L^p estimates for strongly singular convolution operators in $\mathbf{R}^n$

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Let the operator T_K be defined by $T_K f = K * f$, $f \in C_0^\infty(\mathbf{R}^n)$, where K is a distribution with compact support in $\mathbf{R}^n$, locally integrable outside the origin and satisfying the conditions

$$A(\alpha) \qquad |\hat{K}(\xi)| \leqq B(1+|\xi|)^{-n\alpha/2}, \quad \xi \in \mathbf{R}^n,$$

and

$$B(\theta) \qquad \int_{|x|>2|y|^{1-\theta}} |K(x-y)-K(x)|\,dx \leqq B, \quad |y| < b.$$

Here $\hat{K}$ is the Fourier transform of K, B and b denote positive constants and $0 \leqq \alpha \leqq \leqq \theta < 1$.

The conditions $A(0)$ and $B(0)$ are satisfied by the well-known Calderón—Zygmund kernels and in the case $\alpha = \theta = 0$ it is known that T_K can be extended to a bounded linear operator on $L^p(\mathbf{R}^n)$ for $1 < p < \infty$.

In the case $\alpha > 0$ an example of a kernel satisfying the above conditions can be obtained in the following way. Let L be defined by setting

$$\hat{L}(\xi) = \psi(\xi) e^{i|\xi|^a} |\xi|^{-n\alpha/2}, \quad \xi \in \mathbf{R}^n,$$

where $a = (n\alpha(1-\theta)+2\theta)/(n(1-\theta)+2)$ and ψ is a C^∞ function which vanishes near the origin and is equal to one for $|\xi|$ large. Then $L = K + M$, where K satisfies the above conditions and M is an L^1 function with $\hat{M}(\xi) = O(|\xi|^{-N})$, $|\xi| \to \infty$, for all N. In fact it was proved by S. Wainger [7] that $K(x)$ is essentially equal to $c_1 |x|^{-n-\lambda} e^{ic_2|x|^{a'}}$ close to the origin, where $\lambda = n(a-\alpha)/2(1-a)$ and $1/a + 1/a' = 1$. Hence $|\text{grad } K(x)| \leqq \leqq C|x|^{-n-\lambda-1+a'}$, from which it follows that $B(\theta)$ is satisfied. The L^p theory for operators obtained by convolution with kernels of this type has been studied by I. I. Hirschmann [4] and Wainger [7].

C. Fefferman [2] proved that if $0<\alpha=\theta<1$ and K satisfies the above conditions then T_K is bounded on $L^p(\mathbf{R}^n)$ for $1<p<\infty$. Recently Fefferman and E. M. Stein [3] have obtained the following results.

Theorem A. *If* $0<\alpha=\theta<1$ *and* K *satisfies the above assumptions, then* $\hat{K}$ *is a Fourier multiplier for* $H^1(\mathbf{R}^n)$.

Theorem B. *Suppose* m *is a Fourier multiplier for* $H^1(\mathbf{R}^n)$. *Let* δ *be a positive number and assume* $|m(\xi)|\leqq C|\xi|^{-\delta}$. *Then* $|\xi|^\gamma m(\xi)$ *is a Fourier multiplier for* $L^p(\mathbf{R}^n)$ *if* $1<p<\infty$, $|1/p-1/2|\leqq 1/2-\gamma/2\delta$ *and* $\gamma\geqq 0$.

Theorems A and B can be applied to the multipliers $\hat{L}$ defined above.

J.-E. Björk [1] has determined the values of p for which T_K is bounded on $L^p(\mathbf{R}^n)$ in the case when K is a distribution with compact support, which satisfies only condition $A(\alpha)$. Björk has also asked if it is possible to obtain sharp results on the L^p boundedness of T_K in the case $\alpha<\theta$. We shall here use Theorems A and B to prove that the answer to this question is affirmative.

Theorem 1. *If* $0<\alpha<\theta<1$ *and* K *satisfies the above assumptions, then* T_K *can be extended to a bounded linear operator on* $L^p(\mathbf{R}^n)$ *if*

$$|1/p-1/2| \leqq (n\alpha(1-\theta)+2\alpha)/(2n\alpha(1-\theta)+4\theta).$$

It follows from the counterexamples of Wainger [7] for the kernel L defined above that the condition on p in Theorem 1 cannot be relaxed.

Proof of Theorem 1. Choose $\varphi\in C_0^\infty(\mathbf{R}^n)$ such that $\int\varphi(x)\,dx=1$, $\operatorname{supp}\varphi\subset\{x;\ |x|<1\}$ and $\varphi(x)\geqq 0$, $x\in\mathbf{R}^n$. Set $\varphi_\varepsilon(x)=\varepsilon^{-n}\varphi(x/\varepsilon)$, $\varepsilon>0$.

We shall first prove that $K*\varphi_\varepsilon$, $0<\varepsilon\leqq 1$, satisfies conditions $A(\alpha)$ and $B(\theta)$ with constants B and b independent of ε (cf. the argument in [2], pp. 23—24). $A(\alpha)$ is obviously satisfied. To prove that $B(\theta)$ is satisfied, first assume $\varepsilon<|y|^{1-\theta}$. Then

$$\int_{|x|>2|y|^{1-\theta}}|K*\varphi_\varepsilon(x-y)-K*\varphi_\varepsilon(x)|\,dx \leqq$$

$$\leqq \int\varphi_\varepsilon(t)\Big(\int_{|x|>2|y|^{1-\theta}}|K(x-y-t)-K(x-t)|\,dx\Big)\,dt \leqq$$

$$\leqq \int\varphi_\varepsilon(t)\Big(\int_{|x|>|y|^{1-\theta}}|K(x-y)-K(x)|\,dx\Big)\,dt \leqq C,$$

if $|y|$ is sufficiently small, where the last inequality follows from writing

$$K(x-y)-K(x) = \sum_{i=1}^m \big(K(x-iy/m)-K(x-(i-1)y/m)\big), \tag{1}$$

using the triangle inequality and applying the condition $B(\theta)$ for K.

If $\varepsilon \geqq |y|^{1-\theta}$ then

$$\int_{|x|>3\varepsilon} |K*\varphi_\varepsilon(x-y)-K*\varphi_\varepsilon(x)|\,dx \leqq$$

$$\leqq \int \varphi_\varepsilon(t)\Big(\int_{|x|>3\varepsilon} |K(x-y-t)-K(x-t)|\,dx\Big)\,dt \leqq \int_{|x|>2\varepsilon} |K(x-y)-K(x)|\,dx \leqq C$$

and

$$\int_{|x|\leqq 3\varepsilon} |K*\varphi_\varepsilon(x-y)-K*\varphi_\varepsilon(x)|\,dx \leqq 2\int_{|x|\leqq 4\varepsilon} |K*\varphi_\varepsilon(x)|\,dx \leqq$$

$$\leqq C\,\varepsilon^{n/2}\|K*\varphi_\varepsilon\|_2 \leqq C\,\varepsilon^{n/2}\|\varphi_\varepsilon\|_2 = C.$$

In proving the theorem we may therefore assume that $K\in L^1(\mathbf{R}^n)$, as long as our estimates depend only on the constants in $A(\alpha)$ and $B(\theta)$.

We set $a=\big(n\alpha(1-\theta)+2\theta\big)/\lambda$ and $\beta=n(\theta-\alpha)/\lambda$, where $\lambda=n(1-\theta)+2$, so that

$$\beta = na/2-n\alpha/2. \tag{2}$$

Also let G_β denote the Bessel kernel defined by $\hat{G}_\beta(\xi)=(1+|\xi|^2)^{-\beta/2}$. Then there exist finite measures μ and ν on $\mathbf{R}^n$ so that

$$\hat{K}(\xi) = (1+|\xi|^2)^{\beta/2}(G_\beta*K)\hat{\ }(\xi) = |\xi|^\beta\hat{\mu}(\xi)(G_\beta*K)\hat{\ }(\xi)+\hat{\nu}(\xi)(G_\beta*K)\hat{\ }(\xi)$$

(see [6], p. 133). We shall prove that convolution with $G_\beta*K$ defines a bounded operator on $H^1(\mathbf{R}^n)$. It then follows from Theorem B that T is bounded on $L^p(\mathbf{R}^n)$ if $|1/p-1/2|\leqq 1/2-\beta/na$, which gives our theorem.

First choose $\varphi\in C_0^\infty(\mathbf{R}^n)$ so that $\varphi\geqq 0$, $\operatorname{supp}\varphi\subset\{x;\ 1/2<|x|<2\}$ and $\sum_{k=0}^\infty \varphi(2^kx)=1$, $0<|x|<1$. Also set $\varphi_k(x)=\varphi(2^kx)$ and $\psi(x)=\sum_{k=0}^\infty \varphi_k(x)$. We shall use the decomposition

$$G_\beta*K = (\psi G_\beta)*K+\big((1-\psi)G_\beta\big)*K. \tag{3}$$

$(1-\psi)G_\beta$ is smooth and rapidly decreasing at infinity and hence the last term in (3) is an L^1 function. It follows from $A(\alpha)$ and (2) that $(\psi G_\beta)*K$ satisfies $A(a)$ and to use Theorem A it suffices to prove that $(\psi G_\beta)*K$ satisfies $B(a)$. We set $G_{\beta,k}=\varphi_kG_\beta$ so that $(\psi G_\beta)*K=\sum_{k=0}^\infty G_{\beta,k}*K$.

We also fix y, $|y|<\min(b, 1/2)$, and set

$$I_k = \int_{|x|>2|y|^{1-a}} |G_{\beta,k}*K(x-y)-G_{\beta,k}*K(x)|\,dx.$$

We shall estimate each I_k separately and consider three cases.

Case 1. $2^{-k}<|y|/2$.

A change of the order of integration yields

$$I_k \leqq \int G_{\beta,k}(t)\Big(\int_{|x|>2|y|^{1-a}} |K(x-y-t)-K(x-t)|\,dx\Big)\,dt \leqq$$

$$\leqq \int G_{\beta,k}(t)\,dt\int_{|x|>|y|^{1-a}} |K(x-y)-K(x)|\,dx \leqq$$

$$\leqq C2^{-k\beta}\sum_{i=1}^m \int_{|x|>|y|^{1-a}} \big|K(x-iy/m)-K\big(x-(i-1)y/m\big)\big|\,dx.$$

Choosing m as the least positive integer such that $2|y/m|^{1-\theta}<|y|^{1-a}-|y|$ we see that each term in the above sum is majorized by

$$\int_{|x|>2|y/m|^{1-\theta}} |K(x-y/m)-K(x)|\,dx$$

and hence $I_k \leqq C2^{-k\beta}m$. From the choice of m it follows that $m \leqq C|y|^{1-(1-a)/(1-\theta)}$, which yields $I_k \leqq C2^{-k\beta}|y|^{1-(1-a)/(1-\theta)}$. Hence

$$\sum_{2^{-k}<|y|/2} I_k \leqq C|y|^{\beta+1-(1-a)/(1-\theta)}.$$

From the choice of a and β it follows that $\beta+1-(1-a)/(1-\theta)=0$ and the above sum is majorized by a constant.

Case 2. $|y|/2 \leqq 2^{-k} < |y|^{(1-\theta)/(1-\alpha)}$. (4)

Setting

$$\delta_k = 5|y|^{2(1-\theta)/\lambda}2^{-kn(1-\alpha)(1-\theta)/\lambda}$$

we obtain the inequalities

$$\delta_k > 5\cdot 2^{-k} \tag{5}$$

and

$$\delta_k < 5|y|^{1-\theta} \tag{6}$$

as a consequence of (4). We split the integral defining I_k in two parts J_1 and J_2, the first obtained by integrating over the set $\{x;\ |x|\leqq\delta_k\}$ and the second by integrating over $\{x;\ |x|>\delta_k\}$. Arguing as above we then have

$$J_2 \leqq \int G_{\beta,k}(t)\,dt \int_{|x|>\delta_k-2^{-k+1}} |K(x-y)-K(x)|\,dx \leqq C2^{-k\beta}m,$$

where m is now defined to be the least positive integer such that

$$2|y/m|^{1-\theta} < \delta_k - 2^{-k+2}.$$

It follows that $m \leqq C|y|\delta_k^{-1/(1-\theta)}$ and hence, using the definition of δ_k, we conclude that

$$J_2 \leqq C|y|^{n(1-\theta)/\lambda}2^{kn(1-\theta)/\lambda}. \tag{7}$$

To estimate J_1 we first use the Schwarz inequality and get

$$J_1 \leqq 2\int_{|x|\leqq 2\delta_k} |G_{\beta,k}*K(x)|\,dx \leqq C\,\delta_k^{n/2}\|G_{\beta,k}*K\|_2 = C\,\delta_k^{n/2}\|\hat{G}_{\beta,k}\hat{K}\|_2. \tag{8}$$

We have $\phi_k(\xi) = 2^{-kn}\phi(2^{-k}\xi)$ and it follows that

$$|\phi_k(\xi)| \leqq C2^{-kn}, \quad |\xi| \leqq 2^k,$$

and

$$|\phi_k(\xi)| \leqq C2^{k(N-n)}|\xi|^{-N}, \quad |\xi| > 2^k,$$

where N denotes a large positive integer. For $|\xi|\leqq 2^k$ we have

$$|\phi_k*\hat{G}_\beta(\xi)| \leqq \left|\int_{|\xi-t|\leqq 3\cdot 2^k} \phi_k(\xi-t)\hat{G}_\beta(t)\,dt\right| + \left|\int_{|\xi-t|>3\cdot 2^k} \phi_k(\xi-t)\hat{G}_\beta(t)\,dt\right| \leqq$$

$$\leqq C2^{-kn}\int_{|t|\leqq 2^{k+2}} \hat{G}_\beta(t)\,dt + C\int_{|t|>2^{k+1}} |t|^{-\beta-N}\,dt\,2^{k(N-n)} \leqq C2^{-k\beta}.$$

Using the condition $A(\alpha)$ we get

$$\Big(\int_{|\xi|\leqq 2^k} |\hat{G}_{\beta,k}(\xi)\hat{K}(\xi)|^2\,d\xi\Big)^{1/2} \leqq C\,2^{-k\beta}\Big(\int_{|\xi|\leqq 2^k} |\hat{K}(\xi)|^2\,d\xi\Big)^{1/2} \leqq$$
$$\leqq C\,2^{-k\beta}\Big(\int_{|\xi|\leqq 2^k} |\xi|^{-n\alpha}\,d\xi\Big)^{1/2} \leqq C\,2^{-k(\beta+n\alpha/2-n/2)} = C\,2^{-kn(a-1)/2}.$$

We also have

$$\Big(\int_{|\xi|\geqq 2^k} |\hat{G}_{\beta,k}(\xi)\hat{K}(\xi)|^2\,d\xi\Big)^{1/2} \leqq C\,2^{-kn\alpha/2}\Big(\int_{|\xi|\geqq 2^k} |\hat{G}_{\beta,k}(\xi)|^2\,d\xi\Big)^{1/2} \leqq$$
$$\leqq C\,2^{-k(n\alpha/2+\beta-n/2)} = C\,2^{-kn(a-1)/2}.$$

It follows that

$$\|\hat{G}_{\beta,k}\hat{K}\|_2 \leqq C\,2^{-kn(a-1)/2},$$

and inserting this estimate in (8) we obtain

$$J_1 \leqq C|y|^{n(1-\theta)/\lambda}2^{kn(1-\theta)/\lambda}. \tag{9}$$

From the estimates (7) and (9) it follows that $\sum I_k \leqq C$, where the summation is taken over all k satisfying (4).

Case 3. $|y|^{(1-\theta)/(1-\alpha)} \leqq 2^{-k}$. (10)

In this case we choose $\delta_k = 4\cdot 2^{-k(1-\alpha)}$ and split I_k in two parts J_1 and J_2 as above. We have $\delta_k/2 \geqq 2|y|^{1-\theta}$ and it follows that

$$J_2 \leqq \int G_{\beta,k}(t)\,dt \int_{|x|>\delta_k-2^{-k+1}} |K(x-y)-K(x)|\,dx \leqq$$
$$\leqq C\,2^{-k\beta}\int_{|x|>\delta_k/2} |K(x-y)-K(x)|\,dx \leqq C\,2^{-k\beta}.$$

Arguing as above we also have

$$J_1 \leqq C\,\delta_k^{n/2}\,2^{-kn(a-1)/2},$$

and using the definitions of a and δ_k we get $J_1 \leqq C2^{-k\beta}$. Hence $I_k \leqq C2^{-k\beta}$ and $\sum I_k \leqq C$, where the summation is taken over all k satisfying (10).

We have proved that $(\psi G_\beta)*K$ satisfies $B(a)$ and this completes the proof of the theorem.

We finally remark that the method in [5], pp. 162—163, can be used to prove that the above theorem has the following consequence.

Corollary. *If* $0<\alpha<\theta<1$ *then* T_K *can be extended to a bounded linear operator from* $L^p(\mathbf{R}^n)$ *to* $L^q(\mathbf{R}^n)$ *for all* K *satisfying the above assumptions if and only if* $p \leqq q$ *and*

$$\alpha/2 \geqq 1/p-1/q+a\max(1/2-1/p,\ 1/q-1/2,\ 0),$$

where $a=(n\alpha(1-\theta)+2\theta)/(n(1-\theta)+2)$.

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