Uniform convergence of random Fourier series

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1. Introduction

We study the uniform convergence of the random trigonometric series

$$\sum_{n=0}^{\infty} a_n \eta_n \cos(nt+\Phi_n) \tag{1.1}$$

where $\{\eta_n e^{i\Phi_n}\}$ is a sequence of independent complex valued random variables (η_n and Φ_n are real). Various additional conditions are put on $\{\eta_n e^{i\Phi_n}\}$ (and $\{a_n\}$) in the different results obtained. This work was motivated by our desire to prove the following theorem.

Theorem 1.1. *Let $\{\eta_n e^{i\Phi_n}\}$ be independent symmetric complex valued random variables, $E|\eta_n|^2=1$ and $\liminf_{n\to\infty} E|\eta_n|>0$. Let $\{a_n\}\in l^2$ and assume that a_n is non-increasing ($a_n\downarrow$). Then*

$$\sum_{n=2}^{\infty} \frac{\left(\sum_{k=n}^{\infty} a_k^2\right)^{1/2}}{n(\log n)^{1/2}} < \infty \tag{1.2}$$

is a necessary and sufficient condition for the uniform convergence a.s. of the series (1.1).

The sufficient part of this theorem was obtained by Salem and Zygmund [7] in the case where Φ_n is a real number and $\{\eta_n\}$ a Rademacher sequence (a Rademacher sequence is a sequence of independent random variables $\{\varepsilon_n\}$ where $\varepsilon_n=\pm 1$ each with probability 1/2) and extended to independent symmetric $\{\eta_n e^{i\Phi_n}\}$ by Kahane [4]. In fact for sufficiency neither the condition $a_n\downarrow$ nor $\liminf_{n\to\infty} E|\eta_n|>0$ is needed, on the other hand symmetry is not needed for necessity. Theorem 1.1 was obtained for random trigonometric series that are also stationary Gaussian

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processes in [6] but the proof makes critial use of Slepian's lemma and this lemma cannot be used (as far as we can see) in the non-Gaussian case. The proof of Theorem 1.1 is given in Section 5.

The necessary part of Theorem 1.1 is a consequence of some new conditions for a Rademacher series

$$\sum a_n \varepsilon_n \cos nt, \tag{1.3}$$

($\{\varepsilon_n\}$ is a Rademacher sequence) to be unbounded on all intervals of $[0, 2\pi]$. The conditions have much greater generality than $a_n\downarrow$. The proof utilizes results of Salem and Zygmund [7] on lower bounds for the supremum of partial sums of (1.3). It is given in Section 2.

In section 3 we show that as long as $\{\eta_n e^{i\Phi_n}\}$ is independent and symmetric, the function Φ_n does not affect the uniform convergence of the series (1.1). These results utilize and extend Billard's theorem ([4] pg. 49 Theorem 2) in which the same observation is applied to Rademacher and Steinhaus series.

In Section 4 we use a recent result of Hoffman—Jørgensen [2] to show that if (1.3) does not converge uniformly then (1.1) is unboudned a.s. under very general conditions on $\{\eta_n e^{i\Phi_n}\}$ (when $\{a_n\}$ is held fixed). This enables us to extend the results of Section 2 to the series (1.1). Theorem 4.3 shows the failure of uniform convergence for a much wider class of series than is considered in Theorem 1.1.

Section 5 applies some recent results of Jain and Marcus [3] and Fernique [1] to our problem. Sufficient conditions that are sharper than (1.2) are obtained and a conjecture is made on a necessary and sufficient condition for the uniform convergence a.s. of (1.1) when $\{\eta_n e^{i\Phi_n}\}$ satisfy the hypothesis of Theorem 1.1.

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2. Discontinuity of Rademacher processes

A Rademacher sequence is a sequence of independent random variables $\varepsilon_j = \pm 1$ each with probability 1/2. In this section we specialize to the process

$$X(t) = \sum_{j=0}^{\infty} a_j \varepsilon_j \cos jt, \tag{2.1}$$

$t \in [0, 2\pi]$, $a_j^2 \in l^2$. With no loss of generality we take $a_j \geqq 0$. Since the series converges a.s. for each fixed value of t equality in (2.1) is meaningful.

Lemma 2.1. *Consider* (2.1). *Define* $n(k) = 2^{2^k}$, $k = 0, 1, \ldots,$ *and*

$$R_k = \sum_{n(k) \leqq j < n(k+1)} a_j^2 \tag{2.2}$$

$$T_k = \sum_{n(k) \leqq j < n(k+1)} a_j^4 \tag{2.3}$$

Suppose

$$\frac{T_k}{R_k^2} = O\big(n(k)^{-\theta}\big) \quad \textit{for some} \quad \theta > 0, \tag{2.4}$$

and

$$\sum_{k=1}^{\infty} (2^k R_k)^{1/2} = \infty. \tag{2.5}$$

Then the sample paths of (2.1) *are a.s. unbounded on every subinterval of* $[0, 2\pi]$.

Proof. Let j be an integer such that $\theta - 2^{-j+2} \geqq \varepsilon_1 > 0$ for some number ε_1. One of the sums

$$\sum_{n=1}^{\infty} (2^{jn+l} R_{jn+l})^{1/2} = \infty \tag{2.5a}$$

where $l=0, 1, \dots, j-1$. To simplify the notation, assume that this happens when $l=0$.

Let

$$X_k(t) = \sum_{n(k) \leqq j < n(k+1)} a_j \varepsilon_j \cos nt$$

and

$$M_k(\alpha, \beta) = \max_{\alpha \leqq t \leqq \beta} X_k(t)$$

for $0 \leqq \alpha \leqq \beta \leqq 2\pi$. Following Salem and Zygmund [7], in particular see the proof of (4.7.1), we define

$$J_k \equiv J_k[\alpha, \beta] = \frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} e^{\lambda X_k(t)}\, dt.$$

In this reference, it is shown that

$$E(J_k) \geqq e^{1/4\, \lambda^2 R_k - \lambda^4 T_k} \left(1 - \frac{\lambda^2 T_k^{1/2}}{(\beta - \alpha)^{1/2}}\right) \tag{2.6}$$

$$E(J_k^2) \leqq e^{1/2\, \lambda^2 R_k} \left[1 + \frac{a \lambda^2 T_k^{1/2}}{\beta - \alpha} + \frac{a T_k}{(\beta - \alpha)^2 R_k^2}\, e^{3/2\, \lambda^2 R_k}\right] \tag{2.7}$$

here $\lambda \geqq 0$ is an arbitrary constant which will be specified later and $a > 0$ is some fixed constant. In [7], 4.24, we find the well known (by now) inequality

$$P\{J_k \geqq \delta E(J_k)\} \geqq (1 - \delta)^2 \frac{E^2(J_k)}{E(J_k^2)}. \tag{2.8}$$

Note that

$$e^{\lambda M_k(\alpha, \beta)} \geqq J_k$$

therefore by (2.6) and (2.8) we have

$$P\left\{M_k(\alpha, \beta) \geqq \frac{1}{4} \lambda R_k - \lambda^3 T_k + \frac{1}{\lambda} \log\left(1 - \frac{\lambda^2 T_k^{1/2}}{(\beta - \alpha)^{1/2}}\right) + \frac{1}{\lambda} \log \delta\right\} \geqq (1 - \delta)^2 \frac{E^2(J_k)}{E(J_k^2)}. \tag{2.9}$$

In all that follows $\alpha, \beta, \lambda, \delta$ will depend on k. We set $\delta = e^{-(1/8)\lambda^2 R_k}$ and $\lambda^2 = (16^2 \varepsilon^2 \log n(k))/R_k$ where $\varepsilon \leqq (\varepsilon_1/3 \cdot 16^2)^{1/2}$. Furthermore α and β will always satisfy

$$(\beta-\alpha)^{-2} \frac{T_k}{R_k^2} = O\big(n(k)^{-\varepsilon_1}\big). \tag{2.10}$$

Therefore for $n(k) \geqq N$, for some N sufficiently large,

$$\begin{aligned} M_k(\alpha, \beta) &\geqq \frac{1}{4}\lambda R_k - \lambda^3 T_k - \frac{2\lambda T_k^{1/2}}{(\beta-\alpha)^{1/2}} + \frac{1}{\lambda}\log \delta(k) = \\ &= \frac{1}{8}\lambda R_k - (16\varepsilon)^2 \log n(k) \left(\frac{T_k}{R_k^2}\right) \lambda R_k - \frac{2}{(\beta-\alpha)^{1/2}} \left(\frac{T_k}{R_k^2}\right)^{1/2} \lambda R_k \geqq \\ &\geqq \frac{1}{16}\lambda R_k = \varepsilon\big(\log n(k) R_k\big)^{1/2} \end{aligned} \tag{2.11}$$

on a set of measure greater than

$$(1-\delta)^2 \frac{E^2(J_k)}{E(J_k^2)}. \tag{2.12}$$

We now use (2.6) and (2.7) to obtain a lower bound for (2.12). In order to do this we must estimate the sizes of the various terms in (2.6) and (2.7). These are given below (the C_i, $i = 1, 2, 3$ are constants).

$$\frac{\lambda^2 T_k^{1/2}}{(\beta-\alpha)^{1/2}} = C_1 \frac{\log n(k)}{(\beta-\alpha)^{1/2}} \left(\frac{T_k}{R_k^2}\right)^{1/2} = O\big(n(k)^{-\varepsilon_1/4}\big)$$

$$\frac{a\lambda^2 T_k^{1/2}}{\beta-\alpha} = C_2 \frac{\log n(k)}{\beta-\alpha} \left(\frac{T_k}{R_k^2}\right)^{1/2} = O\big(n(k)^{-\varepsilon_1/4}\big)$$

$$e^{-\lambda^4 T_k} = e^{-C_3 (\log n(k))^2 T_k/R_k^2} = 1 - O\big(n(k)^{-\theta/2}\big)$$

$$\frac{aT_k}{(\beta-\alpha)^2 R_k^2} C^{(3/2)\lambda^2 R_k} \leqq C_1 n(k)^{-\varepsilon_1} e^{(3/2)16^2 \varepsilon^2 \log n(k)} = C_1 n(k)^{-(\varepsilon_1 - (3/2)16^2 \varepsilon^2)}.$$

Since $\theta > \varepsilon_1$, we have that if (2.10) is satisfied

$$P\{M_k(\alpha, \beta) \geqq \varepsilon 2^{k/2} R_k^{1/2}\} \geqq 1 - Cn(k)^{-\varepsilon'} \tag{2.13}$$

for k sufficiently large, where C is a constant and $\varepsilon' = \varepsilon_1/4$.

We now proceed to show that (2.1) is unbounded on any interval in $[0, 2\pi]$. Any such interval, call it I, contains a subinterval of length $n(jk)^{-1}$ with initial point an integral multiple of $n(jk)^{-1}$ as long as we choose k sufficiently large. Let

$[\alpha, \beta] \subset I$ denote this subinterval, $\alpha-\beta=n(jk)^{-1}$, the value of α is unimportant. It follows from (2.13) that for $k \geqq k_0$ sufficiently large

$$P\{M_{j(k+1)}(\alpha, \beta) \geqq \varepsilon 2^{j(k+1)/2} R_{j(k+1)}^{1/2}\} \geqq 1-Cn\big(j(k+1)\big)^{-\varepsilon'}$$

since (2.10) is satisfied i.e.

$$\frac{1}{(\beta-\alpha)^2} \frac{T_{j(k+1)}}{R_{j(k+1)}^2} \leqq C'n\big(j(k+1)\big)^{-\theta} n(jk)^2 = C'n\big(j(k+1)\big)^{-[\theta-2-j+1]} \leqq$$

$$\leqq C'n\big(j(k+1)\big)^{-\varepsilon_1}.$$

(We assume $k \geqq k_0$ is sufficiently large so that $T_k/R_k^2 \leqq C'n(k)^{-\theta}$, for some constant C', see (2.4)).

The meaning of (2.14) is that on a set of measure close to 1, $X_{j(k+1)}(t)$ is larger than a term of a divergent sequence (see (2.5a)) on a subinterval of I. We will show that the maxima of the processes $X_{jk}(t)$, $k=k_0, k_0+1, \ldots$ occur on top of each other (in some sense) and consequently, with probability close to 1,

$$\sum_{k=k_0}^{\infty} X_{jk}(t) \tag{2.15}$$

is bounded on I. It then follows that since the $X_k(t)$ are independent and symmetric and since "unbounded" is a tail event that (2.1) is unbounded a.s. on I. (Alternately, we could use the contraction principle, Kahane [4], p. 18). We proceed to show that with probability close to 1 (2.15) is unbounded on I.

Bernstein's inequality ([7] (4.2.3)) states that if P is a trigonometric polynomial of degree n and M is the maximum of $|P|$ then max $|P'| \leqq nM$. In order to use this inequality we must show that $\overline{M}_k = \max_{t\in[0,2\pi]} |X_k(t)|$ is not much bigger than $M_k(\alpha, \beta)$. Define $H(k)=[R_k \log n(k)]^{1/2}$. From the proof of (4.31) [7] we obtain that for some constant C'' independent of k

$$P\{\overline{M}_k \leqq C''H(k)\} \geqq \big(1-n(k)^{-1}\big). \tag{2.16}$$

Let $(\Omega, \mathscr{F}, P)$ be the measure space for $X(t)$, $\omega \in \Omega$. Let

$$A_{j(k+1)} = \big\{\omega : M_{j(k+1)}(\alpha, \beta) \geqq \varepsilon H\big(j(k+1)\big),\ \overline{M}_{j(k+1)} \leqq C''H\big(j(k+1)\big)\big\}.$$

Using (2.13) and (2.16) we obtain

$$P(A_{j(k+1)}) > 1-Cn\big(j(k+1)\big)^{-\varepsilon'} \tag{2.17}$$

for some absolute constant C (not necessarily the same as the one in (2.13)).

Each path in $A_{j(k+1)}$ exceeds $\varepsilon H\big(j(k+1)\big)$. It follows from Bernstein's inequality that it exceeds $(\varepsilon/2)H\big(j(k+1)\big)$ over a subinterval of (α, β) of lenght at least

$$\delta \geqq \frac{\varepsilon}{2C''} n\big(j(k+1)\big)^{-2}.$$

Let $m=[\varepsilon/(2C'')]+1$ ([ ] denotes integral part). Recall that $\beta-\alpha=n(jk)^{-1}$. Divide $[\alpha,\beta]$ into intervals of length $1/(2m)\,n(j(k+1))^{-2}$ and label these intervals I_l, $l=1,\dots,2mn(jk)^{-1}n(j(k+1))^2$. Each path in $A_{j(k+1)}$ has the property that it exceeds $(\varepsilon/2)H(j(k+1))$ for all $t\in I_l$ for at least one value of l. Let

$$B_p=\{\omega:\min_{t\in I_l}X_{j(k+1)}<(\varepsilon/2)H(j(k+1)),\quad l=1,\dots,p-1;$$
$$\inf_{t\in I_p}X_{j(k+1)}(t)\geqq(\varepsilon/2)H(j(k+1))\}\tag{2.18}$$

i.e. the first I_l over which $X_{j(k+1)}(t)$ is not less than $(\varepsilon/2)H(j(k+1))$. Let $[\alpha_1,\beta_1]$ be a subinterval of length $(1/2m)n(j(k+1))^{-2}$. By (2.13)

$$P\{M_{j(k+2)}(\alpha_1,\beta_1)\geqq\varepsilon H(j(k+2))\}\geqq 1-Cn(j(k+2))^{-\varepsilon'}\tag{2.19}$$

because

$$(\beta_1-\alpha_1)^{-2}\left(\frac{T_{j(k+2)}}{R^2_{j(k+2)}}\right)\leqq(2m)^2C'n(j(k+1))^4n(j(k+2))^{-\theta}=$$
$$=(2m)^2C'n(j(k+2))^{-[\theta-2-(j-2)]}\leqq\text{Const}\,n(j(k+2))^{-\varepsilon_1}.$$

It is important to observe that neither C' nor m depends on k so (2.19) holds for $k\geqq k_0$. Also by (2.16)

$$P\{\overline{M}_{j(k+2)}\leqq C''H(j(k+2))\}\geqq 1-n(j(k+2))^{-1}.$$

We have

$$P\{\max_{t\in[\alpha\ \beta]}(X_{j(k+1)}(t)+X_{j(k+2)}(t))>(\varepsilon/2)H(j(k+1))+\varepsilon H(j(k+2))\}=$$
$$=\sum_p P\{X_{j(k+1)}(t)\varepsilon B_p\}P\{M_{j(k+2)}(I_p)\geqq\varepsilon H(j(k+2))\}\geqq$$
$$\geqq P(A_{j(k+1)})P(A_{j(k+2)})\geqq 1-C[n(j(k+1))^{-\varepsilon'}+n(j(k+2))^{-\varepsilon'}].\tag{2.20}$$

Here $A_{j(k+2)}$ is defined as in (2.17). Note that $N_{j(k+2)}(I_p)$ stands for $M_{j(k+2)}(\gamma,\delta)$ where $I_p=[\gamma,\delta]$. The statement (2.20) follows because $M_{j(k+2)}$ is independent of $X_{j(k+1)}$ and because the bound for $M_{j(k+2)}(I_p)$ depends only on the length of I_p (and hence is the same for all p).

If we divide $[\alpha_1,\beta_1]$ into intervals of length $(1/2m)n(j(k+2))$ we can show by the above argument, that each path in $A_{j(k+1)}\cap A_{j(k+2)}$ is not less than $(\varepsilon/2)[H(j(\mathrm{k}+1))+H(j(k+2))]$ over at least one of these intervals. Iterating the above argument we obtain

$$P\left\{\max_{t\in[\alpha,\beta]}\sum_{l=1}^m X_{j(k+l)}(t)\geqq\frac{\varepsilon}{2}\sum_{l=1}^m H(j(k+l))\right\}\geqq 1-C\sum_{l=1}^m n(j(k+l))^{-\varepsilon'}.\tag{2.21}$$

Thus we have shown that (2.15) is unbounded on I with probability close to 1. This completes the proof.

Note that (2.21) can be used to obtain a lower bound for the maximum of the partial sums of $X(t)$ under hypothesis (2.4), irregardless of whether (2.5) holds.

Lemma 2.2. *Let $a_j\downarrow$ in (2.1) then (2.5) is a sufficient condition for $X(t)$ to be unbounded on all intervals of* $[0, 2\pi]$.

Proof. Consider (2.1) with $a_j\downarrow$ and suppose that (2.5) is satisfied. Form a new series

$$\sum_{j=0}^{\infty} b_j \varepsilon_j \cos nt \tag{2.22}$$

where the b_i are defined as follows:

$$b_j = a_{n(k)+n(k-1)}, \quad n(k) \leqq j \leqq n(k)+n(k-1)$$
$$b_j = a_j, \quad n(k)+n(k-1) \leqq j < n(k+1).$$

Define T_k' and R_k' as in (2.2) and (2.3) but for the series (2.22). We now show that $\{T_k'\}$ and $\{R_k'\}$ satisfy (2.4) and (2.5). For (2.5) we observe that

$$R_{k-1}' + R_k' \geqq \sum_{n(k)-n(k-1)\leqq j<n(k)} a_j^2 + \sum_{n(k)+n(k-1)\leqq j<n(k+1)} a_j^2 \geqq R_k$$

since $a_j\downarrow$. Therefore

$$\sum (2^k R_{k-1}')^{1/2} + \sum (2^k R_k')^{1/2} \geqq \sum (2^k R_k)^{1/2} = \infty$$

so $\{R_k'\}$ satisfies (2.5). For (2.4) let $a_{n(k)+n(k-1)}=a$ and $n(k-1)=N$, then

$$T_k' = Na^4 + \sum a_j^4$$
$$R_k' = Na^2 + \sum a_j^2$$

where the sum is taken over $n(k)+n(k-1)\leqq j<n(k+1)$. We have

$$\frac{T_k'}{(R_k')^2} \leqq \frac{1}{n(k-1)} = n(k)^{-1/2},$$

since

$$\frac{Na^4 + \sum a_j^2}{(Na^2+\sum a_j^2)^2} = \frac{1}{N}\frac{a^4 + \frac{1}{N}\sum a_j^4}{\left(a^2+\frac{1}{N}\sum a_j^2\right)^2}$$

and

$$\left(a^2+\frac{1}{N}\sum a_j^2\right)^2 \geqq a^4 + \frac{2}{N}a^2 \sum a_j^2 \geqq a^4 + \frac{2}{N}\sum a_j^4.$$

For the least inequality we use that $a\geqq a_j$ for $n(k)+n(k-1)\leqq j<n(k+1)$.

It follows from Lemma 2.1 that (2.22) is unbounded on all intervals of $[0, 2\pi]$. By the contraction principle [4], p. 18; (2.1) is also.

Lemma 2.3. *Let $a_j\downarrow$ and R_k be as defined in* (2.2) *then*

$$\sum (2^k R_k)^{1/2} < \infty \tag{2.23}$$

if and only if

$$\sum \frac{(\sum_{k=n}^{\infty} a_k^2)^{1/2}}{n(\log n)^{1/2}} < \infty. \tag{2.24}$$

Proof. By a change of variables (2.24) holds if and only if

$$\sum (2^k \sum_{n=k}^{\infty} R_k)^{1/2} < \infty. \tag{2.25}$$

Therefore (2.24) implies (2.23). For the reverse implication we have

$$\sum_{k=1}^{\infty} (2^k R_k)^{1/2} \geqq \sum_{k=1}^{\infty} (2^k \sum_{n=k}^{\infty} R_n)^{1/2} - \sum_{k=1}^{\infty} (2^k \sum_{n=k+1}^{\infty} R_n)^{1/2} =$$

$$= \sum_{k=1}^{\infty} (2^k \sum_{n=k}^{\infty} R_n)^{1/2} - \frac{1}{\sqrt{2}} \sum_{k=1}^{\infty} (2^{k+1} \sum_{n=k+1}^{\infty} R_n)^{1/2} \geqq$$

$$\geqq \left(1 - \frac{1}{\sqrt{2}}\right) \sum_{k=1}^{\infty} (2^k \sum_{n=k}^{\infty} R_k)^{1/2}.$$

Theorem 2.4. *Let $a_j\downarrow$, then* (2.24) *is a necessary and sufficient condition for the uniform convergence of* (2.1).

Proof. Sufficiency is given in [7] (5.1.5), necessity by the previous three lemmas. Let

$$S_j = (\sum_{k=2^j}^{2^{j+1}-1} a_k^2)^{1/2}$$

Paley and Zygmund [8] (see also [7] (5.2.2) ff and [4] pg 73) have shown that $\sum S_j < \infty$ is necessary for the uniform convergence of (2.1). This condition is different from ours even when $a_j\downarrow$. In the following example the series in (2.24) diverges but $\sum S_j < \infty$. As above let $n(k) = 2^{2^k}$. Take $a_j = (k^3 n(k))^{-1/2}$, $n(k) \leqq j < n(k+1)$, then $S_j = 2^{j/2}(k^3 n(k))^{-1/2}$, $2^k \leqq j < 2^{k+1}$.

$$\sum_{j=2^N}^{\infty} S_j = \sum_{k=N}^{\infty} \sum_{j=2^k}^{2^{k+1}-1} 2^{j/2}(k^3 n(k+1))^{-1/2} \leqq \sqrt{2} \sum_{k=N}^{\infty} k^{-3/2} < \infty.$$

Recall (2.24) is satisfied if and only if (2.25) is satisfied. The series in (2.25) is

$$\sum_{k=1}^{\infty} (2^k \sum_{j=2^k}^{\infty} S_j^2)^{1/2} \geqq \sum_{k=1}^{\infty} 2^{k/2} (\sum_{j=2^k}^{2^{k+1}-1} S_j^2)^{1/2} \geqq 1/2 \sum_{k=1}^{\infty} 2^{k/2} k^{-3/2} = \infty.$$

It is quite easy to find examples where $\sum S_j = \infty$ and condition (2.4) is not satisfied.

Remark. In [7] (5.5.1) there is a result like Theorem 2.4 dealing with the case $a_j\downarrow$ but additional conditions are placed on the $\{a_j\}$.

3. Equivalence classes of uniformly convergent random Fourier series

We show that for random Fourier series involving independent symmetric random variables the distribution of the phase has no effect on the uniform convergence of the series. Consider

$$\sum a_n \eta_n \cos(nt+\Phi_n) \tag{3.0}$$

where $\{\eta_n e^{i\Phi_n}\}$ is a sequence of independent symmetric complex valued random variables (η_n and Φ_n are real and not necessarily independent of each other). Any such sequence can be written in the form $\{|\eta_n| e^{i\Phi_n'}\}$ (Φ_n' being the obvious modification of Φ_n). Note that $\{|\eta_n|\}$ is also an independent sequence.

Theorem 3.1. *Let $\{|\eta_n| e^{i\Phi_n}\}$ and $\{|\eta_n| e^{i\theta_n}\}$ be sequences of independent complex valued random variables as described above. The uniform convergence a.s. of*

$$\sum a_n |\eta_n| \cos(nt+\Phi_n) \tag{3.1}$$

implies the uniform convergence a.s. of

$$\sum a_n |\eta_n| \cos(nt+\theta_n). \tag{3.2}$$

Proof. Let $(\Omega_1, \mathscr{F}_1, P_1)$ be the probability space for $\{|\eta_n| e^{i\Phi_n}\}$ and introduce an independent Rademacher sequence $\{\varepsilon_n\}$ defined on $(\Omega_2, \mathscr{F}_2, P_2)$. Consider

$$\sum a_n \varepsilon_n |\eta_n| \cos(nt+\Phi_n) \tag{3.3}$$

on the product space $(\Omega_1 \times \Omega_2, \mathscr{F}_1 \times \mathscr{F}_2, P_1 \times P_2)$. Since $\{|\eta_n| e^{i\Phi_n}\}$ are symmetric (3.3) is stochastically similar to (3.1) and consequently converges uniformly a.s. (with respect to $P_1 \times P_2$). It follows that for each $\omega_1 \in \bar{\Omega}_1$ where $\bar{\Omega}_1 \subset \Omega_1$, $P(\bar{\Omega}_1)=1$

$$\sum a_n \varepsilon_n |\eta_n(\omega_1)| \cos(nt+\Phi_n(\omega_1)) \tag{3.4}$$

converges uniformly a.s. (P_2) on Ω_2.

Let $\{\Psi_n\}$ be a sequence of independent random variables each one uniformly distributed on $[0, 2\pi]$ with probability space $(\Omega_3, \mathscr{F}_3, P_3)$. By Billard's theorem ([4], pg. 49, Theorem 2),

$$\sum a_n |\eta_n(\omega_1)| \cos(nt+\Phi_n(\omega_1)+\Psi_n) \tag{3.5}$$

converges uniformly a.s. (P_3), for each $\omega_1 \in \bar{\Omega}_1$.

Since the Ψ_n are uniformly distributed on $[0, 2\pi]$ (3.5) is stochastically similar to

$$\sum a_n |\eta_n(\omega_1)| \cos(nt+\theta_n(\omega_1)+\Psi_n). \tag{3.6}$$

Applying Billard's theorem again we see that

$$\sum a_n \varepsilon_n |\eta_n| \cos(nt+\theta_n)$$

converges uniformly a.s. $(P_1 \times P_2)$. Since $\{|\eta_n| e^{i\theta_n}\}$ is symmetric the theorem follows.

For emphasis we add the following:

Corollary 3.2. *Let $\{\eta_n\}$ be independent, symmetric real valued random variables $\{\alpha_n\}$ real numbers; then if*

$$\sum a_n \eta_n \cos(nt + \alpha_n)$$

converges uniformly a.s. so does

$$\sum a_n |\eta_n| \cos(nt + \Phi_n)$$

for every independent symmetric $\{|\eta_n| e^{i\Phi_n}\}$.

Let $\{\eta_n e^{i\Phi_n}\}$ be independent symmetric complex valued random variables; then if

$$\sum a_n \eta_n \cos(nt + \Phi_n)$$

converges uniformly a.s. so does

$$\sum a_n \varepsilon_n \eta_n \cos(nt + \alpha_n)$$

for any sequence $\{\alpha_n\}$ of real numbers ($\{\varepsilon_n\}$ is a Rademacher sequence independent of $\{\eta_n\}$).

Proof. This is simply a restatement of Theorem 3.1. We need only note that since $\{|\eta_n|\}$ are independent $\{\varepsilon_n \eta_n\}$ is a sequence of independent symmetric random variables.

4. Undbounded random trigonometric series

We want to infer from the unboundedness a.s. of the series (2.1) that

$$\sum a_n \eta_n \cos(nt + \Phi_n) \tag{4.1}$$

is also unbounded a.s. where the $\{a_n\}$ are the same and $\{\eta_n e^{i\Phi_n}\}$ are independent complex valued random variables (η_n and Φ_n real). In this section $\eta_n e^{i\Phi_n}$ is not assumed to by symmetric unless specifically stated. By unbounded we mean that the partial sums of (4.1) are unbounded.

A recent theorem of Hoffmann—Jørgensen ([2], Theorem 5.7) says that certain series of continuous functions multiplied by a Rademacher sequence is bounded a.s. (or uniformly convergent a.s.) if they are bounded a.s. (or uniformly convergent a.s.) when multiplied by any other sequence of independent identically distributed random variables. We state this theorem specialized to our problem and add an observation that enables us to use it when the random variables are not identically distributed.

Theorem 4.1. *Let $\{\eta_n\}$ be independent nondegenerate random variables such that*

$$P(|\eta_n-\eta_n'|>a)>\delta \tag{4.2}$$

for some $a, \delta>0$ uniformly in $n(\{\eta_n'\}$ is an independent copy of $\{\eta_n\})$. Then if

$$\sum a_n\eta_n \cos nt \tag{4.3}$$

is bounded a.s.

$$\sum a_n\varepsilon_n \cos nt \tag{4.4}$$

converges uniformly a.s., where $\{\varepsilon_n\}$ is a Rademacher sequence.

Statement (4.2) *is satisfied if either*

$$\{\eta_n\} \text{ are identically distributed} \tag{4.5}$$

or

$$E\eta_n^2 \leqq M, \quad \liminf_{n\to\infty} E|\eta_n-E\eta_n| = C>0 \quad (M, C \text{ are numbers}). \tag{4.6}$$

Proof. In Theorem 5.7 [2] it is shown that the boundedness a.s. of (4.3) implies the boundedness a.s. of (4.4) under condition (4.5). The only use of the provision "identically distributed" was to show that for some N

$$(\mu_n*\ldots*\mu_n)[-1,1]<1/2 \tag{4.7}$$

for all n, when μ_n is the measure corresponding to the symmetric random variable $\eta_n-\eta_n'$ and the convolution is taken N times. (Of course under (4.5) all the μ_n are the same.) However (4.7) will also be true under condition (4.2) as can be seen by considering the random walk $\xi_{n,1}+\ldots+\xi_{n,N}$ where $\xi_{n,j}$ are independent copies of $\eta_n-\eta_n'$. Therefore Hoffmann—Jørgensen's theorem follows under the more general hypothesis (4.2). We use Theorem 3, pg. 49 [4] to show that if (4.4) is bounded a.s. it converges uniformly a.s.

We now show that (4.6) implies (4.2). Let $(\Omega_1, \mathscr{F}_1, P_1)$ and $(\Omega_2, \mathscr{F}_2, P_2)$ be the probability spaces of $\{\eta_n\}$, $\{\eta_n'\}$ and E_1, E_2 the corresponding expectation operators. Let E be expectation on the product space $(\Omega_1\times\Omega_2, \mathscr{F}_1\times\mathscr{F}_2, P_1\times P_2)$. Note that

$$E|\eta_n-E\eta_n| \leqq E|\eta-\eta_n'| \leqq 2E|\eta_n-E\eta_n|. \tag{4.8}$$

The right side of (4.8) follows from the triangle inequality. The left side from Jensen's inequality as follows:

$$E_2|\eta_n-\eta_n'| \geqq |\eta_n-E_2\eta_n'| = |\eta_n-E\eta_n|$$

$$E_1E_2|\eta_n-\eta_n'| \geqq E_1|\eta_n-E\eta_n| = E|\eta_n-E\eta_n|.$$

(Of course $E_1E_2=E$.)

By [7] (4.2.4)

$$P[|\eta_n-\eta_n'|>\theta E|\eta_n-\eta_n'|] \geqq (1-\theta)^2 \frac{E^2|\eta_n-\eta_n'|}{E|\eta_n-\eta_n'|^2}.$$

Therefore

$$P[|\eta_n - \eta_n'| > \theta C] \geqq \frac{(1-\theta)^2 C^2}{4M}.$$

This completes the proof.

Remark 1. Theorem 5.7 [2] is proved in a much more general setting than is given here. The modification given in our Theorem 3.1 also applies in the general setting.

Remark 2. When $\{\eta_n\}$ are not identically distributed some condition like (4.2) is necessary. Otherwise we can easily obtain counterexamples. Choose any sequence $\{a_n\}$ so that (4.2) is unbounded a.s. (use Lemma 2.1 or 2.2) and let $\{b_n\} \in l^1$. Pick any symmetric $\{\eta_n\}$ such that $E\eta_n^2 = 1$ and

$$\sum P(a_n \eta_n > b_n) < \infty.$$

By the Borel—Cantelli lemma (4.3) is absolutely convergent a.s.

Theorem 4.2. *Let $\{\eta_n e^{i\Phi_n}\}$ be independent complex valued random variables. If either*

$$\{\eta_n e^{i\Phi_n}\} \text{ is identically distributed} \tag{4.9}$$

or

$$E\eta_n^2 \leqq M, \quad \liminf_{n\to\infty} E|\eta_n e^{i\Phi_n} - E\eta_n e^{i\Phi_n}| > 0, \tag{4.10}$$

then the boundedness a.s. of (4.1) *implies the uniform convergence a.s. of* (4.4).

Proof. Let $\{\eta_n' e^{i\Phi_n'}\}$ be an independent copy of $\{\eta_n e^{i\Phi_n}\}$. If (4.1) is bounded a.s. so is

$$\sum a_n[(\eta_n \cos \Phi_n - \eta_n' \cos \Phi_n') \cos nt + (\eta_n \sin \Phi_n - \eta_n' \sin \Phi_n') \sin nt]. \tag{4.11}$$

This series can be put in the form

$$\sum a_n \xi_n \cos (nt + \theta_n) \tag{4.12}$$

where

$$\xi_n = |\eta_n e^{i\Phi_n} - \eta_n' e^{i\Phi_n'}| \tag{4.13}$$

and $\{\xi_n e^{i\theta_n}\}$ is independent and symmetric. By Theorem 3, pg. 49 [4] (4.12) converges uniformly a.s. By our Corollary (3.2)

$$\sum a_n \varepsilon_n \xi_n \cos nt \tag{4.14}$$

also converges uniformly a.s., where $\{\varepsilon_n\}$ is a Rademacher sequence independent of $\{\xi_n\}$.

We now use Theorem (4.1) on (4.14). Therefore (4.4) converges uniformly if $\{\varepsilon_n \xi_n\}$ is identically distributed which is the case under (4.9). Since $E\varepsilon_n \xi_n = 0$ and $E\xi_n^2 \leqq 4E\eta_n^2$ using (4.6) we see that (4.4) converges uniformly a.s. if

$$\liminf_{n\to\infty} E|\xi_n| > 0. \tag{4.15}$$

By (4.8), (4.15) and (4.10) are equivalent.

In [4] Chapter 8 §4, a condition under which the series (4.4) does not converge uniformly a.s. is extended to series of the form (4.1) under the additional conditions that $\{\eta_n e^{i\Phi_n}\}$ is symmetric, $E\eta_n^2 = 1$ and $E\eta_n^4 \leqq C'$, (C' is a constant). This result is contained in Theorem 4.2. To see this we use [7] (4.2.4)

$$P[\eta_n^2 > \delta] \geqq (1-\delta)^2 \frac{(E\eta_n^2)^2}{E\eta_n^4} \geqq \frac{(1-\delta)^2}{C} \equiv \alpha.$$

Consequently $E|\eta_n| \geqq \delta^{1/2}\alpha$ and $\liminf_{n\to\infty} E|\eta_n| > 0$. The random variables η_n do not have to have a fourth moment; they needn't even have a second moment as long as (4.2) is satisfied.

Finally we state our most general condition on the coefficients $\{a_n\}$ that imply (4.1) is unbounded a.s.

Theorem 4.3. *Consider the series* (4.1) *with* $\{\eta_n e^{i\Phi_n}\}$ *independent. If* $\{\eta_n e^{i\Phi_n}\}$ *satisfies either* (4.9) *or* (4.10) *and if* $\{a_n\}$ *satisfies* (2.4) *and* (2.5) *then* (4.1) *is unbounded a.s.*

Proof. The proof follows from Lemma 2.1 and Theorem 4.2.

5. Some Conditions for the Uniform Convergence of Random Fourier Series

We define

$$\sigma^2(h) = \sum a_n^2 \sin^2 \frac{nh}{2} \tag{5.1}$$

and

$$I(\sigma) = \int_0 \frac{\sigma(u)}{u(\log 1/u)^{1/2}}\, du.$$

The following theorem is obtained in [3].

Theorem 5.1. *Consider the series* (3.0) *where* $\{\eta_n e^{i\Phi_n}\}$ *is independent and symmetric,* $E\eta_n^2 = 1$ *and* $\{a_n\} \in l^2$. *If* $I(\sigma) < \infty$ *the series converges uniformly a.s.*

It is shown in [5], Theorem 1 (see also [6] that (2.24) implies $I(\sigma) < \infty$; therefore this theorem can be used to supply the sufficiency part of Theorem 1.1. The necessary part comes from Theorem 4.3.

It is interesting to see where the function $\sigma^2(h)$ comes from. Let

$$X(t) = \sum_{n=0}^{\infty} a_n[\eta_n \cos nt + \eta_n' \sin nt] \tag{5.2}$$

where $\{a_n\} \in l^2$, $\{\eta_n\}$ is independent and symmetric, $E\eta_n^2 = 1$ and $\{\eta_n'\}$ is an independent copy of $\{\eta_n\}$. Since $\{E|a_n^2\eta_n^2|\} \in l^2$ the series converges for each t a.s. hence $X(t)$ is well defined. Notice that

$$\sigma^2(h) = E|X(t+h) - X(t)|^2.$$

That is, $X(t)$ is a weakly stationary process with increments variance $\sigma^2(h)$. However, as we have shown in Section 3, it doesn't matter whether the series (3.0) are stationary or not; therefore we simply define $\sigma^2(h)$ as a function of the coefficients as in (5.1).

If $\{\eta_n e^{i\Phi_n}\}$ satisfies the hypothesis of Theorem 1.1 we will say it has property A. When $\{\eta_n e^{i\Phi_n}\}$ has property A all the conditions we have given for uniform convergence or unboundedness a.s. of the series (3.0) depend only on the coefficients $\{a_n\}$. In fact we know of no examples of coefficients $\{a_n\}$ for which (3.0) converges uniformly for some $\{\eta_n e^{i\Phi_n}\}$ with property A and not for every other one having property A. Nevertheless by placing further conditions on the random variables $\{\eta_n e^{i\Phi_n}\}$ we can obtain much stronger conditions for uniform convergence. In order to show this we need the following definitions.

Let $f(h)$, $h \in [0, 2\pi]$ be a positive continuous real valued function and let

$$m(y) = \lambda\{h \in [0, 2\pi] : f(h) < y\}$$

where λ is Lebesgue measure. Let $\bar{f}(h)$ be the generalized inverse of m given by $\bar{f}(h) = \sup\{y : m(y) < h\}$. The function $\bar{f}$ is called the "monotone rearrangement" of f. Let $\bar{\sigma}$ be the monotone rearrangement of σ.

A random variable is called subgaussian if for any real λ

$$E[e^{\lambda\xi}] \leqq e^{\lambda^2\sigma^2/2}$$

where $\sigma^2 = EX^2$. Both a zero mean normal random variable and $\varepsilon_n = \pm 1$ each with probability 1/2 are subgaussian.

Theorem 5.2. *Let $\{\eta_n e^{i\Phi_n}\}$ be independent and symmetric, $E|\eta_n|^2 = 1$, $\{a_n\} \in l^2$ and assume that either*

$$\varepsilon_n|\eta_n| \text{ is subgaussian } (\{\varepsilon_n\} \text{ is a Rademacher sequence independent of } |\eta_n|) \tag{5.3}$$

or

$$|\eta_n| \leqq M \text{ a.s. where } M \text{ is a constant independent of } n. \tag{5.4}$$

Then if $I(\bar{\sigma}) < \infty$, (3.0) converges uniformly a.s.

Proof. This theorem is proved in [3] (Theorem 3.2), for series of the form (5.2) when the random variables $\{\eta_n\}$ and $\{\eta_n'\}$ are independent and subgaussian. When this series converges uniformly a.s. it is easy to see that both the cosine series and the sine series converges uniformly a.s. ([3] Lemma 4.1). Therefore if (5.3) is satisfied $I(\bar{\sigma})<\infty$ implies the uniform convergence of $\sum a_n \varepsilon_n |\eta_n| \cos nt$. This result extends to the series (3.0) by Corollary 3.2.

Suppose (5.4) is satisfied. Let $(\Omega_1, \mathscr{F}_1, P_1)$ be the probability space of $\{|\eta_n|\}$ and $(\Omega_2, \mathscr{F}_2, P_2)$ be the probability space of $\{\varepsilon_n\}$ and consider

$$\sum a_n \varepsilon_n |\eta_n| \cos nt \tag{5.5}$$

on $(\Omega_1 \times \Omega_2, \mathscr{F}_1 \times \mathscr{F}_2, P_1 \times P_2)$. For each $\omega_1 \in \bar{\Omega}_1$, $\bar{\Omega}_1 \subset \Omega_1$, $P(\bar{\Omega}_1)=1$,

$$\sum a_n \varepsilon_n |\eta_n(\omega_1)| \cos nt$$

converges uniformly a.s. (P_2) by the contraction principle ([4], pg. 19, Theorem 5). By Fubini's theorem (5.5) converges uniformly a.s. and the theorem follows by Corollary 3.2.

We know [3] that $I(\sigma)<\infty$ implies $I(\bar{\sigma})$ finite and that the converse is false. The reason that $I(\bar{\sigma})$ is so important is a consequence of a remarkable theorem of Fernique [1] which gives us:

Theorem 5.3. *Let $\{\eta_n e^{i\Phi_n}\}$ be independent and symmetric $E|\eta_n|^2=1$, $\{a_n\}\in l^2$ and suppose that $\varepsilon_n|\eta_n|$ is a normal random variable; then $I(\bar{\sigma})<\infty$ is necessary and sufficient for the uniform convergence of* (3.0).

Proof. Fernique's theorem together with line (2.8) in [3] says that $I(\bar{\sigma})<\infty$ is necessary and sufficient for sample path continuity of a stationary Gaussian process. By the proof of Theorem 5.2 (5.5) converges uniformly a.s. for $\varepsilon_n|\eta_n|$ under our hypothesis. Therefore by Corollary 3.2 so does (3.0).

This leads us to the following:

Conjecture. Let $\{\eta_n e^{i\Phi_n}\}$ satisfy the hypothesis of Theorem 1.1, then $I(\bar{\sigma})<\infty$ is necessary and sufficient for the uniform convergence of (3.0).

The results of Section 4 show that for the necessary part of the conjecture one need only show that $I(\bar{\sigma})=\infty$ implies that the series (2.1) does not converge uniformly.

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