Weak sequential convergence in the dual of a Banach space does not imply norm convergence

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We shall prove that for every infinite-dimensional Banach space E there is a sequence in E', the dual space, which tends to 0 in the weak topology $\sigma(E', E)$ but not in the norm topology. This is well known for separable or reflexive Banach spaces. See also [3] for other examples. The theorem has its main applications in the theory of holomorphic functions on infinite-dimensional topological vector spaces (TVS).

Let l^∞ be the Banach space of all complex-valued, bounded functions on the natural numbers $\mathbf{N}$; $z=(z_j)_{j=1}^\infty$ denotes a point in l^∞. Let c_0 be the Banach space $c_0=\{z\in l^\infty;\, z_j\to 0 \text{ as } j\to\dots\infty\}$, $c=\{z\in l^\infty;\, \lim_{j\to\infty} z_j \text{ exists}\}$ and $l^1=\{z\in c_0,\, \sum_{j=1}^\infty |z_j|<<\infty\}$. Let $L(F, F_1)$ denote the set of all bounded linear mappings from F into F_1 and let $\mathfrak{H}(F)$ denote the set of Gâteaux-analytic, locally bounded functions on F, where F and F_1 are locally convex TVS. See [5]. A set $B\subset F$ is called *bouding* if $\sup_{z\in B} |f(z)|<\infty$ for every $f\in\mathfrak{H}(F)$. Put $\mathfrak{H}_b(F)=\{f\in\mathfrak{H}(F);\, f \text{ is bounded on bounded subsets of } F\}$.

Theorem. *To every infinite-dimensional Banach space E there exist $\varphi_j\in E'$ such that $\|\varphi_j\|=1$ and $\lim_{j\to\infty}\varphi_j(z)=0$ for every $z\in E$.*

Corollary 1. *No neighbourhood of $0\in F$, where F is a locally convex* TVS, *is a bounding set.*

Proof. See [2].

Corollary 2. *$\mathfrak{H}_b(E)\neq\mathfrak{H}(E)$ for very infinite-dimensional Banach space E.*

Proof. Se [2].

Proof of the Theorem. Let $F\subset E$ be a separable, infinite-dimensional subspace. From [1] and [2] it follows that there are $z^{(j)}\in F$ and $\psi_j\in E'$ such that $\|\psi_j\|=1$, $\|z^{(j)}\|=1$, $\psi_j(z^{(j)})=1$ and $\lim_{j\to\infty}\psi_j(z)=0$ for every $z\in F$. Let $\psi\in L(E, l^\infty)$ be the mapping $\psi(z)=(\psi_1(z), \psi_2(z), \ldots, \psi_j(z), \ldots)$. Put $D=\psi(B_E)$ where B_E is the closed unit ball in E. We shall say that E has *property A* if there are linear functionals as in the theorem. We recall that l^∞ has property A which follows from the fact that there is $\varphi\in L(l^\infty, l^2(B))$, where card $B=$card $\mathbf{R}$ and $l^2(B)$ is the Hilbert space on B, such that φ is onto [6]. In the rest of the proof we shall prove that if there is no $\varphi_0\in L(l^\infty, l^\infty)$ such that $\varphi_0(D)$ is separable and not compact then D is like the unit ball in l^∞ in the sense that we may use a technique to prove that E has property A which is similar to that used to prove that l^∞ has property A. More explicitly, if there is no φ_0 as above Lemma 6 gives that X_n in the Proposition may be taken such that (X_n) is not dominated by a geometric series and then the sequence of mappings (φ_n) in the Proposition and Lemma 4 replace $\varphi\in L(l^\infty, l^2(B))$. If, on the other hand, there is $\varphi_0\in L(l^\infty, l^\infty)$ such that $\varphi_0(D)$ is separable but not compact then it follows trivially that E has property A.

Definition 1. Put, for $z\in l^\infty$ and $M\subset\mathbf{N}$, supp $z=\{j\in\mathbf{N}; z_j\neq 0\}$ and $\mathrm{Proj}_{[M]}\, z= =(z'_j)_{j\in\mathbf{N}}$ where $z'_j=z_j$ if $j\in M$ and $z'_j=0$ if $j\notin M$. Let $l^\infty(M)=\{z\in l^\infty; z_j=0$ if $j\notin M\}$.

Definition 2. Put, for $z\in l^\infty$ and $M\subset\mathbf{N}$, $N_M(z)=\overline{\lim}_{j,k\to\infty, j,k\in M} |z_j-z_k|$ ($N_M(z)= =0$ if M is finite).

Definition 3. A set $A\subset l^\infty$ is called a 1-*set* if for all finite subsets $\{a^{(1)}, \ldots, a^{(k)}\}$ of A the vector of components $(a_j^{(1)}, \ldots, a_j^{(k)})\in\mathbf{C}^k$ assumes exactly the values $(\pm 1, \pm 1, \ldots, \pm 1)$ for all possible 2^k choices of signs.

Definition 4. Let $\{a^{(k)}\}_{k=1}^\infty$ be a 1-set and r a positive integer. Let $\{M_j(r, \{a^{(k)}\}); j=1, 2, \ldots, 2^{r-1}\}$ be the partitioning of $\mathbf{N}$ into 2^{r-1} disjoint parts such that $(a_s^{(1)}, \ldots \ldots, a_s^{(r-1)})=(a_l^{(1)}, \ldots, a_l^{(r-1)})$ if and only if $s, l\in M_j(r, \{a^{(k)}\})$ for some j. Put $M(1, \{a^{(k)}\})=\mathbf{N}$.

We note that $\|\sum_{k=1}^\infty \lambda_k a_k\|\geq\frac{1}{2}\sum_{k=1}^\infty|\lambda_k|$ if $\{a_k\}$ is a 1-*set*. In fact, C. O. Kiselman has proved that the constant 1/2 can be replaced by $2/\pi$ and this is best possible.

Lemma 1. *If E does not have property A there exist an infinite set $V\subset\mathbf{N}$ and a number $\varepsilon>0$ such that for every infinite $U\subset V$,* $\sup_{z\in D} N_U(z)>\varepsilon$.

Proof. Assume that the lemma is false. Then there are infinite sets U_j such that $U_j\subset U_{j-1}\not\subset U_j$ and $\sup_{z\in D} N_{U_j}(z)<2^{-j}$. There is an infinite set $U\subset\mathbf{N}$ such that $U\cap\complement U_j$ is finite for every $j\in\mathbf{N}$. Hence $\sup_{z\in D} N_U(z)=0$ which is a contradiction.

We may assume that $V=\mathbf{N}$. Let $e\in l^\infty$ be $\{1, 1, \ldots, 1, \ldots\}$.

Lemma 2. *There exist an index set B, $H_k\subset B$, $\varphi\in L(l^\infty, l^2(B))$ where $l^2(B)$ is the Hilbert space on B, $C_1>0$, $C_2>0$ and a 1-set $\{a^{(k)}\}_{k=1}^\infty\subset l^\infty$ such that* card $B=$

$=\operatorname{card}\mathbf{R}$, $B\setminus H_k$ *is finite,*

$$H_k\subset H_{k-1}\subset\ldots\subset H_0=B,\quad \|\varphi\|<C_2,\quad \|\mathrm{Proj}_{[H_{k-1}\setminus H_k]}\varphi(a^{(k)})\|>C_1,$$

$$\sum_j \|\mathrm{Proj}_{[H_k]}\varphi(\mathrm{Proj}_{[M_j(k+1,\{a^{(r)}\})]}e\|<10^{-4}\cdot k^{-k-1}\cdot\varepsilon\cdot C_1$$

and $\varphi(e)=\varphi(z)=0$ *if* $z\in c_0$. *Here* ε *is the constant in Lemma 1.*

Proof. From [6] it follows that there is $\varphi_1\in L(l^\infty, l^2(B))$ such that φ_1 is onto. Since $\operatorname{card}B>\operatorname{card}\mathbf{N}$ it follows there are $C_1>0$, $b^{(k)}\in l^\infty$ and $H_k\subset B$ such that $H_k\subset H_{k-1}$, $B\setminus H_k$ is finite,

$$\|b^{(k)}\|<\tfrac12\quad\text{and}\quad \|\mathrm{Proj}_{[H_{k-1}\setminus H_k]}\varphi_1(b^{(k)}\|>C_1.$$

Let $\{a^{(k)}\}_{k=1}^\infty$, be a 1-*set* and $F\subset l^\infty$ the subspace generated by $a^{(k)}$ and c. Then $a\in F$ if and only if $a=\sum_{k=1}^\infty\lambda_k a^{(k)}+x$ where $x\in c$ and $\lambda=(\lambda_1,\ldots,\lambda_k,\ldots)\in l^1$. Let $\psi\in L(F, l^\infty)$ be the mapping defined by $\psi(a^{(k)})=b^{(k)}$ and $\psi(x)=0$ if $x\in c$. We have $\|\psi\|\leqq 1$ because

$$\Big\|\sum_{k=1}^\infty\lambda_k\cdot a^{(k)}+x\Big\|\geqq\tfrac12\sum_{k=1}^\infty|\lambda_k|\quad\text{if}\quad x\in c.$$

But l^∞ has the norm preserving extension property hence ψ can be extended to $\psi_1\in L(l^\infty, l^\infty)$ such that $\|\psi_1\|=\|\psi\|$. Put $\varphi=\varphi_1\circ\psi_1$.

Assume now that we have found $(j_k)_{k=1}^s\subset\mathbf{N}$, where $s\in\mathbf{N}$, and $(H_k')_{k=1}^s\subset B$ such that $H_{k-1}'\supset H_k'$, $H_k'=H_{j_{p_k}}$ for some $j_{p_k}\in\mathbf{N}$,

$$\|\mathrm{Proj}_{[H_{k-1}'\setminus H_k']}\varphi(a^{(j_k)})\|>C_1\quad\text{if}\quad k\leqq s$$

and

$$\sum_l\|\mathrm{Proj}_{[H_k']}\varphi(\mathrm{Proj}_{[M_l(k+1,\{a^{(j_n)}\})]}e)\|<\frac{C_1\cdot\varepsilon}{10^4\cdot k^{k+1}}\quad\text{if}\quad k\leqq s.$$

Choose now $j_{s+1}\in\mathbf{N}$ and then $H_{s+1}'=H_{j_{p_{s+1}}}$ for some $j_{p_{s+1}}\in\mathbf{N}$ such that

$$\|\mathrm{Proj}_{[H_s'\setminus H_{s+1}']}\varphi(a^{(j_{s+1})})\|>C_1$$

and

$$\sum_l\|\mathrm{Proj}_{[H_{s+1}']}\varphi(\mathrm{Proj}_{[M_l(s+2,\{a^{(j_n)}\})]}e)\|<\frac{C_1\cdot\varepsilon}{10^4\cdot(s+1)^{s+2}}$$

which of course is possible according to the construction of φ and the fact that a vector in $l^2(B)$ is "small" outside a finite subset of B. Hence $\{a^{(j_k)}\}_{k=1}^\infty$ and (H_k') have the desired properties. Q.E.D.

Lemma 3. *Let $j\in\mathbf{N}$ be a fixed number and φ, $\{a^{(k)}\}_{k=1}^{\infty}$ and (H_k) be as in Lemma 2 and $z\in l^{\infty}$ be such that $\|z\|<2$ and*

$$\|\mathrm{Proj}_{[H_j\setminus H_k]}\varphi(z)\| > C_3 > \frac{2^{-j}\cdot\varepsilon\cdot C_1}{5}$$

for some $k>j$. Then there exist an infinite set $V\subset\mathbf{N}$, $\varphi'\in L(l^{\infty}(V),l^{\infty})$, a 1-set $\{b^{(r)}\}_{r=1}^{\infty}\subset l^{\infty}(V)$ and $h_{s,r}\in\mathbf{C}$ such that

$$\Big\|\mathrm{Proj}_{[V]}z-he-\sum_{r=1}^{j+1}\sum_s h_{r,s}\,\mathrm{Proj}_{[M_s(r,\{b^{(l)}\})]}b^{(r)}\Big\| < \frac{\varepsilon\cdot j^{-j}\cdot C_1}{10^4\cdot C_2},$$

$$\varphi'(\mathrm{Proj}_{[M_s(r,\{b^{(l)}\})]}b^{(r)}) = \mathrm{Proj}_{[M_s(r,\{a^{(l)}\})]}a^{(r)} \quad \text{if} \quad r\leqq j$$

$$\varphi'(e)=e,\quad \varphi'(z)\in c_0 \quad \text{if} \quad z\in c_0,\quad \varphi'(b^{(j+r)})=a^{(k+r)}r>1$$

and

$$\|\mathrm{Proj}_{[H_j\setminus H_k]}\varphi\circ\varphi'(z)\| > \frac{C_3}{32}.$$

Proof. It is an immediate consequence of the definition of $N_{M_s(j+1,\{a^{(l)}\})}(z)$ that there exist an infinite set $V_s\subset M_s(j+1,\{a^{(l)}\})$, a 1-point $\delta^s\in l^{\infty}(V_s)$, $p_{1,s}\in\mathbf{C}$ and $h_{j+1,s}\in\mathbf{C}$ such that

$$|h_{j+1,s}| = \tfrac{1}{2}N_{M_s(j+1,\{a^{(l)}\})}(z),$$

$$\|\mathrm{Proj}_{[V_s]}(z-p_{1,s}\cdot e-h_{j+1,s}\cdot\delta^s)\| < \frac{\varepsilon\cdot j^{-j}\cdot C_1}{10^4\cdot C_2} \quad \text{and} \quad N_{V_s}(z-h_{j+1,s}\delta^s)=0.$$

Put $V=\bigcup_s V_s$ and $a_0^{(j+1)}=\sum_s\delta^s$. It is obvious that we can take $\{a_0^{(r)}\}_{r=j+2}^{\infty}\subset\subset l^{\infty}(V)$ such that $\{a_0^{(r)}\}_{r=1}^{\infty}$ is a 1-*set* in $l^{\infty}(V)$ where $a_0^{(r)}=\mathrm{Proj}_{[V]}a^{(r)}$ if $r\leqq j$. Since $N_{M_s(j+1,\{a^{(l)}\})}(z-h_{j+1,s}a_0^{(j+1)})=0$ it follows from the definition that we can find $h_{j,s}\in\mathbf{C}$ and $p_{2,s}\in\mathbf{C}$ such that

$$N_{M_s(j,\{a_0^{(l)}\})}\big(\underbrace{z-\sum_t h_{j+1,t}\cdot\mathrm{Proj}_{[M_t(j+1,\{a_0^{(l)}\})]}a_0^{(j+1)}-h_{j,s}a_0^{(j)}}_{b_s}\big)=0$$

and

$$\|\mathrm{Proj}_{[M_s(j,\{a_0^{(l)}\})]}(z-b_s-p_{2,s}e\| < \frac{\varepsilon\cdot j^{-j}\cdot C_1}{10^4\cdot C_2}.$$

In the same way we may continue and after $j+1$ steps we get that there are $h_{r,s}\in\mathbf{C}$ and $h\in\mathbf{C}$ such that

$$\text{(1)}\qquad \Big\|\mathrm{Proj}_{[V]}\big(z-he-\sum_{r=1}^{j+1}\sum_s h_{r,s}\cdot\mathrm{Proj}_{[M_s(r,\{a_0^{(l)}\})]}a_0^{(r)}\Big\| < \frac{\varepsilon\cdot j^{-j}\cdot C_1}{10^4\cdot C_2}.$$

$|h_{r,s}|<2$ because $\{a_0^{(l)}\}_{l=1}^{\infty}$ is a 1-set and because $\|z\|<2$.

In the same manner it follows that there are $h'\in\mathbf{C}$, $h'_{r,s}\in\mathbf{C}$ and $z'\in l^\infty$ such that $|h'|<2$, $|h'_{r,s}|<2$,

$$z = z'+h'e+\sum_{r=1}^{j}\sum_s h'_{r,s}\cdot \mathrm{Proj}_{[M_s(r,\{a^{(l)}\})]}a^{(r)}$$

and

$$\varlimsup_{\substack{t\to\infty\\ t\in M_s(j+1,\{a^{(l)}\})}} |z'_t| = \tfrac{1}{2}N_{M_s(j+1,\{a^{(l)}\})}(z).$$

Lemma 2 gives that

$$\|\mathrm{Proj}_{[H_j\setminus H_k]}\varphi(z')\| > C_3-\frac{C_1\cdot\varepsilon\cdot j^{-j}}{10^4} > \frac{C_3}{2}.$$

Let $\{d^{(l)}\}_{l=1}^\infty\subset l^\infty(V)$ be a 1-*set* such that there are disjoint infinite sets $U_{j,l}^{(s)}\subset \subset M_l(j+1,\{d^{(r)}\})$ such that $\bigcup_{s=1}^4 U_{j,l}^{(s)}=M_l(j+1,d^{(r)})$ and

$$\{\mathrm{Proj}_{[U_{j,l}^{(s)}]}d^{(r)}\}_{r=j+1}^\infty$$

is a 1-*set* for every l and s. Let $z''=\{z''_t\}_{t\in V}$ be such that $z''_t=(i)^s\cdot|h_{j+1,l}|$ if $t\in U_{j,l}^{(s)}$. Let $F\subset l^\infty(V)$ be the subspace which is generated by z'', $d^{(l)}$, $\mathrm{Proj}_{[M_s(r,\{d^{(l)}\})]}d^{(r)}$ $1\leqq r\leqq j$ and c. $b\in F$ if and only if

$$b = \gamma_0\cdot z''+x+\sum_{r=1}^{j}\sum_s\gamma_{r,s}\cdot\mathrm{Proj}_{[M_s(r,\{d^{(l)}\})]}d^{(r)}+\sum_{k=j+1}^{\infty}\lambda_k d^{(k)}$$

where $x\in c$ $\gamma_0\in\mathbf{C}$, $\gamma_{r,s}\in\mathbf{C}$ and $\lambda=\{\lambda_{j+1},\dots,\lambda_{j+k},\dots\}\in l_1$. Let $\psi\in L(F,l^\infty)$ be the mapping $\psi(z'')=z'/2$, $\psi(e)=e$, $\psi(d^{(r)})=a^{(r)}$, $\psi(z)=0$ if $z\in c_0$ and

$$\psi(\mathrm{Proj}_{[M_s(r,\{d^{(l)}\})]}d^{(r)} = \mathrm{Proj}_{[M_s(r,\{a^{(l)}\})]}a^{(r)},$$

$r\leqq j$. It is easy to check that $\|\psi\|=1$. But to has the norm preserving extension property hence ψ can be extended to $\psi_1\in L(l^\infty(V),l^\infty)$ such that $\|\psi_1\|=1$. We may assume that

$$\|\mathrm{Proj}_{[H_j\setminus H_k]}\varphi\circ\psi_1(\mathrm{Proj}_{[\bigcup_l U_{j,l}^{(2)}\cup U_{j,l}^{(4)}]}z'')\| > \frac{C_3}{8}.$$

(Since otherwise

$$\|\mathrm{Proj}_{[H_j\setminus H_k]}\varphi\circ\psi_1(\mathrm{Proj}_{[\bigcup_l U_{j,l}^{(1)}\cup U_{j,l}^{(3)}]}z'')\| > \frac{C_3}{8}.\Big)$$

Hence there is $t=(t_1,t_2,\dots)$ where $t_j=1$ or -1 such that

$$(2)\quad \Big\|\mathrm{Proj}_{[H_j\setminus H_k]}\varphi\circ\psi_1\big(\mathrm{Proj}_{[\bigcup_l U_{j,l}^{(2)}\cup U_{j,l}^{(4)}]}\big(\textstyle\sum_s t_s h_{j+1,s}\,\mathrm{Proj}_{[M_s(j+1,\{b_0^{(l)}\})]}b_0^{(j+)}\big)\Big\| > \frac{C_3}{16},$$

where $b_0^{(r)}=d^{(r)}$ if $r\neq j+1$ and $b_0^{(j+1)}=\{x_t\}_{t\in V}$ is such that $x_t=-1$ if $t\in U_{l,j}^{(2)}$, $x_t=1$ if $t\in U_{,j}^{(4)}$, and $x_t=0$ elsewhere.

Let $G\subset l^\infty(V)$ be the subspace which is generated by c, $\{a_0^{(r)}\}_{r=1}^\infty$ and

$$\mathrm{Proj}_{[M_s(r,\{a_0^{(l)}\})]}a_0^{(r)} \quad\text{if}\quad r\leqq j+1.$$

As above it follows that there is $J \in L(G, l^\infty(V))$ which can be extended to $J_1 \in L(l^\infty(V), l^\infty(V))$ such that $\|J\| = \|J_1\| = 1$, $J(e) = e$, $J(z) = 0$ if $z \in c_0$, $J(a_0^{(r)}) = b_0^{(r)}$ if $r \neq j+1$,

$$J(\mathrm{Proj}_{[M_s(r,\{a_0^{(i)}\})]} a_0^{(r)} = \mathrm{Proj}_{[M_s(r,\{b_0^{(i)}\})]} b_0^{(r)} \quad \text{if} \quad r \leqq j$$

and

$$J(\mathrm{Proj}_{[M_s(j+1,\{a_0^{(i)}\})]} a_0^{(j+1)}) = t_s \cdot \mathrm{Proj}_{[M_s(j+1,\{b_0^{(i)}\})]} b_0^{(j+1)}.$$

But then (1) and (2) give that $\varphi' = \psi_1 \circ J_1$ and $b^{(r)} = a_0^{(r)}$ have the properties in the lemma. Q.E.D.

Proposition 1. *There are $\varphi_n \in L(l^\infty, l^2(B))$, $H_n \subset B$, $X_n > 0$ and $z^{(n)} \in D$ such that $X_n > C_1 \cdot \varepsilon \cdot 2^{-n}$ where C_1 is the constant in Lemma 2 and ε the constant in Lemma 1, $B \setminus H_n$ is finite*

$$H_n \subset H_{n-1} \subset \ldots \subset H_0 = B, \quad \sup_{z \in D} \|\mathrm{Proj}_{[H_{k-1}]} \varphi_n(z)\| \leqq X_k$$

and

$$\|\mathrm{Proj}_{[H_{k-1} \setminus H_k]} \varphi_n(z^{(k)})\| > X_k \cdot 10^{-2} \quad \text{if} \quad k \leqq n.$$

Proof. Let φ, $\{a^{(k)}\}_{k=1}^\infty$ and H_k be as in Lemma 2. Put $X_1 = \sup_\varphi \sup_{z \in D} \|\varphi \circ \varphi'(z)\|$ where φ' satisfies the following conditions.

a) There is an infinite set $V \subset \mathbf{N}$ such that $\varphi' \in L(l^\infty(V), l^\infty)$ and $\|\varphi'\| = 1$.

b) There are a 1-*set* $\{b^{(k)}\}_{k=1}^\infty \subset l^\infty(V)$ and $j \in \mathbf{N}$ such that $\varphi'(b^{(k)}) = a^{(j+k)}$ if $k > 1$.

c) $\varphi'(e) = e$ and $\varphi'(z) = 0$ if $z \in c_0$.

Take $\varphi_0' \in L(l^\infty(V), l^\infty)$ and $z^{(1)} \in D$ such that $\|\varphi \circ \varphi_0'(z^{(1)})\| > 32/50 X_1$ and such that φ_0' satisfies conditions a)—c) for a 1-*set* $\{b_0^{(k)}\}_{k=1}^\infty \subset l^\infty(V_0)$ and $j_0 \in \mathbf{N}$. We may assume that

$$\|\mathrm{Proj}_{[B \setminus H_{j_0}]} \varphi \circ \varphi'(z^{(1)})\| > \frac{32}{50} X_1,$$

since otherwise we just have to take a bigger j_0, omit finitely many $b_0^{(k)}$ and renumber. Now, if we assume $X_1 > C_1 \cdot \varepsilon$, a direct application of Lemma 3, where $\varphi \circ \varphi_0'$ correspond to φ and $(H_{j_0+k})_{k=1}^\infty$ correspond to $(H_k)_{k=1}^\infty$ in Lemma 3, we get that there are an infinite set $V_1 \subset V_0$, a 1-*set* $\{b_{(1)}^{(k)}\}_{k=1}^\infty \subset l^\infty(V_1)$ and $\varphi_1' \in L(l^\infty(V_1), l^\infty(V_0))$ $h^{(1)}$ and $h_1^{(1)} \in \mathbf{C}$ such that

$$\text{(1)} \qquad \|\mathrm{Proj}_{[V_1]}(z^{(1)} - h^{(1)} e - h_1^1 b_{(1)}^{(1)})\| < \frac{\varepsilon \cdot C_1}{10^4 \cdot C_2}$$

and $\psi_1 = \varphi_0' \circ \varphi_1'$ satisfies a)—c) with V, $\{b^{(k)}\}_{k=1}^\infty$ and j replaced by V_1, $\{b_{(1)}^{(k)}\}_{k=1}^\infty$ and $j_0 + 1$. That $X_1 > C_1 \cdot \varepsilon$ follows because there is $z^\circ \in D$ such that $N_{\mathbf{N}}(z^\circ) > 2 \cdot \varepsilon$, according to Lemma 1, hence there are an infinite set $V' \subset \mathbf{N}$ and a 1-*set* $\{a_0^{(k)}\}_{k=1}^\infty \subset l^\infty(V')$ such that $N_{V'}(z^\circ - h \cdot a_0^{(1)}) = 0$ for some $h \in \mathbf{C}$ such that $|h| > \varepsilon$. Assume now that there are, for every $t<n$, an infinite set $V_t\subset V_{t-1}$, $\psi_t\in L(l^\infty(V_t), l^\infty(V_{t-1}))$, $\varphi_t\in L(l^\infty(V_t, l^2(B))$, a *1-set* $\{b_{(t)}^{(k)}\}_{k=1}^\infty\subset l^\infty(V)$, $z^{(t)}\in D$, $j_t\in\mathbf{N}$, $X_t>C_1\cdot\varepsilon\cdot 2^{-t}$, $h^{(t)}\in\mathbf{C}$ and $h_{r,s}^{(t)}\in\mathbf{C}$ such that

$$\|\mathrm{Proj}_{[H_{j(t-1)}\setminus H_{j(t)}]}\varphi_t(z^{(t)})\| > \frac{X_t}{50}, \quad \text{where} \quad j(t) = \sum\nolimits_{r=1}^t j_r,$$

$$\sum\nolimits_s \|\mathrm{Proj}_{[H_{j(t)}]}\varphi_t(\mathrm{Proj}_{[M_s(t+1,\{b_{(t)}^{(l)}\})]}e)\| < \frac{\varepsilon\cdot t^{-t}\cdot C_1}{10^4\cdot C_2}$$

$$\psi_t(\mathrm{Proj}_{[M_s(r,\{b_{(t)}^{(l)}\})]}b_{(t)}^{(r)}) = \mathrm{Proj}_{[M_s(r,\{b_{(t-1)}^{(l)}\})]}b_{(t-1)}^{(r)} \quad \text{if} \quad r<t,$$

$$\psi_t(b_{(t)}^{(r)}) = b_{(t-1)}^{(j_t+r)} \quad \text{if} \quad r>t,$$

$$\left\|\mathrm{Proj}_{[V_t]}\left(z^{(t)}-h^{(t)}e-\sum\nolimits_{r=1}^t\sum\nolimits_s h_{r,s}^{(t)}\cdot\mathrm{Proj}_{[M_s(r,\{b_{(t)}^{(l)}\})]}b_{(t)}^{(r)}\right)\right\| < \frac{\varepsilon\cdot t^{-t}\cdot C_1}{10^4\cdot C_2}.$$

Then it follows, since $|h^{(t)}|$, $|h_{r,s}^{(t)}|<2$ because $\sup_{z\in D}\|z\|=1$, that

$$\|\mathrm{Proj}_{[H_{j(k-1)}\setminus H_{j(k)}]}\varphi_t(z^{(k)})\| > \frac{X_k}{100} \quad \text{if} \quad k\leqq t.$$

Put $X_n=\sup_\varphi\sup_{z\in D}\|\mathrm{Proj}_{[H_{j(n-1)}]}\varphi_{n-1}\circ\varphi'(z)\|$ where φ' satisfies the following conditions:

a′) There is an infinite set $V\subset V_{n-1}$ such that $\varphi'\in L(l^\infty(V), l^\infty(V_{n-1}))$ and $\|\varphi'\|=1$.

b′) There are a *1-set* $\{c^{(k)}\}_{k=1}^\infty\subset l^\infty(V)$ and $j\in\mathbf{N}$ such that $\varphi'(c^{(k)})=b_{(n-1)}^{(j+k)}$ if $k>n$ and

$$\varphi'(\mathrm{Proj}_{[M_s(r,\{c^{(l)}\})]}c^{(r)}) = \mathrm{Proj}_{[M_s(r,\{b_{(n-1)}^{(l)}\})]}b_{(n-1)}^{(r)} \quad \text{if} \quad r\leqq n-1.$$

c′) $\varphi'(e)=e$ and $\varphi'(z)=0$ if $z\in c_0$.

Take now $\varphi_0\in L(l^\infty(V_0), l^\infty(V_{n-1}))$, $j_0\in\mathbf{N}$ and $z^{(n)}\in D$ such that

$$\|\mathrm{Proj}_{[H_{j(n-1)}\setminus H_{j(n-1)+j_0}]}\varphi_{n-1}\circ\varphi_0(z^{(n)})\| > \frac{32}{50}X_n$$

and such that φ_0 satisfies the conditions a′)—c′) for a *1-set* $\{c_0^{(k)}\}_{k=1}^\infty\subset l^\infty(V_0)$. The existence of j_0 follows as before. Now, if we assume $X_n>C_1\cdot\varepsilon\cdot 2^{-n}$, an application of Lemma 3, where $\varphi_{n-1}\circ\varphi_0$ correspond to φ and $(H_{j(k)})_{k=1}^{n-1}\cup(H_{j(n-1)+j_0+k})_{k=0}^\infty$ correspond to $(H_k)_{k=1}^\infty$ in the Lemma, give that there are an infinite set $V_n\subset V_0$, a *1-set* $\{b_{(n)}^{(k)}\}_{k=1}^\infty\subset l^\infty(V_n)$ and $\varphi_0'\in L(l^\infty(V_n), l^\infty(V_0))$ such that $\psi_n=\varphi_0\circ\varphi_0'$, $\varphi_n==\varphi_{n-1}\circ\psi_n$, $V_n, j_n=j_0+1$, $\{b_{(n)}^{(k)}\}_{k=1}^\infty$ and $z^{(n)}$ satisfy the conditions 1)—5) for some $h^{(n)}\in\mathbf{C}$ and $h_{r,s}^{(n)}\in\mathbf{C}$. $X_n>C_1\cdot\varepsilon\cdot 2^{-n}$ because there is $s\in\mathbf{N}$ such that

$$\|\mathrm{Proj}_{[H_{j(n-1)}]}\varphi_{n-1}(\mathrm{Proj}_{[M_s(n,\{b_{(n-1)}^{(l)}\})]}b_{(n-1)}^{(n)})\| > C_1\cdot 2^{-n}$$

and because $\sup_{z\in D} N_{M_s(n,\{b^{(l)}_{(n-1)}\})}(z) > 2\cdot\varepsilon$

$$\|\mathrm{Proj}_{[H_{j(k-1)}\setminus H_{j(k)}]}\varphi_n(z^{(k)})\| > \frac{X_k}{100}, \quad \text{if} \quad k \leqq n,$$

because of b′) and 5). Hence φ_n, $H_{j(n)}$, X_n and $z^{(n)}$ have the desired properties. QED

Lemma 4. *E has property A if there exists to every given $t\in\mathbf{N}$ a mapping $\varphi_t\in L(l^\infty, \mathbf{C}^t)$ and to every given $\gamma>0$ a number $C_\gamma\in\mathbf{N}$ such that $\sup_{z\in D}|\mathrm{Proj}_{[n]}\varphi_t(z)|\geqq 1$ for every $n\in\{1, 2, \dots, t)$ and such that for every $z\in D$ and $t\in\mathbf{N}$ $|\mathrm{Proj}_{[n]}\varphi_t(z)|\geqq\gamma$ for at most C_γ different $n\in\{1, \dots, t\}$.*

Proof. Assume the lemma is false. It is easy to see that we may assume without loss of generality that $\sup_t\sup_n\sup_{z\in D}|\mathrm{Proj}_{[n]}\varphi_t(z)|<2$. It is well known and easily seen that there are uncountably many $g_\alpha\in G=U_1\times U_2\times\dots\times U_t\times\dots$, where $U_t=\{1, 2, \dots, t\}$, such that if $\alpha_1\neq\alpha_2$ $\mathrm{Proj}_{[j]}g_{\alpha_1}=\mathrm{Proj}_{[j]}g_{\alpha_2}$ for at mos finitely many j. Let $\varphi\in L(E, l^\infty(G))$ be the mapping $\varphi=\{\varphi_1\circ\psi, \varphi_2\circ\psi, \dots, \varphi_t\circ\psi, \dots\}$ where $\psi\in L(E, l^\infty)$ is the mapping in the beginning of the proof of the theorem. Since $\{g_\alpha\}_\alpha$ is uncountable it follows from the argument in the proof of Lemm a1 if E does not have property A that there are $\{\alpha_j\}_{j=1}^\infty\subset\{\alpha\}$, $\varepsilon_0>0$, infinite sets $V_j\subset V_{j-1}\subset\mathbf{N}$ and $z^{(j)}\in D$ such that

$$\lim_{\substack{t\to\infty \\ t\in\mathrm{Proj}_{[V_j]}g_{\alpha_j}}} \varphi(z^{(j)}) = \varepsilon_0.$$

Let $\psi_1\in L(l^\infty, \mathbf{C})$ be such that $\|\psi_1\|<1/\varepsilon_0$, $\psi_1(x^{(j)})=\frac{1}{2}$ and $\psi(z)=0$ if supp $z\cap V_j$ is finite for some $j\in\mathbf{N}$ where $x^{(j)}=(x_n^{(j)})_{n=1}^\infty$, $x_n^{(j)}=\varepsilon_0$ if $n\in V_j$ and $x_n^{(j)}=0$ if $n\in\complement V_j$. Let $\psi_{1,j}\in L(l^\infty(g_{\alpha j}), \mathbf{C})$ correspond to ψ_1. From the proof of Lemma 1 it follows, since $|\psi_{1,j}(\varphi(z^{(j)}))|=\frac{1}{2}$, that if E does not have property A there are a subsequence $\{j_k\}_{k=1}^\infty\subset\mathbf{N}$, $\delta>0$ and $z\in D$ such that $|\psi_{1,j_k}(\varphi(z))|>\delta$ for every $k\in\mathbf{N}$. Hence there are, to every $r\in\mathbf{N}$, $t_r\in\mathbf{N}$ and $Y_r\subset\mathbf{N}$ such that Y_r contains r elements and $|\mathrm{Proj}_{[n]}\varphi_{t_r}(\psi(z))|>\delta\cdot\varepsilon_0$ if $n\in Y_r$ since otherwise there are $z^{(j)}\in l^\infty$ such that supp $z^{(j_1)}\cap\text{supp } z^{(j_2)}=\emptyset$ if $j_1\neq j_2$ and $|\psi_1(z^{(j)})|>1$ for all $j\in\mathbf{N}$ which is impossible. Hence we get a contradiction if $r>C_{\gamma\cdot\varepsilon_0}$. QED.

Lemma 5. *If E does not have property A there exist to every $\gamma>0$ a number $C_\gamma\in\mathbf{N}$ and number $T_\gamma\in\mathbf{N}$ such that if $t\geqq C_\gamma$ and $\sup_{z\in D}\sum_{k=1}^{j_n} N_{M_k^{(n)}}(z)\geqq 1$, where $M_k^{(n)}\subset\mathbf{N}$, for every $n\in\{1, \dots, t\}$ then there are $z^{(\gamma)}\in D$ and $V\subset\{1, 2, \dots, t\}$ such that $\sum_{n\in V}\sum_{k=1}^{j_n} N_{M_k^{(n)}}(z^{(\gamma)})>2^{T_\gamma-\gamma T_\gamma}$ and such that V contains 2^{T_γ} elements.*

Proof. It is easily seen that it is enough to prove the lemma if $\sum_{k=1}^{j_n} N_{M_k^{(n)}}(z)$ is replaced by $|\mathrm{Proj}_{[n]}\varphi_t(z)|$ where $\varphi_t\in L(l^\infty, \mathbf{C}^t)$. From Lemma 4 it follows easily that there is to every $\gamma>0$ a number $P_\gamma\in N$ such that if $t\geqq P_\gamma$, $\varphi_t\in L(l^\infty, \mathbf{C}^t)$ and $\sup_{z\in D}|\mathrm{Proj}_{[n]}\varphi_t(z)|\geqq 1$ for every $n\in\{1, \dots, t\}$ then there are $T\in\mathbf{N}$ and $z\in D$ such that $|\mathrm{Proj}_{[n]}\varphi_t(z)| \geqq 2^{-\gamma\cdot T}$ for at least 2^T different $n \in \{1, \ldots, t\}$ wher$T$e perhaps depends on the choice of φ_t. Assume that $t \geqq (P_\gamma)^3$ and let T_0 be the biggest $T \in \mathrm{N}$ such that $2^T \leqq P_\gamma$. It follows from above that either there is $z^{(1)} \in D$ such that $|\mathrm{Proj}_{[n]}\varphi_t(z^{(1)})| \geqq 2^{-\gamma\cdot T_0}$ for at least 2^{T_0} different $n \in \{1, \ldots, t\}$ or there are $z^{r,s} \in D$ and disjoint sets $V_s^r \subset \{1, \ldots, t\}$, where $1 \leqq r \leqq T_0 - 1$ and $1 \leqq s \leqq j_r \in \mathrm{N}$ such that $\bigcup_{r=1}^{T_0-1} \bigcup_{s=1}^{j_r} V_s^r$ contains more than $t - P_\gamma$ elements, V_s^r contains 2^r elements or is emty and $|\mathrm{Proj}_{[n]}\varphi_t(z^{r,s})| \geqq 2^{-r\cdot\gamma}$ if $n \in V_s^r$. Hence there is

$$\varphi_1 \in L\Big(\mathrm{C}^{\bigcup_{s=1}^{j_1} V_s^1}, \mathrm{C}^{j_1}\Big)$$

such that $\|\varphi_1\| = 1$ and $|\mathrm{Proj}_{[s]}\varphi_{t,1}(z^{1,s})| \geqq 2^{1-\gamma}$ for every $s \in \{1, \ldots, j_1\}$ where $\varphi_{t,1} = \varphi_1 \circ \varphi_t$. But then it follows from above that there are $z^{r,s,1} \in D$ and disjoint sets $V_s^{r,1} \subset \{1, \ldots, t\}$ where $2 \leqq r \leqq T_0$ and $s \leqq i_r \in \mathrm{N}$ such that $\bigcup_{r=2}^{T_0} \bigcup_s V_s^{r,1}$ contains more than $t - P_\gamma - 2\cdot P_\gamma$ elements $V_s^{r,1}$ contains 2^r elements or is empty and $\sum_{n \in V_s^{r,1}} |\mathrm{Proj}_{[n]}\varphi_t(z^{r,s,1})| \geqq 2^{r - r\gamma}$. Since $t - \sum_{k=1}^{T_0} k \cdot P_\gamma > 0$ we can repeat this argument and we get at most T_0 steps that there are $z^{(1)} \in D$ and $V \subset \{1, \ldots, t\}$ such that $\sum_{n \in V} |\mathrm{Proj}_{[n]}\varphi_t(z^{(1)})| \geqq 2^{T_0 - \gamma T_0}$ where V contains 2^{T_0} elements. Hence $C_\gamma = (P_\gamma)^3$, $T_\gamma = T_0$, $z^{(\gamma)} = z^{(1)}$ and V have the properties in the lemma. QED.

Lemma 6. *If E does not have property A then, for every $\gamma > 0$, $\sup_{z \in D} \sum_{k=1}^{j_n} N_{M_k^n}(z) > 2^{-\gamma\cdot n}\cdot j_n$ if n is big enough wehre $M_k^{(n)} \subset \mathrm{N}$ are infinite sets such that for fixed n, $M_k^{(n)}$ are disjoint and $1 \leqq k \leqq j_n \leqq 2^n$.*

Proof. Take $\gamma_0 < \gamma$ and take C_{γ_0} and T_{γ_0} as in Lemma 5. Let $l_n \in \mathrm{N}$ be the greatest integer l such that $\sum_{r=1}^{l} C_{\gamma_0} \cdot 2^{(r-1)T_{\gamma_0}} \leqq j_n$. Repeated applications of Lemma 5 give, since $\sup_{z \in D} N_{M_k^n}(z) > \varepsilon$, that

$$\sup_{z \in D} \sum_{k=1}^{j_n} N_{M_k^n}(z) > 2^{-\gamma_0\cdot T_{\gamma_0}\cdot l_n} 2^{l_n\cdot T_{\gamma_0}} \varepsilon > 2^{-n\gamma_0} \cdot \frac{j_n \cdot \varepsilon}{C_{\gamma_0}} 2^{-T_{\gamma_0}} > 2^{-\gamma\cdot n} j_n$$

if n is big enough because $2^{l_n\cdot T_{\gamma_0}} \leqq 2^n$, hence $l_n \leqq n/T_{\gamma_0}$ and because $C_{\gamma_0} \cdot 2^{(l_n+1)T_{\gamma_0}} > j_n$. Q.E.D.

Proof of the theorem, continued. We shall use the notation in the proposition and its proof. It is easy to see that we may assume that X_n decreases. There is $\delta > 0$ such that $X_n < (1-2\delta)^n \cdot C_1$ if n is big since if $(X_n)_{n=1}^\infty$ is not dominated by a geometric series there is, to every $t \in \mathrm{N}$, $n_t \in \mathrm{N}$ such that $X_{n_t}/X_{n_t+t} < 1 + 1/t$ hence

$$\left(\frac{100}{X_{n_t+t}} \varphi_t' \circ \mathrm{Proj}_{[Hn_t - \setminus Hn_t + t - 1]} \varphi_{n_t+t}\right)_{t=1}^\infty \quad \text{and} \quad c_\gamma = \frac{200}{\gamma^2}$$

have the same properties as $(\varphi_t)_{t=1}^{\infty}$ and C_γ in Lemma 4 for a suitable choice of $\varphi_t' \in L(l^2(H_{n_t-1} \setminus H_{n_t+t-1}), \mathbf{C}^t)$ which is impossible. Divide now, for each n, $\{1, 2, \ldots, 2^{n-1}\}$ into $[2/\delta]+1$ disjoing parts $U_{r,n}$ ([ ] denotes the integer part) such that

$$(1) \qquad \left(1-r\cdot\frac{\delta}{2}\right)^n \cdot \|\mathrm{Proj}_{[H_{n-1}]}\varphi_{n-1}(b_{(n-)}^{(n)})\| \leqq$$

$$\leqq \|\mathrm{Proj}_{[H_{n-1}]}\varphi_{n-1}(\mathrm{Proj}_{[M_j(n,\{b_{(n-1)}^{(k)}\})]}b_{(n-1)}^{(n)})\| \leqq$$

$$\leqq \left(1-(r-1)\frac{\delta}{2}\right)^n \|\mathrm{Proj}_{[H_{n-1}]}\varphi_{n-1}(b_{(n-1)}^{(n)})\| \quad \text{if} \quad j\in U_{r,n}.$$

Since $\|\mathrm{Proj}_{[H_{n-1}]}\varphi_{n-1}(b_{(n-1)}^{(n)})\| > C_1$ it follows that there is $r_n \in \mathbf{N}$ such that $(1-r_n\delta/2)^n \geqq 1/4^n$ and

$$(2) \qquad \left\|\mathrm{Proj}_{[H_{n-1}]}\left(\sum\nolimits_{s\in U_{r_n,n}}\varphi_{n-1}(\mathrm{Proj}_{[M_s(n,\{b_{(n-1)}^{(l)}\})]}b_{(n-1)}^{(n)}\right)\right\| \geqq \frac{\delta}{3}\|\mathrm{Proj}_{[H_{n-1}]}\varphi_{n-1}(b_{(n-1)}^{(n)})\|.$$

Lemma 6 gives that to every $\gamma>0$ there is $z^{(\gamma)}\in D$ such that

$$(3) \qquad \sum\nolimits_{s\in U_{r_n,n}} N_{M_s(n,\{b_{(n-1)}^{(l)}\})}(z^{(\gamma)}) > 2^{-\gamma\cdot n}\cdot j_n,$$

if n is large, where j_n is the number of elements in $U_{r_n,n}$. But then the proposition and the proof of Lemma 3 give that

$$x_n > \frac{\delta}{3}\cdot\left(\frac{1-\dfrac{r_n\cdot\delta}{2}}{1-\dfrac{(r_n-1)\cdot\delta}{2}}\right)^n\cdot\|\mathrm{Proj}_{[H_{n-1}]}\varphi_{n-1}(b_{(n-1)}^{(n)})\|\cdot 2^{-\gamma\cdot n-1}\cdot j_n$$

according to (1), (2) and (3). But since $1-r_n\cdot\delta/2\geqq 1/4$ it follows that $r_n\cdot\delta\leqq 3/2$ hence that

$$\frac{1-\dfrac{r_n\cdot\delta}{2}}{1-\dfrac{(r_n-1)\cdot\delta}{2}} \geqq 1-\frac{2\delta}{1+2\delta}$$

hence if γ is samll and n is big enough it follows that $X_n > (1-2\delta)^n\cdot C_1$, because $\|\mathrm{Proj}_{[H_{n-1}]}\varphi_{n-1}(b_{(n-1)}^{(n)})\| > C_1$, which is a contradiction. Q.E.D.

Added in proof

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The Theorem has been proved independently by A. Nissenzweig.

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