of L/K finite Galois extension. Let M be a finite Galois extension of K containing L. Since we know already that that (ii) implies (i), it suffices to prove that D semisimple implies D_M semisimple. So there is no restriction in assuming L/K Galois.

We shall now show that V_L as a left $L\langle \Delta \rangle$-module has no radical. Since V_L is an artinian $L\langle \Delta \rangle$-module, this will show that V_L is semisimple as a $L\langle \Delta \rangle$-module (cf. [5], §6, n° 4, Théorème 4.)

For any $\sigma \in \mathrm{Gal}\,(L/K)$ we have

(1) $$D_L \circ (\sigma \otimes 1) = (\sigma \otimes 1) \circ D_L,$$

where

$$\sigma \otimes 1 \colon V_L \to V_L$$

is the K-linear automorphism, defined by

$$(\sigma \otimes 1)\,(a \otimes v) = \sigma(a) \otimes v \quad (a \in L,\ v \in V).$$

In order to prove (1), we notice that

$$\big(D_L \circ (\sigma \otimes 1)\big)\,(a \otimes v) = D_L\big(\sigma(a) \otimes v\big) = \vartheta\sigma(a) \otimes v + \sigma(a) \otimes Dv,$$

and

$$\big((\sigma \otimes 1) \circ D_L\big)\,(a \otimes v) = (\sigma \otimes 1)\,(\vartheta a \otimes v + a \otimes Dv) = \sigma(\vartheta a) \otimes v + \sigma(a) \otimes Dv.$$

Hence (1) will follow from the relation

$$\vartheta\sigma = \sigma\vartheta$$

(both members operating on L), and this relation holds, for $\sigma^{-1}\vartheta\sigma$ is a derivation on L extending ϑ to L.

Let W be any L-subspace of V_L invariant under D_L (or — what comes down to the same — a $L\langle \Delta \rangle$-submodule of V_L). When $\sigma \in \mathrm{Gal}\,(L/K)$, it follows from (1) that $(\sigma \otimes 1)\ (W)$ is also an L-subspace of V_L invariant under D_L; if, moreover, W is maximal in the set of subspaces of V_L invariant under D_L and different from V_L, then $(\sigma \otimes 1)(W)$ is also maximal. So it follows at once that $R = \mathrm{Rad}\,(V_L)$ is an L-subspace of V_L invariant under all $\sigma \otimes 1$, $\sigma \in \mathrm{Gal}\,(L/K)$. It is well known (cf. [7], Chap. X) that in these circumstances R "comes" from a subspace of V i.e. there exists a K-subspace U of V invariant under D such that $R=L\otimes_K U$. Since we have assumed D to be semisimple, there is a complementary invariant subspace U': $V=U+U'$ direct sum, whence

$$V_L = U_L + U'_L$$

direct sum, and U_L is the radical of V_L (considered as a left $L\langle\Delta\rangle$-module). Now it follows from ([5], §6, Prop. 3, Cor. 3) that $U_L=0$. This concludes the proof.

We can now state our principal results:

Theorem I. *If D: $V\to V$ is a differential operator, then there exist maps S, N: $V\to V$ satisfying:*

(i) *S is a semisimple operator.*
(ii) *N is a nilpotent K-linear map.*
(iii) *$D=S+N$.*
(iv) *S and N commute.*

Moreover, there is only one pair S, N of maps satisfying these conditions.

Theorem II. *D: $V\to V$ is a semisimple if and only if D_L: $V_L\to V_L$ is diagonalizable over a finite extension L of K.*

"If" is obvious from the preceding proposition and our observation that diagonalizable differential operators are semisimple.

We conclude this section by a remark on eigenvectors. As an immediate result of the theorems we have:

Corollary. *If D: $V\to V$ is a differential operator, then there exists a finite extension $K\subset L$ and an eigenvector of D_L i.e. an element $v\in V_L$ different from 0 and an element $a\in L$ such that $D_L v=av$.*

It would be interesting to have a direct proof of the corollary. For in that case the proofs of both theorems could be simplified considerably.

2. A splitting lemma

In this section we prove a fundamental lemma which is well-known in the literature (cf. [6], Chap. IV, Theorem 11, 1).

In addition to the notations introduced earlier we shall write

q: positive integer.
D_q: k-linear map $\Lambda\to\Lambda$ satisfying $D_q(ae)=(t^q(d/dt)a)e+aD_qe$ all $a\in\mathfrak{O}$, $e\in\Lambda$.
φ: $\Lambda\to\bar{\Lambda}=\Lambda/t\Lambda$, canonical map.
δ: $\bar{\Lambda}\to\bar{\Lambda}$, k-linear map induced by D_q.

Lemma. *Let $\bar{\Lambda}$ be the direct sum of two k-subspaces $\bar{\Lambda}_1, \bar{\Lambda}_2$ which are invariant under δ, and suppose that the restrictions $\delta_1=\delta|_{\bar{\Lambda}_1}$, $\delta_2=\delta|_{\bar{\Lambda}_2}$ don't have common eigenvalues (respectively, in the case $q=1$, $|\alpha-\beta|\neq 1, 2, \ldots$, if α is an eigenvalue of δ_1 and β an eigenvalue of δ_2).*

Then Λ is the direct sum of two free submodules Λ_1 and Λ_2, invariant under D_q, and such that $\bar{\Lambda}_1=\varphi(\Lambda_1)$, $\bar{\Lambda}_2=\varphi(\Lambda_2)$.

Moreover, Λ_1 and Λ_2 are uniquely determined by these properties.

Proof. Let $(e_1, \ldots, e_n)$ be a basis of Λ such that $(\varphi(e_1), \ldots, \varphi(e_r))$ (resp. $(\varphi(e_{r+1}), \ldots, \varphi(e_n))$) is a basis of $\bar{\Lambda}_1$ (resp. of $\bar{\Lambda}_2$). When

$$\operatorname{Mat}(D_q, (e_1, \ldots, e_n)) = \begin{pmatrix} P & Q \\ R & S \end{pmatrix}$$

(P, Q having r lines and P, R r columns), we have

$$\operatorname{Mat}(\delta, (\varphi(e_1), \ldots, \varphi(e_n))) = \begin{pmatrix} P_0 & Q_0 \\ R_0 & S_0 \end{pmatrix},$$

where P_0 etc., is obtained from P by replacing every element by its class mod $t\mathfrak{O}$. Since $\bar{\Lambda}_1$ and $\bar{\Lambda}_2$ are invariant by δ we have

$$R_0=Q_0=0,$$

and P_0, S_0 have no common characteristic value (resp., ... if $q=1$), for these are the matrices of δ_1 resp. δ_2.

We try to find another $\mathfrak{O}$-basis $(f_1, \ldots, f_n)$ of Λ, where

$$f_i = \textstyle\sum_{j=1}^{n} t_{ji} e_j,$$

and the matrix $T=(t_{ji})$ has the following structure

$$T = I_n + \begin{pmatrix} 0 & X \\ Y & 0 \end{pmatrix}$$

(I_n indicates the $n\times n$ identity matrix), and such that

$$\operatorname{Mat}(D_q, (f_1, \ldots, f_n)) = \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix}.$$

If we succeed in finding T in such a way that X and Y have their elements in $t\mathfrak{O}$, we see that $\varphi(e_i)=\varphi(f_i)$ and that the submodules of Λ_1 resp. Λ_2 generated by $(f_1, \ldots, f_r)$ resp. $(f_{r+1}, \ldots, f_n)$ have the desired properties.

In order to find such X, Y we develop all matrices in series of powers of t, the coefficients being matrices with elements in k,

$$\begin{aligned} P &= P_0 + P_1 t + \dots \\ S &= S_0 + S_1 t + \dots \\ Q &= \quad Q_1 t + \dots \\ R &= \quad R_1 t + \dots \\ X &= \quad X_1 t + \dots \\ Y &= \quad Y_1 t + \dots . \end{aligned}$$

Since

$$\mathrm{Mat}\,(D_q, (f_1, \dots, f_n)) = T^{-1}\,\mathrm{Mat}\,(D_q, (e_1, \dots, e_n))\,T + T^{-1} t^q \frac{d}{dt} T,$$

we must have

$$\begin{pmatrix} P & Q \\ R & S \end{pmatrix} T + t^q \frac{d}{dt} T = T \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix},$$

which leads to the following set of equations

$$\begin{aligned} P + QY &= U \\ S + RX &= V \\ t^q \frac{d}{dt} X + PX - XV + Q &= 0 \\ t^q \frac{d}{dt} Y + SY - YU + R &= 0 \end{aligned}$$

for the unknown matrices X, Y, U, V. These equations admit a solution if X and Y can be found satisfying

$$\text{(1)} \qquad \begin{aligned} t^q \frac{d}{dt} X + PX - XS - XRX + Q &= 0 \\ t^q \frac{d}{dt} Y + SY - YP - YQY + R &= 0. \end{aligned}$$

First suppose that $q > 1$ and insert the power series developments into the equation (1). Comparing coefficients of t we find

$$P_0 X_1 - X_1 S_0 + Q_1 = 0.$$

Now, since P_0 and S_0 have no eigenvalue in common, this equation in X_1 can be solved (uniquely). When $m > 1$ and $X_1, \dots, X_{m-1}$ have already been found, then equating coefficients of t^m in (1) we find that $P_0 X_m - X_m S_0$ is equal to an expression containing known matrices and $X_0, X_1, \ldots, X_{m-1}$. It is then possible to find X_m. The equation for Y can be treated in the same way. So the "existence part" of the lemma has been proved in the case $q>1$.

Finally, when $q=1$, we have to solve in the m-th step the equation

$(P_0+m\cdot I_r)X_m - X_m S_0 =$ expression containing $X_0, \ldots, X_m$ and known matrices. Now P_0+mI_r and S_0 have no eigenvalue in common in virtue of the hypothesis on δ_1 and δ_2 in this case.

In order to prove the uniqueness of the splitting we suppose that another direct sum decomposition

$$\Lambda = \Lambda_1' + \Lambda_2'$$

is given, and that it has the properties of the lemma. Let $(g_1, \ldots, g_n)$ be an $\mathfrak{O}$-basis of Λ such that $(g_1, \ldots, g_r)$ is a basis of Λ_1' and $(g_{r+1}, \ldots, g_n)$ is a basis of Λ_2'. Since Λ_1' and Λ_2' are invariant under D_q, we have

$$\operatorname{Mat}(D_q(g_1, \ldots, g_n)) = \begin{pmatrix} Z & 0 \\ 0 & W \end{pmatrix},$$

where X (resp. W) is a matrix with elements in $\mathfrak{O}$. On the other hand, $(f_1, \ldots, f_n)$ and $(g_1, \ldots, g_n)$ being both bases of Λ, there exists a relation

$$(g_1, \ldots, g_n) = (f_1, \ldots, f_n)\begin{pmatrix} E & F \\ G & H \end{pmatrix},$$

wher the matrix has its elements in $\mathfrak{O}$ and is invertible over $\mathfrak{O}$. Now using the fact that $\varphi(\Lambda_1)=\varphi(\Lambda_1')$, $\varphi(\Lambda_2)=\varphi(\Lambda_2')$, we find

$$F_0 = G_0 = 0.$$

Connecting the two matrices of D_q with respect to both bases, we have

$$\begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix}\begin{pmatrix} E & F \\ G & H \end{pmatrix} + t^q \frac{d}{dt}\begin{pmatrix} E & F \\ G & H \end{pmatrix} = \begin{pmatrix} E & F \\ G & H \end{pmatrix}\begin{pmatrix} W & 0 \\ 0 & Z \end{pmatrix},$$

from which we deduce

(2) $$FZ - UF = t^q \frac{d}{dt} F$$

$$GW - VG = t^q \frac{d}{dt} G.$$

Uniqueness is proved when we have shown that $F=G=0$. Let us show that $F=0$ ($G=0$ can be shown in the same way). We insert the power series developments

$$F = F_1 t + F_2 t^2 + \ldots$$

$$U = U_0 + U_1 t + \ldots$$

$$Z = Z_0 + Z_1 t + \ldots$$

into (2). First assume that $q>1$. Comparing coefficients of t^m we find

$$F_1 Z_0 - U_0 F_1 = 0, \tag{3}$$

and, if $m \geqq 2$,

$$F_m Z_0 - U_0 F_m = L(F_1, \ldots, F_{m-1}), \tag{4}$$

where L is a "linear expression" in $F_1, \ldots, F_{m-1}$ with known matrices as coefficients and without constant term. Now Z_0 is the matrix of δ_2 with respect to $(\varphi(g_{r+1}), \ldots$ $\ldots, \varphi(g_n))$. Hence U_0 and Z_0 have no eigenvalues in common and, consequently, (3) has $F_1=0$ as its unique solution. Then applying (4) for $m=2, 3, \ldots$ sucessively, we deduce $F_2=0$, $F_3=0, \ldots$. This terminates the proof in the case $q>1$. The simple modifications to be made in the case $q=1$ are left to the reader.

3. Existence of a decomposition over an extension of K

Using the notations of the preceding sections we are going to prove:

Proposition. *There exist a finite field extension $K \subset L$ and maps S, N: $V_L \to V_L$ such that*

(a) *S is a diagonalizable differential operator.*
(b) *N is a nilpotent L-linear map.*
(c) $D_L = S + N$.
(d) $[S, N] = 0$.

Remark. It will be evident from the proof that $K \subset L$ is a composition of extensions of the type described in section 1.b. Hence $K \subset L$ itself is an extension of that type.

Proof. Induction on $n = \dim_K V$. The proposition being trivial for $n=1$, we suppose $n>1$, and assume that the proposition holds for differential operators on vector spaces of dimension $<n$.

We use the fact that a cyclic vector for D exists [cf. section 1, c], i.e. there exists $e \in V$ such that $e, De, \ldots, D^{n-1}e$ are linearly independent over K.

Let $a_1, \ldots, a_n \in K$ be such that

$$D^n e + a_1 D^{n-1} e + \ldots + a_n e = 0, \tag{1}$$

and put

$$\sup_{1 \leqq i \leqq n} \left(\frac{-v(a_i)}{i} \right) = \frac{l}{m},$$

where v is the valuation on $K=k((t))(v(t)=1)$, m is a positive integer, $l \in \mathbf{Z}$ and l and m are relatively prime.

This definition is not correct only in the case that all a_i vanish. However, in that case D has a singularity of the "first kind" and the argument of (i) below can be used to obtain the desired result.

Now consider the extension $K'=k((s))$ of K, where s is an m-th root of t, and put $V_{K'}=K' \otimes_K V$, $D_{K'}$ extension of D to $V_{K'}$. In $V_{K'}$ we take as a K'-basis $(f_1, \ldots, f_n)$, where

$$f_i = s^{l(i-1)} D^{(i-1)} e \quad (i = 1, \ldots, n).$$

With respect to this basis we have

$$D_{K'} = \frac{1}{m} s \frac{d}{ds} + s^{-l} \begin{pmatrix} 0 & & & 0 & -b_n \\ 1 & & & & \\ 0 & 1 & & & \\ & & & 0 & \\ 0 & & 0 & 1 & -b_1 \end{pmatrix} + \frac{1}{m} \begin{pmatrix} 0 & & & 0 \\ & l & & \\ & & \ddots & \\ 0 & & & (n-1)l \end{pmatrix}$$

where

$$b_i(s) = s^{il} a_i(s^m) \quad (i = 1, \ldots, n).$$

Let v' be the valuation on $K'(v'(s)=1$, whence $v'(t)=m)$. Then by definition of m and l

$$v'(b_i) = il + mv(a_i) \geqq 0 \tag{2}$$

all i, and equality for at least one value, i_0 say, of i.

Now we distinguish two cases:

(i) $l \leqq 0$. This implies $v(a_i) \geqq 0$ all i, and consequently our D has a singularity of the "first kind" i.e.

$$D = t \frac{d}{dt} + A$$

(with respect to a certain K-basis $(e_1, \ldots, e_n)$ of V) and A has its elements in $\mathfrak{O}$. Since this type of operator has been studied at lenght in the literature (cf. [6], Chapter II and Chapter V, sec. 17), we shall restrict ourselves to some brief indications.

First of all, develop A into a formal power series

$$A = A_0 + A_1 t + A_2 t^2 + \ldots,$$

the A_i being matrices with elements in k. Now if the eigenvalues of A_0 don't differ by positive integers, it can immediately be verified that the equation

$$T^{-1} A T + T^{-1} \vartheta T = A_0,$$

where

$$T = I + T_1 t + T_2 t^2 + \ldots$$

is a formal power series (T_i being matrices with elements in k), has a solution. This implies the following fact.

$$\mathrm{Mat}\,(D, (f)) = A_0,$$

where (f) is the K-basis of V defined by $(f_1, \ldots, f_n) = (e_1, \ldots, e_n)T$. Defining S and N by their matrices with respect to (f):

$$\mathrm{Mat}\,(S, (f)) = A_0^s, \quad \mathrm{Mat}\,(N, (f)) = A_0^n,$$

where A_0^s (resp. A_0^n) denotes the semisimple (resp. the niltpotent) part of A_0, one verifies that S and N have the properties of the proposition. Remark that S is diagonalizable over $k'((t))$, when k' is an extension of k containing all eigenvalues of A_0, so that A_0 can be assumed in Jordan canonical form.

In the general case let $\alpha_1, \ldots, \alpha_r$ be the different eigenvalues of A_0. Then after making a finite extension k' of k and a "constant" change of bases $(e) \to (f)$ (i.e. with matrix elements in k') we can assume A_0 in the form

$$A_0 = \begin{pmatrix} A_0' & 0 \\ 0 & A_0'' \end{pmatrix},$$

where A_0' (resp. A_0') is a square matrix of μ (resp. $n-\mu$) rows, μ being the multiplicity of α_1, and where A_0' has α_1 as its unique eigenvalue. This implies that $\alpha_2, \ldots, \alpha_r$ are the different eigenvalues of A_0''. Taking $(f') = (tf_1, \ldots, tf_\mu, f_{\mu+1}, \ldots, f_n)$ as a new basis, it can easily be verified that

$$\mathrm{Mat}\,(D, (f')) = B_0 + B_1 t + \ldots,$$

where the eigenvalues of B_0 are $\alpha_1 + 1, \alpha_2, \ldots, \alpha_r$.

By repeating this procedure a finite number of times we find in the end a finite extension field k_0 of k and a basis of $V_{k_0((t))}$, such that the constant term in the development of the matrix of D has no eigenvalues which differ by an integer $\neq 0$. Then we are back in the situation treated above.

(ii) $l > 0$. We try to apply the Lemma of section 1. With respect to the basis $(f_1, \ldots, f_n)$ of $V_{K'}$ we have

$$s^l D_{K'} = E = \frac{1}{m} s^{l+1} \frac{d}{ds} + B_0 + sC,$$

where

$$B_0 = \begin{pmatrix} 0 & & & 0 & -b_n(0) \\ 1 & & & & \\ 0 & 1 & & & \\ & & & 0 & \\ 0 & & 0 & 1 & -b_1(0) \end{pmatrix}$$

and C is a matrix with coefficients in $k[[s]]$. Now suppose that B_0 has at least two distinct eigenvalues. Then as in the case (i) we can find a finite extension $k \subset k'$ and a basis $(g_1, \ldots, g_n)$ in $V_{k'((s))}$ such that the extension F of E to $V_{k'((s))}$ has the form

$$F = \frac{1}{m} s^{l+1} \frac{d}{ds} + G + sH,$$

where

$$G = \begin{pmatrix} G' & 0 \\ 0 & G'' \end{pmatrix}$$

is a matrix with elements in k', H a matrix with elements in $k'[[s]]$, G' (resp. G'') a square matrix having r (resp. $n-r$) rows, $1 \leqq r \leqq n-1$, and G' and G'' have no common eigenvalue. Now we apply the lemma of the preceding section to the differential operator F, taking $q=l+1$ and Λ the $k'[[s]]$-module generated by $(g_1, \ldots, g_n)$. Then we conclude as in (i).

Finally we treat case of B_0 having only one eigenvalue β (automatically: $\beta \in k$). Since $b_{i_0}(0) \neq 0$, we have $\beta \neq 0$, and consequently all coefficients of

$$X^n + b_1(0) X^{n-1} + \ldots + b_n(0) = (X-\beta)^n$$

are different from 0. In view of (2) this implies

$$il + mv(a_i) = 0 \quad \text{all} \quad i.$$

Since by definition l and m are relatively prime we have

$$m = 1, \quad s = t, \quad v(a_i) = -il, \quad b_i = t^{il} a_i, \quad \text{all}$$

and

$$D = t\frac{d}{dt} + t^{-l} B_0 + t^{-l+1} C$$

with respect to the basis $(f_1, \ldots, f_n) = (e, t^l De, \ldots, t^{(n-1)l} D^{n-1} e)$ of V. Now define the differential operator D^*: $V \to V$ by

$$D^* = D - \beta t^{-l} I.$$

Obviously it suffices to prove our proposition with D replaced by D^*. It can easily be proved by induction that

$$D^m = \sum_{i=0}^{m} p_i^m t^{-il} D^{*(m-i)},$$

where p_i^m is a polynomial in t and

$$p_i^m(0) = \binom{m}{i} \beta^i, \quad p_0^m = 1.$$

Using these formulae, we see that $e, D^*e, \ldots, D^{*(n-1)}e$ are linearly independent over K and that

$$D^{*n}e + a_1^* D^{*n-1}e + \ldots + a_n^* e = 0,$$

where

$$a_i^* = t^{-il} c_i,$$

and

$$c_i = \sum\nolimits_{h=0}^{i} p_{i-h}^{n-h} b_h.$$

So we see that c_i is an element of $k[[t]]$ and that

$$c_i(0) = \sum\nolimits_{h=0}^{i} (-1)^h \binom{n-h}{i-h} \beta^{i-h} \binom{n}{h} \beta^h = \beta^i \binom{n}{i} \sum\nolimits_{h=0}^{i} (-1)^h \binom{i}{h} = 0,$$

consequently

$$\sup\left(-\frac{v(a_i^*)}{i}\right) < \sup\left(-\frac{v(a_i)}{i}\right). \tag{4}$$

Now we apply the preceding chain of arguments to D^* instead of D. This leads to a splitting in all cases except one, viz. $v(a_i^*) = -l^*i$ and $l^* > 0$. However, by (4), we have $l^* < l$ and the proof can be completed by induction on l.

4. Uniqueness in a special case

In this section we shall prove

Propostion. *Let the differential operator D: $V \to V$ be sum of a diagonalizable operator S and a nilpotent K-linear map N, and let S and N commute. Then S and N are uniquely determined by D.*

We shall first investigate S. Considering V as a $K\langle \Delta \rangle$-module by putting

$$\Delta x = Sx,$$

we see that V is a direct sum of simple submodules which are one-dimensional vector spaces over K (cf. section 1, d). These are characterized up to isomorphism by their types $\varphi \in K/\mathfrak{D}^{\cdot}$. Now let V_φ the isotypical component of V, i.e. the (direct) sum of the submodules of type φ. Then we have

$$V = \sum\nolimits_{\varphi}' V_\varphi \quad (\varphi \in K/\mathfrak{D}^{\cdot})$$

and this is a dircet sum ([5], §3 no. 4, Proposition 9).

As a first step in the proof of the Proposition we show that the subspaces V_φ are uniquely determined by D.

Lemma. *(Same hypotheses as in the Proposition.) For $a \in K$ define*

$$E_a = \{x | x \in V \text{ and } (D-a)^m x = 0 \text{ for some positive integer } m\}.$$

Then

$$KE_a = V_{[a]}$$

and this subspace is invariant under D, S and N (for shortness' sake we write $D-a$ for the map $x \to Dx - ax$ of V into itself).

Proof. (i) $V_{[a]} \subset KE_a$. To see this it suffices to show that $W \subset KE_a$, where W is a one dimensional subspace of V, invariant under S, and of type $[a]$. So we may assume, that $W = Ke$, and $Se = ae$. Hence

$$(D-a)e = (S-a)e + Ne = Ne,$$

and since N and $D-a$ commute

$$(D-a)^2 e = (D-a)Ne = N(D-a)e = N^2 e,$$

etc. This shows $e \in KE_a$, and $W \subset KE_a$ as a consequence.

(ii) We next show

$$KE_a \subset V_{[a]}.$$

It suffices to show that $(D-a)^m x = 0$ for some m implies $x \in V_{[a]}$. Let r be a positive integer such that $N^{r+1} = 0$. Then

$$(S-a)^{m+r} x = ((D-a)-N)^{m+r} x = (D-a)^{m+r} x - \binom{m+r}{1} N(D-a)^{m+r-1} x + \\ + \dots + \binom{m+r}{r} N^r (D-a)^m x = 0.$$

Since S is a diagonalizable we can write

$$x = x_1 + \dots + x_n$$

according to a direct sum decomposition of V into one-dimensional subspaces V_i invariant under S. From this we see that

$$(S-a)^l x_i = 0 \quad (1 \leq i \leq n)$$

for some positive integer l. Now for fixed i suppose $x_i \neq 0$, choose l as small as possible and write $y = (S-a)^{l-1} x_i$. Then

$$y \in V_i, \quad y \neq 0, \quad (S-a)y = 0.$$

Hence, when $S_{|V_i}$ has type φ, we deduce $[a] = \varphi$, which implies that x belongs to the sum of those V_i having type $[a]$, i.e. to $V_{[a]}$.

(iii) The invariance of $V_{[a]}$ under S is evident. In order to show the invariance under N and $D=S+N$ it suffices to prove: If $Sx=bx$, and $[b]=[a]$, then $y=N(cx)\in V_{[a]}$ for all $c\in K$. Now this is obvious for $c=0$, so assume $c\neq 0$. Then we have

$$Sy = SN(cx) = NS(cx) = N((\vartheta c)x+cbx) = (b+c^{-1}\vartheta c)N(cx) = (b+c^{-1}\vartheta c)y,$$

and

$$[b+c^{-1}\vartheta c] = [a],$$

whence $y\in V_{[a]}$.

Proof of the Proposition. Let $D=S'+N'$ be another decomposition of D, satisfying the hypotheses of the Proposition. We can write

$$V = \sum_{\varphi} V'_{\varphi} \quad (\varphi\in K/\mathfrak{D}^{\cdot}),$$

a direct sum of isotypical components with respect to S'. However, by the Lemma we know that

$$V_{[a]} = KE_a = V'_{[a]} \quad (\text{all } a\in K)$$

and it suffices to show that S and S' operate in the same way on these subspaces. Restricting D, S, N, S', N' to $V_{[a]}$, replacing D by $D-a$, S by $S-a$, S' by $S'-a$, and chaning our notations, it suffices to prove the Proposition under the special assumption, that V is the isotypical component (with respect to S and S') corresponding to the type 0. This means that K-bases $(e_1, \dots, e_n)$ and $(e'_1, \dots, e'_n)$ of V exist such that

$$\operatorname{Mat}(S, (e)) = 0 \quad \text{and} \quad \operatorname{Mat}(S', (e')) = 0.$$

Abusing notation we still denote by N (resp. N') the matrix of N (resp. N') with respect to $(e_1, \dots, e_n)$ (resp. $(e'_1, \dots, e'_n)$). Since S and N commute, we see that N is a constant matrix, and the same holds for N'.

When T is the matrix connecting the two bases, i.e.

$$(e'_1, \dots, e'_n) = (e_1, \dots, e_n)T$$

we have

(1) $$N' = T^{-1}NT+T^{-1}\vartheta T.$$

We develop T in a Laurant series

$$T = T_q t^q + T_{q+1}t^{q+1}+\dots,$$

where q is an integer and all T_i constant matrices. From (1) we deduce

$$qT_q t^q + (q+1)T_{q+1}t^{q+1}+\dots = (T_qN'-NT_q)t^q+(T_{q+1}N'-NT_{q+1})t^{q+1}+\dots,$$

and comparing coefficients of equal powers of t we find

$$T_mN'-(N+mI)T_m = 0.$$

Since for $m \neq 0$, N' and $N+mI$ have no common eigenvalues, it follows that $T_m=0$. On the other hand T is invertible, whence $T=T_0$ is a constant non singular matrix. It follows that S and S' annihilate the same K-basis of V, whence $S=S'$. This completes the proof.

5. Proof of the Theorems

As before $K=k((t))$, V a n-dimensional vector space over K and D: $V \to V$ a differential operator. We are going to prove Theorem 1 of section 1. For the time being we replace "semisimple" by "diagonalizable over a finite extension of K".

Applying the Proposition of section 3 we know that there exist a finite extension $K \subset L$, a diagonalizable operator S: $V_L \to V_L$ and a nilpotent L-linear map N: $V_L \to V_L$ such that

$$D_L = S+N \quad [S, N] = SN-NS = 0.$$

When $L \subset M$ is another finite field extension, we can extend D, S, N to $V_M=(V_L)_M$ and find

$$D_M = S_M+N_M, \quad [S_M, N_M] = 0,$$

S_M being diagonalizable over M and N_M a nilpotent M-linear map of V_M. We can take for instance M such that $K \subset M$ is a finite Galois extension containing L. Moreover, it is easy to see that, if $L=k'((\sqrt[m]{t}))$ and $k \subset k'$ finite, then M can be found having the same structure.

From now on we shall assume that $K \subset L$ is a finite Galois extension.

Now we fix once and for all a K-basis $(e_1, \ldots, e_n)$ of V, which shall also be used as a L-basis in V_L. Write

$$A = \mathrm{Mat}(D, (e)),$$

$$B = \mathrm{Mat}(S, (e)),$$

$$C = \mathrm{Mat}(N, (e)),$$

so the elements of A (resp. B, C) are in K (resp. L),

$$A = B+C$$

and

$$A = \sigma B+\sigma C,$$

when $\sigma \in \mathrm{Gal}(L/K)$. Denote by ${}^\sigma S$ the differential operator on V_L having σB as its matrix with respect to $(e_1, \ldots, e_n)$. In the same way let ${}^\sigma N$ be the L-linear map of V_L having σC as its matrix with respect to $(e_1, \ldots, e_n)$. Obviously ${}^\sigma S$ is diagonalizable, ${}^\sigma N$ nilpotent, and $[{}^\sigma S, {}^\sigma N]=0$, whereas $D={}^\sigma S+{}^\sigma N$. Applying the Proposition of section 4 we conclude

$$^{\sigma}S = S, \quad ^{\sigma}N = N.$$

Hence $\sigma B = B$, $\sigma C = C$ all $\sigma \in \mathrm{Gal}\,(L/K)$. This proves that all elements of B and C are in K, so S and N are defined as maps of V, and the properties (i), (ii), (iii) and (iv) of Theorem I are evident.

Finally we prove uniqueness. Suppose (S, N) and (S', N') are two couples of maps of V having the properties (i), (ii), (iii) and (iv) of Theorem I. There exists a finite extension $K \subset L$ such that S and S' extend to diagonalizable differential operators S_L and S'_L of V_L. We then have

$$D_L = S_L + N_L, \quad D_L = S'_L + N'_L$$

(S_L, N_L) and (S'_L, N'_L) having the properties of the Proposition of section 4. We conclude $S_L = S'_L$ and $N_L = N'_L$, whence $S = S', N = N'$. This finishes the proof of the Theorem I, if "semisimple" is replaced by "diagonalizable over a finite extension of K". Next we prove Theorem II. Theorem I as stated in section 1 follows at once from Theorem II and the "diagonalizable" version just proved.

All we have to show is this: if $D\colon V \to V$ is a semisimple differential operator, D is a diagonalizable over a finite extension L of K.

Anyway, by the above arguments we know that $D = S + N$, where S is diagonalizable over a finite extension $K \subset L$, N is a nilpotent K-linear map and S and N commute. We shall show that $D = S$. For this it is sufficient that $D_L = S_L$. So we may assume that S is already diagonalizable over K. When r is a positive integer satisfying $N^r = 0$, we put

$$V_i = \mathrm{Ker}\,(N^{r-i}) \quad (i = 0, 1, \ldots, r).$$

Then we have a decreasing chain of K-linear subspaces of V

$$V = V_0 \supset V_1 \supset \ldots \supset V_r = (0),$$

and since S and D commute with N we have

$$S(V_i) \subset V_i, \quad D(V_i) \subset V_i, \quad NV_i \subset V_{i+1}.$$

Assume that

$$NV_{i+1} = 0$$

which is correct if $i = r-1$. We shall show that $NV_i = 0$. Replacing V by V_i we may assume that $N^2 = 0$. (Notice that $D_{|V_i}$ is semisimple and $S_{|N_i}$ diagonalizable.)

We make some further reductions. Replacing V by an isotypical component of type φ with respect to S, we may assume that V itself is already isotypical of type $\varphi=[a]$. As D-aI is clearly semisimple, we can as well assume that S is isotypical of type 0. So there exists a basis $(e_1, \ldots, e_n)$ of V such that $Se_i=0$ $(1 \leqq i \leqq n)$. Abusing notation we still denote by N the matrix of N with respect to $(e_1, \ldots, e_n)$. Obviously it is a constant matrix. Making a "constant" change of basis we may finally assume that N is composed of blocks along the main diagonal having one of both shapes

$$(0), \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

The proof will be finished when we have shown that the latter block in fact does not appear. So we only have to consider the differential operator

$$D = \vartheta + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

of K^2 into itself and to show that it is not semisimple. If D were semisimple, the subspace $Ke_1\big((e_1, e_2)$ being the canonical basis of $K^2\big)$ would have an invariant complement W, necessarily of dimension 1. It is easy to see that $D_{|W}$ is simple of type 0. So W contains an element $f \neq 0$ which satisfies $Df=0$. When $f=f_1 e_1+f_2 e_2$, we have

$$\vartheta f_1 + f_2 = 0, \quad \vartheta f_2 = 0.$$

But these equations have $f_1=f_2=0$ as unique solution. This is a contradiction.

6. Applications

a. *Jordan form*

Let again be D: $V \to V$ a differential operator and V an n-dimensional vector space over $K=k((t))$. According to Theorem I we can write $D=S+N$, where S is a semisimple differential operator, N a K-linear nilpotent map and S commutes with N. In virtue of Theorem II there exists a finite extension L of K such that S_L is diagonalizable. We may assume $L=k'((\sqrt[m]{t}))$, $k \subset k'$ being a finite extension and m a positive integer. We shall denote $\sqrt[m]{t}$ by s.

We write V_L as the direct sum of its isotypical components $V_{\varphi_1}, \ldots, V_{\varphi_l}$ with respect to S. Now consider one isotypical component V_φ. Let $a \in L$ be a polynomial in $1/s$ such that $[a]=\varphi$. Then there is a L-basis $(e_1, \ldots, e_r)$ in V_φ such that Mat $(S_{|V_\varphi}; (e))=aI$. Write N_e to denote Mat$\big(N_{|V_\varphi}, (e)\big)$. Since $N(V_\varphi) \subset V_\varphi$ (cf. section 4, Lemma) and S and N commute, we have

$$\vartheta N_e = 0.$$

So N_e is a constant nilpotent matrix. By a "constant" change of basis in V_φ which does not affect the matrix of $S_{|V_\varphi}$ we can get

$$N_e = \begin{pmatrix} 0 & * & 0 & & 0 \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & 0 \\ & & & \ddots & * \\ 0 & & & & 0 \end{pmatrix},$$

the asterisks denoting 0 or 1. The matrix representation of $D_{|V_\varphi}$ is

$$\frac{1}{m}s\frac{d}{ds}+\begin{pmatrix} a & * & & 0 \\ & \ddots & \ddots & \\ & & \ddots & * \\ 0 & & & a \end{pmatrix},$$

and proceding in the same way with all isotypical components one gets a matrix representation

$$\frac{1}{m}s\frac{d}{ds}+\begin{pmatrix} \boxed{\begin{matrix} a_1 & * & & 0 \\ & \ddots & \ddots & \\ & & \ddots & * \\ 0 & & & a_1 \end{matrix}} & & & 0 \\ & \boxed{\begin{matrix} a_2 & * & & 0 \\ & \ddots & \ddots & \\ & & \ddots & * \\ 0 & & & a_2 \end{matrix}} & & \\ & & \ddots & \\ 0 & & & \boxed{\begin{matrix} a_l & * & & 0 \\ & \ddots & \ddots & \\ & & \ddots & * \\ 0 & & & a_l \end{matrix}} \end{pmatrix},$$

$a_1, \ldots, a_l$ being polynomials in $1/s$ and $a_i - a_j \in \mathfrak{O}_L^{\cdot}$ implying $i=j$.

This result should be compared with [1], §3, Theorem I.

b. *Characteristic polynomial*

Using the above notations we write $p(\varphi)$ to denote the principal part of a, where $a \in L$ represents φ. We shall show that the *characteristic polynomial.*

$$c(X) = \prod_\varphi (X - p(\varphi))^{\nu(\varphi)},$$

φ running through the different types of S_L and $\nu(\varphi)$ indicating $\dim_L V_\varphi$, has its coefficients in K.

Let $\varphi_1, \ldots, \varphi_l$ be the different types, represented by $a_1, \ldots, a_l$, respectively, as before. Let $K \subset M$ be a finite Galois extension containing L. Then $a_1, \ldots, a_l$ represent the types of S_M. Hence, there is no restriction in assuming $K \subset L$ Galois. When $\sigma \in \mathrm{Gal}(L/K)$, $\sigma(V_{\varphi_i})$ is an isotypical component of V_L. So we have $\sigma(V_{\varphi_i}) = V_{\varphi_j}$ for some j and $\sigma(a_i) - a_j \in \mathfrak{O}_L^{\cdot}$, implying $\sigma(p(\varphi_i)) = p(\varphi_j)$ and

$$\nu(\varphi_i) = \dim_L V_{\varphi_i} = \dim_L \sigma(V_{\varphi_i}) = \dim_L V_{\varphi_j} = \nu(\varphi_j).$$

This shows that $p(\varphi_1), \ldots, p(\varphi_l)$ are permuted by all $\mathrm{Gal}(L/K)$, and that the multiplicities are conserved; consequently $c(X) \in K[X]$.

It is easy to see that $c(X)$ depends only on S (and not on the extension L). Since S is uniquely determined by D there is no objection to calling $c(X)$ the *characteristic polynomial of D*.

c. *Splitting field*

We try to find the smallest extension $K \subset L$ such that S_L is diagonalizable.

In order to simplify the investigations we shall assume here that k is algebraically closed. It is well-known (cf. [7], Chap. IV, §2, Prop. 8) that $[L:K] = m$ implies that $K \subset L$ is a cyclic Galois extension, $L = K((\sqrt[m]{t}))$, and $\sigma\colon s \to \xi s$ is a generator of the Galoisgroup (we have put $s = \sqrt[m]{t}$ and ξ primitive m-th root of unity).

Using the notations of a. and b. we first show the follwing: If $p(\varphi_1), \ldots, p(\varphi_l)$ are in K, then S is diagonalizable over K.

Since k is algebraically closed, we see that $a_1, \ldots, a_l \in K$. As before for any $\tau \in \mathrm{Gal}(L/K)$ we have $\tau(V_{\varphi_i}) = V_{\varphi_j}$ for some $j (1 \leqq j \leqq l)$, implying $\tau(a_i) - a_j \in \mathfrak{O}_L^{\cdot}$. However, $a_i \in K$, thus $\tau(a_i) = a_i$, whence $a_i - a_j \in \mathfrak{O}_L^{\cdot}$ and $i = j$, a_i representing φ_i and all types $\varphi_1, \ldots, \varphi_l$ being different. We have shown that the isotypical components are invariant under $\mathrm{Gal}(L/K)$; consequently they "come from" invariant subspaces of V, i.e. for every i there exists a K-subspace W_i of V, invariant under D, such that $V_{\varphi_i} = L \otimes_K W_i$.

Replacing V by W_i we have reduced our assertion to the special case where S_L is isotypical of type φ, $p(\varphi) \in K$. Replacing S by $S - aI$, where $[a] = \varphi$, we may moreover assume that S_L is isotypical of type 0. This means that an L-basis $(f_1, \ldots, f_n)$ of V_L exists such that $\mathrm{Mat}(S_L, (f)) = 0$. We try to find a K-basis $(e_1, \ldots, e_n)$ of V such that $A = \mathrm{Mat}(S, (e))$ is diagonal. Let $(g_1, \ldots, g_n)$ be any K-basis of V and let T be the matrix (elements in L) connecting to two bases:

$$(f_1, \ldots, f_n) = (g_1, \ldots, g_n) T.$$

Then

$$BT + \vartheta T = 0, \quad \text{where} \quad B = \mathrm{Mat}(S, (g)).$$

Appliyng σ we find

$$B\sigma(T) + \vartheta\sigma(T) = 0.$$

These formulas can be considered as expressing the fact that the columns of T resp. $\sigma(T)$ are fundamental systems of solutions of the same system of differential equations. So there exists a non-singular matrix P, elements in k, satisfying

$$\sigma(T) = TP.$$

As $\sigma^m = 1$, we have $P^m = I$, and there exist a constant non singular matrix R and integers $\nu_1, \ldots, \nu_n$ verifying

$$P = RZR^{-1}, \quad Z = \begin{pmatrix} \xi^{\nu_1} & & 0 \\ & \ddots & \\ 0 & & \xi^{\nu_n} \end{pmatrix}.$$

Define

$$U = TRW, \quad W = \begin{pmatrix} s^{-\nu_1} & & 0 \\ & \ddots & \\ 0 & & s^{-\nu_n} \end{pmatrix}.$$

Then it can easily be verified, that U is invariant under $\mathrm{Gal}(L/K)$ showing that U is a (non-singular) matrix with elements in K. Define

$$(e_1, \ldots, e_n) = (g_1, \ldots, g_n)\, U.$$

Then the matrix A of S with respect to $(e_1, \ldots, e_n)$ satisfies

$$\begin{aligned} A &= U^{-1}BU + U^{-1}\vartheta U \\ &= W^{-1}R^{-1}T^{-1}BTRW + W^{-1}R^{-1}T^{-1}\vartheta(TRW) \\ &= W^{-1}\vartheta W \\ &= \begin{pmatrix} -\nu_1 & & 0 \\ & \ddots & \\ 0 & & -\nu_n \end{pmatrix} \end{aligned}$$

(We have used the relation $BT + \vartheta T = 0$). This proves our assertion.

Proposition. *If k is algebraically closed and if $c(X) \in K[X]$ is the characteristic polynomial of the differential operator D: $V \to V$, a Jordan decomposition* (1) *exists over the splitting field L of $c(X)$ over K.*

Remark 1. Every finite extension of K being Galois and of the type $k((\sqrt[m]{t}))$, it is obviously sufficient to adjoin to K the m-th roots of t, $1 \leqq m \leqq n$. In other words a Jordan decomposition exists over $k((\sqrt[p]{t}))$, where

$$p = l \cdot c \cdot m \cdot \{m \mid 1 \leqq m \leqq n\}.$$

Remark 2. Besides the characteristic polynomial a minimal polynomial $m(X)$ of D can be defined in an obvious way. Again it can be shown that $m(X) \in K[X]$.

and a Jordan decomposition already exists over the splitting field of $m(X)$. However, in general degree $m(X) \leqq n$ is all we know in advance.

Proof. It suffices to show that the semisimple component S of D is diagonalizable over L. S is diagonalizable over some finite extension of M in virtue of Theorem II. On the other hand L contains the principal parts of the types, and so S is already diagonalizable over L as we have just proved.

d. *Invariants*

In [4] invariants $\varrho_1, \varrho_2, \dots$ have been defined for the differential operator D: $V \to V$ in the following way. When $\Lambda \subset V$ is a lattice with respect to $\mathfrak{O}$, q is a positive integer and $D_q = t^{q-1}D$, then

$$\dim_k (\Lambda + D_q(\Lambda) + \dots + D_q^{\nu+1}(\Lambda))/(\Lambda + D_q(\Lambda) + \dots + D_q^{\nu}(\Lambda))$$

is independent of ν when $\nu \to \infty$. The limit does not depend on Λ and is denoted by ϱ_q.

We decribe now the relation to the theory developed here, and we give some indications of proofs.

First suppose that D has a Jordan decomposition (1) over K. We take the basis used in this decomposition as a basis for a lattice Λ. Then it can easily be shown that

$$\varrho_q = -\sum{}' \nu_i v(t^{q-1}a_i),$$

where the summation is over those i, $1 \leqq i \leqq l$, satisfying $v(t^{q-1}a_i) < 0$ (v is the valuation on K). This leads to the formula

$$\varrho_q = \sup_{0 \leqq i \leqq n} (-v(t^{(q-1)i}c_i)),$$

where

$$c(X) = X^n + c_1 X^{n-1} + \dots + c_n$$

is the characteristic polynomial of D and $c_0 = 1$. It can be shown that this formula still holds if D has no Jordan decomposition over K.

It can also be shown that the number r defined by N. Katz ([3], Chap. I, Théorème 1.9) satisfies

$$r = \sup\left(0, \sup_{1 \leqq i \leqq n} \left(\frac{-v(c_i)}{i}\right)\right).$$

In the Jordan decomposition (1) let

$$a_i = a_{i0} + \frac{a_{i1}}{s} + \dots + \frac{a_{i\mu}}{s^\mu}$$

be the term with the highest order pole ($a_{i\mu} \neq 0$), and supopse $s = \sqrt[m]{t}$. Then $r = \mu/m$.

References

1. Turrittin, H. L. Convergent solutions of ordinary linear homogeneous differential equations in the neighborhood of an irregular singular point. *Acta Math.* **93** (1955), pp. 27—66.
2. Wasow, W. Asymptotic expansions for ordinary differential equations: Trends and problems. Proceedings of a Symposium conducted by the Mathematics Research Center, United States Army, at the University of Wisconsin, Madison 1964. Edited by Calvin H. Wilcox. John Wiley and Sons. Inc. 1964.
3. Deligne, P. *Equations différentielles à points singuliers réguliers.* Lecture Notes in Mathematics, **163** (1970).
4. Gérard, R. et Levelt, A. H. M. Invariants mesurant l'irrégularité en un point singulier des systèmes d'équations différentielles linéaires. *Ann. Inst. Fourier,* **23** (1973), pp. 157—195.
5. Bourbaki, N. *Algèbre, Chap. 8. Modules et anneaux semi-simples,* Hermann, 1958.
6. Wasow, W. *Asymptotic expansions for ordinary differential equations.* Interscience Publishers, 1965.
7. Serre, J-P. *Corps locaux,* Hermann, Paris, 1962.

Received June 10, 1974

A. H. M. Levelt
Mathematisch Institut
Toernooiveld
Nijmegen
The Netherlands