Space analogues of some theorems for subharmonic and meromorphic functions

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1. Introduction

Denote points in n dimensional Euclidean space $\mathbf{R}^n$, $n\geqq 3$, by $x=(x_1, x_2, \ldots, x_n)$. Let $r=|x|$ and $x_1=r\cos\theta$, $0\leqq\theta\leqq\pi$. For $r>0$ let $B(r)=\{x:|x|<r\}$, $S(r)=\{x:|x|=r\}$, and $S=S(1)$. For $0\leqq\alpha\leqq\pi$, let $C(\alpha)=S\cap\{x:\theta<\alpha\}$. If E is a set contained in $S(r)$, let ∂E denote the boundary of E relative to $S(r)$. Let H^m denote m dimensional Hausdorff measure on $\mathbf{R}^n$.

If f is defined on a set $E\subset\mathbf{R}^n$, let $\theta(r)$ for $0<r<\infty$ be defined by

$$H^{n-1}(C(\theta(r))) = H^{n-1}(p(S(r)\cap E))$$

where p denotes the radial projection of $\mathbf{R}^n-\{0\}$ onto S. For $0\leqq\theta\leqq\theta(r)$, let

$$\hat{f}(r,\theta) = \sup\int_F f(ry)\,dH^{n-1}y,$$

where the supremum is taken over all measurable sets $F\subset p(S(r)\cap E)$ with

$$H^{n-1}(F) = H^{n-1}(C(\theta)).$$

Given a set $E\subset[0,\infty)$, let

$$\overline{\log\operatorname{dens}}\,E = \limsup_{r\to\infty}\left(\int_{E\cap(1,r)}\frac{dt}{t}\Big/\log r\right)$$

$$\underline{\log\operatorname{dens}}\,E = \liminf_{r\to\infty}\left(\int_{E\cap(1,r)}\frac{dt}{t}\Big/\log r\right).$$

Let u be equal H^n almost every where on $\mathbf{R}^n$ to the difference of two subharmonic functions. By the Riesz representation theorem there is associated with this difference a unique signed Borel measure ν whose total variation on compact sets is finite. Let $\nu=\nu^+-\nu^-$ denote the Jordan decomposition of ν. To simplify matters, we will assume that $\nu^+(B(1))=0$ or equivalently that u is equal H^n almost everywhere in $B(1)$ to a subharmonic function.

From [1, Thm. 2] we see there exist functions u_1 and u_2 subharmonic in $\mathbf{R}^n$ with associated measures $-\nu^-$ and $-\nu^+$ respectively, such that $u_2(0)=0$ and $u=u_1-u_2$, H^n almost everywhere in $\mathbf{R}^n$. For convenience in making the following definitions, we assume that $u=u_1-u_2$ except on the polar set where u_1 and u_2 are both $-\infty$. Otherwise, one may replace u by u_1-u_2 in the definitions.

If f and g are real valued functions on $\mathbf{R}^n$, let

$$(f\vee g)(x) = \max\{f(x), g(x)\}, \quad x\in\mathbf{R}^n.$$

For $0<r<\infty$ let

$$m(r) = \sup\{\hat{u}(r,\theta)\colon 0\leqq\theta\leqq\pi\} = \int_S (u\vee 0)(ry)\,dH^{n-1}y,$$

and

$$T(r) = m(r)+\hat{u}_2(r,\pi) = \int_S (u_1\vee u_2)(ry)\,dH^{n-1}y.$$

We note that $\hat{u}_2(r,\pi)\geqq u_2(0)=0$ for $r>0$ since u_2 is subharmonic. Hence,

$$0\leqq m(r)\leqq T(r) \quad \text{for} \quad 0<r<\infty,$$

and consequently since $u_1\vee u_2$ is subharmonic, either $m(r)\equiv 0$ or $T(r)$ is positive for $r\geqq r_0$ (r_0 large). In this paper we consider only u for which the second possibility occurs.

In analogy with the case for meromorphic functions we define the deficiency δ, order ϱ, and lower order μ of u by

$$\delta = \liminf_{r\to\infty}\frac{m(r)}{T(r)},$$

$$\varrho = \limsup_{r\to\infty}\frac{\log T(r)}{\log r},$$

$$\mu = \liminf_{r\to\infty}\frac{\log T(r)}{\log r}.$$

Observe that $0\leqq\mu\leqq\varrho\leqq\infty$, and $0\leqq\delta\leqq 1$. We remark that if h_1 and h_2 are subharmonic in $\mathbf{R}^n$, h_2 is harmonic in $B(1)$, $h_2(0)=0$, and $u=h_1-h_2$, except on a polar set in $\mathbf{R}^n$, then

$$\liminf_{r\to\infty}\frac{m(r)}{m(r)+\hat{h}_2(r,\pi)}\leqq\delta.$$

Consider for $0<\gamma<\infty$ the ultra-spherical differential equation

$$\frac{d}{d\theta}\left[(\sin\theta)^{n-2}\frac{df}{d\theta}\right] = -\gamma(\gamma+n-2)(\sin\theta)^{n-2}f(\theta), \quad 0<\theta<\pi. \tag{1.1}$$

It is well known and easy to show that (1.1) has two linearly independent solutions ψ_γ, φ_γ, satisfying

(1.2a) $$\lim_{\theta\to 0}\psi_\gamma(\theta) = \psi_\gamma(0) = 1,$$

(1.2b) $$\lim_{\theta\to 0}(\sin\theta)^{n-2}\frac{d\varphi_\gamma}{d\theta} = -1.$$

It follows from (1.1), (1.2a), and (1.2b) that

(1.2c) $$\psi_\gamma(\theta)\frac{d\varphi_\gamma}{d\theta}(\theta) - \varphi_\gamma(\theta)\frac{d\psi_\gamma}{d\theta}(\theta) = -(\sin\theta)^{2-n}, \quad 0 < \theta < \pi.$$

It is also easily shown that

(1.3a) Each ψ_γ has at least one zero in $(0, \pi)$ and if $\alpha = \alpha(\gamma)$ denotes the first zerc of ψ_γ, then ψ_γ is decreasing on $[0, \alpha]$.

(1.3b) If $0 < \tau < \gamma$, then $\psi_\gamma < \psi_\tau$ on $(0, \alpha(\gamma)]$,

(1.3c) $\lim_{\tau\to\gamma}\psi_\tau = \psi_\gamma$ uniformly on conpact subsets of $[0, \pi)$.

It follows from (1.3a) that given γ and δ, $0 \leqq \delta \leqq 1$, there is a unique $\theta_0 = \theta_0(\delta, \gamma)$ with $0 \leqq \theta_0 \leqq \alpha(\gamma)$ and $\psi_\gamma(\theta_0) = 1 - \delta$. In §4 we will prove

Theorem 1. *Let u be as above with deficiency δ, order ϱ, and lower order μ. Given γ, $0 < \gamma < \infty$, let $E(\gamma)$ denote the set of all $r > 0$ such that*

$$H^{n-1}(\{y : u(ry) > 0\} \cap S) \geqq H^{n-1}[C(\theta_0(\delta, \gamma))].$$

Then,

$$\overline{\log\operatorname{dens}}\, E(\gamma) \geqq 1 - \frac{\mu}{\gamma}$$

and

$$\underline{\log\operatorname{dens}}\, E(\gamma) \geqq 1 - \frac{\varrho}{\gamma}.$$

Theorem 1 implies that

$$\limsup_{r\to\infty} H^{n-1}(\{y : u(ry) > 0\} \cap S) \geqq H^{n-1}[C(\theta_0(\delta, \gamma))]$$

whenever $\gamma > \mu$. From (1.3c) it follows that

(1.4) $$\limsup_{r\to\infty} H^{n-1}(\{y : u(ry) > 0\} \cap S) \geqq H^{n-1}[C(\theta_0(\delta, \mu))]$$

for $0 < \mu < \infty$. In §5 we show that (1.4) is sharp. The inequality (1.4) is analogous to a spread conjecture made by Edrei and proved by Baernstein [2] in $\mathbf{R}^2$.

Considering ψ_γ as a function defined on S, we let

$$A(\gamma) = \int_{C(\alpha(\gamma))}\psi_\gamma\, dH^{n-1}.$$

Suppose now that u is subharmonic in $\mathbf{R}^n$, i.e. $u_2 \equiv 0$, and let

$$M(r) = \max\{u(x) : x \in S(r)\}, \quad r > 0.$$

In §6 we prove

Theorem 2. *If u is subharmonic in $\mathbf{R}^n$ with order ϱ, lower order μ, and γ is given, $0 < \gamma < \infty$, let*

$$E_1(\gamma) = \{r : T(r) \geqq A(\gamma) M(r)\}.$$

Then

$$\overline{\log \operatorname{dens}}\, E_1(\gamma) \geqq 1 - \frac{\mu}{\gamma},$$

and

$$\underline{\log \operatorname{dens}}\, E_1(\gamma) \geqq 1 - \frac{\varrho}{\gamma}.$$

We note that Theorem 2 has been obtained by Essén and Shea [7], using a different method. Theorem 2 implies that if $\gamma > \mu$, then

$$\limsup_{r \to \infty} \frac{T(r)}{M(r)} \geqq A(\gamma).$$

Letting $\gamma \to \mu$, we have by (1.3c) that

$$\limsup_{r \to \infty} \frac{T(r)}{M(r)} \geqq A(\mu)$$

when $0 < \mu < \infty$. This result has been obtained and shown to be sharp by Dahlberg [5].

In §7 we prove

Theorem 3. *If u is subharmonic in $\mathbf{R}^n$ with lower order μ, order ϱ, and $0 < \gamma < \infty$, let $E_2(\theta, \gamma)$ denote the set of $r > 0$ for which*

$$H^{n-1}(\{y : u(ry) \geqq \psi_\gamma(\theta) M(r)\} \cap S) \geqq H^{n-1}(C(\theta)),$$

when $0 < \theta \leqq \alpha(\gamma)$. Then

$$\overline{\log \operatorname{dens}}\, E_2(\theta, \gamma) \geqq 1 - \frac{\mu}{\gamma},$$

$$\underline{\log \operatorname{dens}}\, E_2(\theta, \gamma) \geqq 1 - \frac{\varrho}{\gamma}$$

for $0 < \theta \leqq \alpha(\gamma)$.

We note that Theorem 3 can be obtained in $\mathbf{R}^2$ by using a method of Baernstein (see [6, Ch. 8]).

In the proof of Theorems 1—3, we first use a method of the authors [8] to obtain a differential inequality (see (2.6)). Using this inequality, and methods of Essén [6], and Essén and Shea [7], we obtain an integral inequality (see §3). Finally, using this integral inequality and a method of Barry [3, 4] we obtain Theorems 1—3.

2. Spherical Symmetrization

Given a closed set $F\subset\mathbf{R}^n$ define the spherical symmetrization F^* of F as follows: If $F\cap S(r)=\varphi$, then $F^*\cap S(r)=\varphi$. Otherwise,

$$H^{n-1}(F^*\cap S(r)) = H^{n-1}(F\cap S(r))$$

and $F^*\cap S(r)$ is either the point $(r, 0, \dots, 0)$ or the closed cap on $S(r)$ centered at $(r, 0, \dots, 0)$. Let $u=u_1-u_2$ where u_1, u_2, are subharmonic in $B(R)$, $R>0$, with continous second partials. Given t, $-\infty<t<\infty$, let $F(t)=\{x:u(x)\geqq t\}$ and note that $F(t)$ is closed. Define an associated function u^* by letting

$$u^*(x) = \sup\{t\colon x\in F^*(t)\} \quad \text{whenever} \quad x\in B(R).$$

It is easily seen that u^* is symmetric with respect to the x_1 axis, and $\{x:u^*(x)\geqq t\}= =F^*(t)$. It follows that u and u^* are equimeasurable and

$$\hat{u}(r,\theta) = \int_{C(\theta)} u^*(ry)\,dH^{n-1}y \tag{2.1}$$

whenever $r\in(0, R)$, $\theta\in[0, \pi]$. Also for fixed r, $r\in(0, R)$, $u^*(r, \theta)$ is a nonincreasing function of θ on $[0, \pi]$. We note that Gehring [9] has shown that u^* is Lipschitz in $B(R_1)$ whenever $R_1<R$.

Let f be a function defined on $(0, R)$. Define $f_\#$ on (R^{2-n}, ∞) by $f_\#(s)=f(r)$ when $s=r^{2-n}$ and $r\in(0, R)$. Let

$$Lf(r) = (n-2)^2 r^{4-2n} \liminf_{h\to 0}\left[\frac{f_\#(r^{2-n}+h)+f_\#(r^{2-n}-h)-2f_\#(r^{2-n})}{h^2}\right]$$

for $r\in(0, R)$. Note that if f has a second derivative on $(0, R)$, then

$$Lf(r) = r^{3-n}\frac{d}{dr}\left[r^{n-1}\frac{df}{dr}\right], \quad r\in(0, R).$$

Let

$$P(r,\theta) = \hat{u}(r,\theta)+\hat{u}_2(r,\pi)$$

for $r\in(0, R)$ and $\theta\in[0, \pi]$. Given $r_0\in(0, R)$ we shall show that

$$LP(r_0,\theta)\geqq 0 \quad \text{for} \quad 0\leqq\theta\leqq\pi, \tag{2.2a}$$

and

$$LP(r_0,\theta) \geqq -c(\sin\theta)^{n-2}\frac{\partial u^*}{\partial\theta}(r_0,\theta), \tag{2.2b}$$

for almost every θ with respect to one dimensional Lebesgue measure on $[0, \pi]$. Here c is the surface area of the $n-2$ dimensional unit sphere, and for each fixed θ, $LP(r, \theta)=Lf(r)$, where $f(r)=P(r, \theta)$.

To prove (2.2a) let $G(\theta)\subset S$ be such that

(i) $$S\cap\{y: u(r_0 y) > u^*(r_0, \theta)\}\subset G(\theta)\subset S\cap\{y: u(r_0 y)\geqq u^*(r_0, \theta)\},$$

(ii) $$H^{n-1}(G(\theta)) = H^{n-1}(C(\theta)),$$

(iii) $$\hat{u}(r_0, \theta) = \int_{G(\theta)} u(r_0 y)\, dH^{n-1}y = \int_{C(\theta)} u^*(r_0 y)\, dH^{n-1}y,$$

for $\theta\in[0, \pi]$. Let

$$q(r, \theta) = \int_{G(\theta)} u(ry)\, dH^{n-1}y + \hat{u}_2(r, \pi)$$

for $r\in(0, R)$ and $\theta\in[0, \pi]$. Clearly, $q(r, \theta)\leqq P(r, \theta)$, with equality holding at (r_0, θ). Hence for fixed θ,

(2.3) $$LP(r_0, \theta)\geqq Lq(r_0, \theta) = L\Big[\int_{G(\theta)} u_1(r_0 y)\, dH^{n-1}y + \int_{S-G(\theta)} u_2(r_0 y)\, dH^{n-1}y\Big] =$$

$$= \int_{G(\theta)}\left(r^{3-n}\frac{\partial}{\partial r} r^{n-1}\frac{\partial}{\partial r} u_1\right)(r_0 y)\, dH^{n-1}y +$$

$$+ \int_{S-G(\theta)}\left(r^{3-n}\frac{\partial}{\partial r} r^{n-1}\frac{\partial}{\partial r} u_2\right)(r_0 y)\, dH^{n-1}y.$$

Let Δ denote the Laplacian in $\mathbf{R}^n$ and let $\tilde{\Delta}$ be the spherical part of Δ defined by

$$\Delta = r^{1-n}\frac{\partial}{\partial r} r^{n-1}\frac{\partial}{\partial r} + r^{-2}\tilde{\Delta}.$$

Observe that for H^{n-1} almost every $x\in G(\theta)\cap\{y: u(r_0 y) = u^*(r_0, \theta)\}$, we have

$$0 = \tilde{\Delta} u(r_0 x) = \tilde{\Delta} u_1(r_0 x) - \tilde{\Delta} u_2(r_0 x).$$

Using this fact, the subharmonicity of u_1, u_2, and (2.3), we obtain

(2.4) $$LP(r_0, \theta)\geqq -\int_{S\cap\{y: u(r_0 y) > u^*(r_0, \theta)\}} \tilde{\Delta} u_1(r_0 y)\, dH^{n-1}y -$$

$$-\int_{S\cap\{y: u(r_0 y)\leqq u^*(r_0, \theta)\}} \tilde{\Delta} u_2(r_0 y)\, dH^{n-1}y.$$

Now as in [8, §3], we may apply Green's formula for almost every $t\in\mathbf{R}$ to obtain

$$-\int_{S\cap\{y: u(r_0 y) > t\}} \tilde{\Delta} u_1(r_0 y)\, dH^{n-1}y = r_0^{3-n}\int_{S(r_0)\cap u^{-1}(t)} \frac{\partial u_1}{\partial n}\, dH^{n-2}$$

and

$$-\int_{S\cap\{y: u(r_0 y)\leqq t\}} \tilde{\Delta} u_2(r_0 y)\, dH^{n-1}y = -r_0^{3-n}\int_{S(r_0)\cap u^{-1}(t)} \frac{\partial u_2}{\partial n}\, dH^{n-2}$$

where $\partial/\partial n$ is the normal derivative taken into $S(r_0)\cap\{x:u(x)>t\}$. Hence for almost every $t\in\mathbf{R}$,

$$\text{(2.5)}\qquad -\int_{S\cap\{y:u(r_0y)>t\}}\tilde{\Delta}u_1(r_0y)\,dH^{n-1}y-\int_{S\cap\{y:u(r_0y)\leqq t\}}\tilde{\Delta}u_2(r_0y)\,dH^{n-1}y=$$

$$=r_0^{3-n}\int_{S(r_0)\cap u^{-1}(t)}\frac{\partial}{\partial n}(u_1-u_2)\,dH^{n-2}=r_0^{3-n}\int_{S(r_0)\cap u^{-1}(t)}|\tilde{\nabla}u|\,dH^{n-2},$$

where $\tilde{\nabla}$ denotes the spherical gradient of u on $S(r_0)$. Letting $t\to u^*(r_0,\theta)$ from the right through a properly chosen sequence and using (2.4), (2.5), we see that (2.2a) is true.

Let J be the set of $\theta\in[0,\pi]$ where

$$\frac{\partial u^*}{\partial\theta}(r_0,\theta)=-r_0|\tilde{\nabla}u^*(r_0,\theta)|<0.$$

Since $LP(r_0,\theta)\geqq 0$, we see that (2.2b) is valid for almost every $\theta\in[0,\pi]-J$. Let $K=\{u^*(r_0,\theta):\theta\in J\}$. Then in [8, (2.2)] it was shown for almost every $t=u^*(r_0,\theta)\in K$ that

$$\int_{u^{-1}(t)\cap S(r_0)}|\tilde{\nabla}u|\,dH^{n-2}\geqq\int_{\partial C_1(\theta)}|\tilde{\nabla}u^*|\,dH^{n-2}$$

where $C_1(\theta)=\{r_0y:y\in C(\theta)\}$. Note that if $J_1\subset J$ has positive one dimensional Lebesgue measure, then $\{u^*(r_0,\theta):\theta\in J_1\}$ has positive Lebesgue measure. Thus it follows from (2.4), (2.5), and the above inequality that (2.2b) is true.

Let

$$T(r,\theta)=\widehat{(u\vee 0)}(r,\theta)+\hat{u}_2(r,\pi)$$

for $\theta\in[0,\pi]$ and $r\in(0,R)$. For given $r_0\in(0,R)$, let θ_1, $0\leqq\theta_1\leqq\pi$, be such that

$$H^{n-1}(C(\theta_1))=H^{n-1}(\{y:u(r_0y)>0\}\cap S).$$

Note that $P(r,\theta)\leqq T(r,\theta)$ for $\theta\in[0,\pi]$ and $r\in(0,R)$, with equality holding when $r=r_0$, $\theta\in[0,\theta_1]$. Hence, if $\theta\in[0,\theta_1]$, then

$$LT(r_0,\theta)\geqq LP(r_0,\theta).$$

If $\theta\in(\theta_1,\pi]$, then $T(r_0,\theta)=P(r_0,\theta_1)$ and

$$LT(r_0,\theta)\geqq LP(r_0,\theta_1)\geqq 0=\frac{\partial}{\partial\theta}(u\vee 0)^*(r_0,\theta).$$

From these inequalities and (2.2) we obtain

$$\text{(2.6a)}\qquad LT(r,\theta)\geqq 0\quad\text{for}\quad\theta\in[0,\pi],\quad r\in(0,R),$$

$$\text{(2.6b)}\qquad LT(r,\theta)\geqq -c(\sin\theta)^{n-2}\frac{\partial}{\partial\theta}(u\vee\theta)^*(r,0)$$

for almost every $\theta\in[0,\pi]$ when $r\in(0,R)$.

3. Differential and integral inequalities

Let u be as in §2 and observe that

$$T(r,\theta)=\int_{C(\theta)}(u^*\vee 0)(ry)\,dH^{n-1}y+\hat{u}_2(r,\pi)$$

is continuous in $B(R)-\{0\}$, since u^* is Lipschitz in $B(R_1)$ whenever $R_1<R$, and u_2 is subharmonic. This observation and (2.6a) imply for fixed $\theta\in[0,\pi]$ see ([10, Ch. 10, §7]) that $T_\#(s,\theta)$ is a convex function of s on (R^{2-n},∞). Hence for each $h>0$,

$$T_\#(s+h,\theta)+T_\#(s-h,\theta)-2T_\#(s,\theta)\geqq 0 \tag{3.1}$$

when $s\in(R^{2-n}+h,\infty)$.

Given $\tau\in(0,\infty)$ and $\beta\in(0,\alpha(\tau))$, let g be a solution of (1.1) with τ replacing γ and suppose that

$$g'(\theta)=\frac{dg}{d\theta}\leqq 0 \quad\text{on}\quad (0,\beta), \tag{3.2a}$$

(3.2b) $\sigma(r)=-\int_0^\beta T(r,\theta)g'(\theta)\,d\theta$, is a bounded continuous function on $(0,R)$,

(3.2c) $\lim_{\theta\to 0}(\sin\theta)^{n-2}g'(\theta)$ and $\lim_{\theta\to 0}T(r,\theta)g(\theta)$ exist finitely for $r\in(0,R)$.

From (3.1), (3.2), the Fatou lemma, and (2.6b) we obtain

$$L\sigma(r)\geqq-\int_0^\beta LT(r,\theta)g'(\theta)\,d\theta\geqq c\int_0^\beta\frac{\partial}{\partial\theta}(u\vee 0)^*(r,\theta)\,(\sin\theta)^{n-2}g'(\theta)\,d\theta.$$

Since for fixed r, $(u\vee 0)^*(r,\theta)$ is absolutely continuous on $[0,\pi]$, we may integrate the right hand integral twice by parts. Using (3.2c) and (1.1), we obtain

$$0\leqq c\int_0^\beta\frac{\partial}{\partial\theta}(u\vee 0)^*(r,\theta)\,(\sin\theta)^{n-2}g'(\theta)\,d\theta=$$

$$=c(u\vee 0)^*(r,\theta)\,(\sin\theta)^{n-2}g'(\theta)+\tau(\tau+n-2)\,T(r,\theta)g(\theta)|_0^\beta+$$

$$+\tau(\tau+n-2)\sigma(r)=-h(r)+\tau(\tau+n-2)\sigma(r)$$

for $r\in(0,R)$. Hence,

$$L\sigma(r)\geqq-h(r)+\tau(\tau+n-2)\sigma(r)\geqq 0 \tag{3.3}$$

when $r\in(0,R)$. From (3.3) and (3.2b) we deduce that $\sigma_\#$ is convex on (R^{2-n},∞). Thus σ is a convex function of r^{2-n} on $(0,R)$. It follows that the left and right hand derivatives of σ exist at each $r\in(0,R)$ (denoted by $\sigma'(r)$, $\sigma'_+(r)$), and $r^{n-1}\sigma'_-(r)$ is a nondecreasing fuction on $(0, R)$. Moreover,

$$L\sigma(r) = r^{3-n}\frac{d}{dr}[r^{n-1}\sigma'_-(r)]$$

except possibly on a set of Lebesgue measure zero in $(0, R)$. Since we have (3.2b), we also see that σ is nondecreasing on $(0, R)$. Hence, the left and right hand derivatives of σ are nonnegative.

We now argue as in [7]. Fix $R_1\in(0, R)$ and let

$$\Phi(r) = \int_r^{R_1}\frac{h(t)}{t^{1+\tau}}\,dt, \quad r\in(0, R_1].$$

From (3.3) we obtain

$$\Phi(r) \geqq \tau(\tau+n-2)\int_r^{R_1}\frac{\sigma(t)}{t^{1+\tau}}\,dt - \int_r^{R_1}\frac{\frac{d}{dt}[t^{n-1}\sigma'_-(t)]}{t^{n+\tau-2}}\,dt.$$

Integrating the second integral twice by parts, we obtain

$$\Phi(r) \geqq -t^{1-\tau}\sigma'_-(t)-(\tau+n-2)t^{-\tau}\sigma(t)\big|_r^{R_1}. \tag{3.4}$$

Next we use a method of Barry [3, 4]. Let

$$\Psi(r) = r^{\tau}[\Phi(r)+R_1^{1-\tau}\sigma'_-(R_1)+(\tau+n-2)R_1^{-\tau}\sigma(R_1)]$$

for $r\in(0, R_1]$. From (3.4) we have

$$\Psi(r) \geqq r\sigma'_-(r)+(\tau+n-2)\sigma(r), \quad r\in(0, R_1]. \tag{3.5}$$

Assume that

(3.6a) h is continuous on $(0, R_1]$,

(3.6b) $\sigma\vee 0\not\equiv 0$ on $(0, R_1)$.

Then since σ is nondecreasing on $(0, R_1)$, there exists r_1, $0<r_1<R_1$, such that σ is positive on $[r_1, R_1]$. From (3.5) and (3.6a) it follows that Ψ is positive with a continuous derivate on $[r_1, R_1)$. Using (3.5) and (3.3) we obtain

$$r\Psi'(r) = \tau\Psi(r)-h(r) \geqq \tau r\sigma'_-(r)+\tau(\tau+n-2)\sigma(r)-h(r) \geqq \tau r\sigma'_-(r) \geqq 0$$

when $r\in[r_1, R_1)$.

Let

$$\Gamma = \{r: h(r) \leqq 0\}.$$

Observe from the above inequality that

$$r\Psi'(r) \geqq \tau\Psi(r) \quad \text{for} \quad r\in\Gamma\cap[r_1, R_1].$$

Thus

$$\tau\int_{\Gamma\cap[r_1,R_1]}\frac{dr}{r} \leqq \int_{\Gamma\cap[r_1,R_1]}\frac{\Psi'(r)}{\Psi(r)}\,dr \leqq \int_{r_1}^{R_1}\frac{\Psi'(r)}{\Psi(r)}\,dr = \log\left[\frac{\Psi(R_1)}{\Psi(r_1)}\right].$$

Using (3.5) it follows that

$$\tau\int_{\Gamma\cap[r_1,R_1]}\frac{dr}{r} \leqq \log\left(R_1\sigma'_-(R_1)+(\tau+n-2)\sigma(R_1)\right) - \tag{3.7}$$

$$-\log\left(r_1\sigma'_-(r_1)+(\tau+n-2)\sigma(r_1)\right).$$

4. Proof of Theorem 1

Let $u=u_1-u_2$, H^n almost everywhere, be as in Theorem 1 with order ϱ, lower order μ, and deficiency δ. From Fubini's Theorem we see that it sufficies to prove Theorem 1 for u_1-u_2. Hence we assume that $u=u_1-u_2$ off of a polar set. Define $T(r,\theta)$, $r\in(0,\infty)$, $\theta\in(0,\pi)$, relative to u as in §2. Observe that $T\geqq 0$ in $\mathbf{R}^n-\{0\}$, since $u_2(0)=0$ and u_2 is subharmonic. Also, $T(r)=T(r,\pi)$ is nondecreasing on $(0,\infty)$, and by assumption $T(r)>0$ for sufficiently large r, say $r\geqq r_0$.

Let γ, $0<\gamma<\infty$, and $\theta_0=\theta_0(\delta,\gamma)$ be as in Theorem 1. We assume that $\mu<\gamma$ and $0<\delta\leqq 1$, since otherwise the first part of Theorem 1 is trivially true. Let τ satisfy, $\mu<\tau<\gamma$. Note that

$$\limsup_{r\to\infty}\frac{\hat{u}_2(r,\pi)}{T(r)} = 1-\delta = \psi_\gamma(\theta_0) < \psi_\tau(\theta_0),$$

thanks to (1.3b). Hence for sufficiently large r, say $r\geqq r_0$, we have

$$\hat{u}_2(r,\pi) < \psi_\tau(\theta_0)\,T(r)+\frac{1}{2}\,[\psi_\gamma(\theta_0)-\psi_\tau(\theta_0)]\,T(r) \leqq \tag{4.1}$$

$$\leqq \psi_\tau(\theta_0)\,T(r)+\frac{1}{2}\,[\psi_\gamma(\theta_0)-\psi_\tau(\theta_0)]\,T(r_0).$$

There exist nonincreasing sequences $\{v_j\}$, $\{w_j\}$ of subharmonic functions in $\mathbf{R}^n$, with continuous second partial derivatives and pointwise limits u_1, u_2, respectively. Let $p_j=(v_j-w_j)\vee 0$ and put

$$T_j(r,\theta) = \hat{p}_j(r,\theta)+\hat{w}_j(r,\pi),\quad r\in(0,\infty),\quad \theta\in[0,\pi].$$

As in §3 we see that T_j is continous in $\mathbf{R}^n-\{0\}$ and for fixed $\theta\in[0,\pi]$ that $T_j[r,\theta]$ is convex as a function of r^{2-n} on $(0,\infty)$. Since

$$0 \leqq T_j(r,\theta)-T(r,\theta) \leqq \hat{v}_j(r,\pi)-\hat{u}_1(r,\pi)+\hat{w}_j(r,\pi)-\hat{u}_2(r,\pi), \tag{4.2}$$

it follows from the subharmonicity of the above functions, and Dini's Theorem that T_j converges to T uniformly on compact subsets of $\mathbf{R}^n-\{0\}$.

With $g=\psi_\tau$, $\theta_0=\theta_0(\delta,\gamma)$, define σ_j and h_j relative to p_j as in §3 with $\beta=\theta_0$. Let σ be the corresponding quantity for u. From (1.3a) and (1.3b) we see that $g=\psi_\tau$ satisfies (3.2a). Also (1.1) and (1.2a), imply that $\lim_{\theta\to 0} g'(\theta)=0$. Using this fact, and the fact that T_j is continous in $\mathbf{R}^n-\{0\}$, we find (3.2b) and (3.2c) are true with T_j, σ_j, replacing T, σ, and $R>0$ arbitrary. Moreover (3.6) is true with h_j, σ_j, replacing h, σ, provided $R_1\geqq r_0$, as we see from (4.2).

Since T_j converges uniformly to T on compact subsets of $\mathbf{R}^n-\{0\}$, it follows that σ_j converges uniformly to σ on compact subsets of $(0,\infty)$. Hence σ is nondecreasing, convex as a function of r^{2-n} on $(0,\infty)$, and at each $r\in(0,\infty)$ where $\sigma'_-(r)=\sigma'_+(r)$, we have $\lim_{j\to\infty}\sigma'_{j-}(r)=\sigma'_-(r)$ (see [11, p. 46, Lemma 1]). Also $\sigma(r_0)>0$ since $T(r_0)>0$.

We note that

$$h_j(r) = -cp_j^*(r,\theta_0)(\sin\theta_0)^{n-2}\psi'_\tau(\theta_0)+\tau(\tau+n-2)[\hat{w}_j(r,\pi)- \\ -T_j(r,\theta_0)\psi_\tau(\theta_0)],\quad r\in(0,\infty). \tag{4.3}$$

Let $K_j=\{r: h_j(r)\leqq 0\}$ and let K be the set of $r>0$ where

$$H^{n-1}(\{y:(u\vee 0)(ry)>0\}\cap S) < H^{n-1}(C(\theta_0)).$$

Let r_1, R_1, be fixed points where the left and right hand derivatives of σ are equal, and $r_0<r_1<R_1$. If $r\in K\cap[r_1,R_1]$, then $\lim_{j\to\infty}p_j^*(r,\theta_0)=0$, since p_j converges pointwise to $u\vee 0$ off of a polar set. Since for $r\in K\cap[r_1,R_1]$, we have

$$\lim_{j\to\infty} T_j(r,\theta_0) = T(r,\theta_0) = T(r),$$

it follows from (4.1), (4.3), that $r\in K_j\cap[r_1,R_1]$ for sufficiently large j. Hence by the Fatou lemma,

$$\int_{K\cap[r_1,R_1]}\frac{dr}{r} \leqq \liminf_{j\to\infty}\int_{K_j\cap[r_1,R_1]}\frac{dr}{r}.$$

We now replace Γ, σ, in (3.7) by K_j, σ_j. Letting $j\to\infty$ in (3.7) and using the above inequality, it follows that

$$\tau\int_{K\cap[r_1,R_1]}\frac{dr}{r} \leqq \log\big(R_1\sigma'_-(R_1)+(\tau+n-2)\sigma(R_1)\big)- \\ -\log\big(r_1\sigma'_-(r_1)+(\tau+n-2)\sigma(r_1)\big). \tag{4.4}$$

Next since $r^{n-1}\sigma'(r)$ is nondecreasing on $(0,\infty)$, we have

$$\sigma(2R_1)\geqq\sigma(2R_1)-\sigma(R_1) = \int_{R_1}^{2R_1}\sigma'_-(r)\,dr \geqq 2^{1-n}R_1\sigma'_-(R_1).$$

From this inequality and (4.4), we obtain

$$\tau \frac{\int_{K\cap[r_1,R_1]} \frac{dr}{r}}{\log R_1} \leqq \frac{\log[(2^{n-1}+\tau+n-2)\sigma(2R_1)]}{\log R_1} - \frac{\log[r_1\sigma'_-(r_1)+(\tau+n-2)\sigma(r_1)]}{\log R_1}.$$

Letting $2R_1\to\infty$, through a properly chosen sequence and observing that $\sigma(2R_1)\leqq \leqq T(2R_1)$, we get

$$\tau \, \underline{\log \text{dens}}\, K \leqq \mu.$$

Hence,

$$\underline{\log \text{dens}}\,[(0,\infty)-K] \geqq 1-\frac{\mu}{\tau}. \tag{4.5}$$

Letting $\tau\to\gamma$, we obtain the first part of Theorem 1. The proof of the second part of Theorem 1 is similar. We omit the details.

5. Some examples

We now show that (1.4) with $\delta\in(0,1]$ and $\mu\in(0,\infty)$ is sharp. Let ψ_μ, φ_μ, be solutions to (1.1) and satisfy (1.2) with $\mu=\gamma$. Let

$$u(r,\theta) = r^\mu[\psi_\mu(\theta_0)\varphi_\mu(\theta)-\varphi_\mu(\theta_0)\psi_\mu(\theta)]$$

when $r\in(0,\infty)$, $0\leqq\theta\leqq\theta_0=\theta_0(\delta,\mu)$, and

$$u(r,\theta)=0$$

for $r\in(0,\infty)$, $\theta\in(\theta_0,\pi)$. Using (1.2c) we find that φ_μ/ψ_μ is decreasing on $(0,\theta_0)$ and consequently $u(r,\theta)>0$ whenever $r\in(0,\infty)$, $\theta\in(0,\theta_0)$. Using (1.2), one can verify that $u=u_1-u_2$ in $\mathbf{R}^n-\{0\}$, where u_1, u_2 are subharmonic in $\mathbf{R}^n$ and satisfy

(i) The measure associated with u_1 is concentrated on $\{y: y_1=r\cos\theta_0, 0<r<\infty\}$,

(ii) The measure associated with u_2 is concentrated on the positive x_1 axis,

(iii) $u_2(0)=0$ and $u_2=-\infty$ on the positive x_1 axis.

From (1.2) and Green's second identity, it follows that

$$r^{n-1}\frac{d\hat{u}_2}{dr}(r,\pi) = -c\lim_{\theta\to 0}(\sin\theta)^{n-2}\int_0^r \frac{\partial u}{\partial\theta}(s,\theta)s^{n-3}\,ds = c\psi_\mu(\theta_0)\,(\mu+n-2)^{-1}r^{\mu+n-2}.$$

Thus,

$$\mu(\mu+n-2)\hat{u}_2(r,\pi) = c\psi_\mu(\theta_0)r^\mu,$$

where c is as in (2.2b). From (1.1) and (1.2) we see that

$$\mu(\mu+n-2)m(r) = cr^{\mu}\left[(\varphi_\mu(\theta)\psi'_\mu(\theta_0)-\psi_\mu(\theta_0)\varphi'_\mu(\theta))\,(\sin\theta)^{n-2}\right]_0^{\theta_0} = c(1-\psi_\mu(\theta_0))r^{\mu}.$$

Hence u has lower order μ and

$$\frac{\hat{u}_2(r,\pi)}{T(r)} = \psi_\mu(\theta_0) = 1-\delta.$$

By suitably redefining u in $B(1)$, we obtain a function which satisfies the hypotheses of Theorem 1 and for which equality holds in (1.4). Hence (1.4) is sharp.

6. Proof of Theorem 2

Given $\tau\in(0,1)$ let ψ_τ and φ_τ denote solutions of (1.1) as in §1 with $\gamma=\tau$. By (1.2a) we have

(6.1) $$(\sin\alpha(\tau))^{n-2}\psi'_\tau(\alpha(\tau)) =$$
$$= -\tau(\tau+n-2)\int_0^{\alpha(\tau)}\psi_\tau(\theta)\,(\sin\theta)^{n-2}\,d\theta = -c^{-1}\tau(\tau+n-2)A(\tau)$$

where A is as in §1. Let

$$g(\theta) = \varphi'_\tau(\alpha(\tau))\psi_\tau(\theta)-\psi'_\tau(\alpha(\tau))\varphi_\tau(\theta) \quad\text{for}\quad \theta\in(0,\pi),$$

and note that g is a solution to (1.1) with $\gamma=\tau$. We claim that

(6.2a) $$\lim_{\theta\to 0}(\sin\theta)^{n-2}g(\theta) = 0 \quad\text{and}\quad \lim_{\theta\to 0}(\sin\theta)^{n-2}g'(\theta) = \psi'_\tau(\alpha(\tau)),$$

(6.2b) $$g' < 0 \quad\text{on}\quad (0,\alpha(\tau)) \quad\text{and}\quad g'(\alpha(\tau)) = 0,$$

(6.2c) $$g(\alpha(\tau)) = -(\sin\alpha(\tau))^{2-n}.$$

Statement (6.2a) follows from (1.1) and (1.2). Using (1.3a) and (1.2c) we see that $\psi'_\tau<0$ on $(0,\alpha(\tau)]$ and that φ'_τ/ψ'_τ is decreasing on $(0,\alpha(\tau)]$. Thus (6.2b) follows. Letting $\theta=\alpha(\tau)$ in (1.2c) we obtain (6.2c).

Now let u be as in Theorem 2 with order ϱ and lower μ. Then u is subharmonic in $\mathbf{R}^n$ (i.e. $u_2\equiv 0$) and $T(r)>0$ for $r\geqq r_0$. Assume that $u\geqq 0$ since otherse we can consider $u\vee 0$. Let v_j be as in §4, where now $w_j\equiv 0$. Put $\beta=\alpha(\tau)$ and define T_j, σ_j and T, σ relative to v_j and u as in §3.

Observe that, for $r\in(0,\infty)$ and $\theta\in[0,\alpha(\tau)]$,

$$0 \leqq T_j(r,\theta) \leqq cM(r,v_j)\int_0^{\theta}(\sin\zeta)^{n-2}\,d\zeta \leqq k(\sin\theta)^{n-1}M(r,v_j)$$

where k is a positive constant. From this observation and (6.2) we see that (3.2) is valid with T_j, σ_j replacing T,σ. Let h_j correspond to v_j as in §3 and note that by (6.1) and (6.2) we have

$$(\sin\alpha(\tau))^{n-2}h_j(r) = \tau(\tau+n-2)\left[T_j(r,\alpha(\tau))-A(\tau)M(r,v_j)\right].$$

Hence h_j is continuous and as in §4 we see that σ and σ_j are nondecreasing convex functions of r^{2-n} on $(0, \infty)$ which are positive for $r \geqq r_0$.

Let

$$K_j = \{r : h_j(r) \leqq 0\},$$

$$K = \{r : T(r, \alpha(\tau)) < A(\tau)M(r, u)\},$$

and let $r_1 < R_1$ be such that $\sigma(r_1) > 0$, $\sigma'_-(r_1) = \sigma'_+(r_1)$ and $\sigma'_-(R_1) = \sigma'_+(R_1)$.

Arguing as in §4 we obtain (4.4) and then (4.5). Hence

$$\overline{\log\text{dens}}\,\{r : T(r) \geqq A(\tau)M(r,u)\} \geqq$$

$$\overline{\log\text{dens}}\,\{r : T(r,\alpha(\tau)) \geqq A(\tau)M(r,u)\} \geqq 1-\frac{\mu}{\tau},$$

and the first part of theorem 2 is valid with $\gamma=\tau$. The proof of the second part is similar. We omit the details.

7. Proof of Theorem 3

Given $\tau \in (0, \infty)$ and $\theta_1 \in (0, \alpha(\tau)]$, let

$$g(\theta) = -\varphi_\tau(\theta_1)\psi_\tau(\theta) + \psi_\tau(\theta_1)\varphi_\tau(\theta) \quad \text{for} \quad \theta \in (0, \pi)$$

and note that g is a solution to (1.1). We assert that

$$\text{(7.1a)} \qquad \lim_{\theta\to 0}(\sin\theta)^{n-2}g(\theta) = 0 \quad \text{and} \quad \lim_{\theta\to 0}(\sin\theta)^{n-2}g'(\theta) = -\psi_\tau(\theta_1),$$

$$\text{(7.1b)} \qquad g' < 0 \quad \text{on} \quad (0, \theta_1) \quad \text{and} \quad g(\theta_1) = 0,$$

$$\text{(7.1c)} \qquad g'(\theta_1) = -(\sin\theta_1)^{2-n}.$$

Statements (7.1a) and (7.1c) follow immediately from (1.1) and (1.2). Using (1.3a) and (1.2c) we see that $\psi'_\tau < 0$ and that φ'_τ/ψ'_τ is desreasing on $(0, \theta_1]$. Thus, using (1.2c),

$$g'(\theta) = -\psi'_\tau(\theta)\psi_\tau(\theta_1)\left(\frac{\varphi_\tau(\theta_1)}{\psi_\tau(\theta_1)} - \frac{\varphi'_\tau(\theta)}{\psi'_\tau(\theta)}\right) \leqq$$

$$\leqq -\psi'_\tau(\theta)\psi_\tau(\theta_1)\left[\frac{\varphi_\tau(\theta_1)}{\psi_\tau(\theta_1)} - \frac{\varphi'_\tau(\theta_1)}{\psi'_\tau(\theta_1)}\right] = -\frac{\psi'_\tau(\theta)}{\psi'_\tau(\theta_1)}(\sin\theta_1)^{2-n} < 0 \quad \text{for} \quad \theta \in (0, \theta_1].$$

Thus (7.1b) is valid.

Let $\{v_j\}$ be as in §4 and let h_j correspond to v_j as in §3 with $\beta=\theta_1$. Let K denote the set of $r>0$ such that

$$H^{n-1}(\{y: u(ry) \geqq \psi_\tau(\theta_1) M(r, u)\} \cap S) < H^{n-1}(C(\theta_1))$$

and let

$$K_j = \{r: h_j(r) \leqq 0\}.$$

From (7.1) we find that

$$K_j = \{r: v_j^*(r, \theta_1) \leqq \psi_\tau(\theta_1) M(r, v_j)\}.$$

Since $\{v_j\}$ is a nonincreasing sequence with pointwise limit u, it follows for $r_1<R_1$, as in §4, that

$$\int_{K\cap(r_1, R_1)} \frac{dr}{r} \leqq \liminf_{j\to\infty} \int_{K_j\cap(r_1, R_1)} \frac{dr}{r}.$$

Arguing as in §4 we obtain

$$\overline{\log \operatorname{dens}}\,[(0, \infty) - K] \geqq 1 - \frac{\mu}{\tau}$$

which is the first half of the conclusion of Theorem 3 with $\theta=\theta_1$ and $\gamma=\tau$. The proof of the second half is similar. We omit the details.

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