Intersection properties of weak analytically uniform classes of functions

Otto Liess

§ 1. Introduction

1. The notation is standard. **N**, **Z**, **R**, $\mathbf{R}^+$, **C** denote respectively the natural, integral, real, real positive, and complex numbers. $\operatorname{Re}\zeta$ and $\operatorname{Im}\zeta$ is the real and the imaginary part of $\zeta\in\mathbf{C}^n$. $|\ |$ is the usual norm in $\mathbf{C}^n$.

By $\mathscr{E}(t)$, $t\in\mathbf{R}^+\cup\{+\infty\}$, we denote the C^∞ functions defined for $|x|<t$, $x\in\mathbf{R}^n$, and by $C_0^\infty(t)$ the C^∞ functions with support in $|x|<t$. $\mathscr{E}(t)$ is endowed with the usual topology.

If X is a linear topological space, then X' stands for its dual. The Fourier—Borel transform of $u\in\mathscr{A}'(\mathbf{C}^n)$ is denoted $\hat{u}$ or $\mathscr{F}u$. Further, $\mathscr{E}'[B]$, $B<t$ is defined by $\mathscr{E}'[B]=\{u\in\mathscr{E}'(t);\ \operatorname{supp} u\subset\{|x|<B\}\}$. Finally, $\bar\partial$ denotes the Cauchy—Riemann operator, and, for a multi-index α, D^α is the corresponding derivation.

2. Consider $q=(q_j\colon \mathbf{C}^n\to\mathbf{R}^+)_{j\in N}$ a sequence of functions, and consider the following three properties:

(i) $q_j(\zeta)\geqq q_{j+1}(\zeta)$, $\forall j\in\mathbf{N}$, $\forall\zeta\in\mathbf{C}^n$.

(ii) $|q_j(\zeta)-q_j(\eta)|\leqq(1/j)|\zeta-\eta|$, $\forall\zeta$, $\forall\eta\in\mathbf{C}^n$.

(iii) For every $j\in\mathbf{N}$ and every $\tau>0$, there exist $k\in\mathbf{N}$ and $c_j>0$ such that $q_j(\zeta)\geqq\tau q_k(\zeta)-c_j$.

The set of all sequences q which satisfy (i) and (ii) will be denoted M_d (and called decreasing sequences of majorant functions), and the subset in M_d of those sequences which also satisfy (iii), will be denoted M_{sd} (strongly decreasing sequences of majorant functions).

Definition 1.1. For every $q\in M_d$ and every $t\in\mathbf{R}^+\cup\{+\infty\}$ we define a space of C^∞ functions, denoted $\mathscr{E}_q^w(t)$, and called the *weak analytically uniform (A.U.) space* associated with q, in the following way:

$f\in\mathscr{E}_q^w(t)$ if and only if, for every $B<t$ and every $b\geqq 0$, there exist a Radon measure μ and $j\in\mathbf{N}$ such that:

$$\int d|\mu| < \infty$$

and

(1) $$f(x) = \int_{\mathbf{C}^n} \exp i\langle x, \zeta\rangle\, d\mu(\zeta)/\exp\left(q_j(\zeta)+B|\mathrm{Im}\,\zeta|+b\ln(1+|\zeta|)\right),$$

for $|x|\leqq B$.

A space $X\subset\mathscr{E}(t)$ is called a *weak A.U. space in* $\mathscr{E}(t)$, if it is the weak A.U. space associated with some $q\in M_d$.

Examples of weak A.U. spaces are the Gevrey classes Γ^δ (in the notations from [6]), and they appear, more generally, in the local noncharacteristic Cauchy problem for constant coefficient partial differential operators, as the spaces of natural Cauchy data.

Definition 1.2. A weak A.U. space $\mathscr{E}_q^w(t)$ is called *nonquasianalytic*, if there exists a subspace $\mathscr{D}_q^w(t)$ in $\mathscr{E}_q^w(t)$ with the following properties:

(a) $\mathscr{D}_q^w(t)\subset\left(\mathscr{E}_q^w(t)\cap C_0^\infty(t)\right)$.

(b) For every compact K in $|x|<t$ and every neighborhood V of K in $|x|<t$, there exists $v\in\mathscr{D}_q^w(t)$ which is identically one in K, and which vanishes outside V.

(c) The elements from $\mathscr{D}_q^w(t)$ are multipliers for $\mathscr{E}_q^w(t)$.

The main result of this paper is the following

Theorem 1.3. *Consider* $q\in M_{sd}$ *and* $t\in\mathbf{R}^+\cup\{+\infty\}$. *The intersection of all nonquasianalytic weak A.U. spaces in* $\mathscr{E}(t)$ *which contain* $\mathscr{E}_q^w(t)$ *is* $\mathscr{E}_q^w(t)$.

This theorem extends a result of T. Bang, which states that the intersection of all nonquasianalytic Denjoy—Carleman classes gives the real analytic functions.

It is convenient to separate the proof of Theorem 1.3. in two distinct parts. In order to speak about the first, we need an order relation in M_d.

Definition 1.4. Consider $q^1, q^2\in M_d$. We say that $q^1\succeq q^2$, if for every j there exists j' such that $q_j^1(\zeta)\geqq q_{j'}^2(\zeta)$.

The arguments from the first step in the proof of Theorem 1.3. will then give the following abstract intersection theorem:

Theorem 1.5. *Consider* $q^0\in M_{sd}$ *and suppose that* $K\subset M_d$ *is a subset with the following properties:*

(iv) $q\in K$ *implies* $q\preceq q^0$,

(v) *every countable subset of* K *has a majorant in* K,

(vi) *for every function* $I\colon K\to\mathbf{N}$ *there exists* $j\in\mathbf{N}$ *such that*

$$\sup_{q\in K} q_{I(q)}(\zeta) \geqq q_j^0(\zeta).$$

Then

$$\mathscr{E}^{w}_{q^0}(t) = \bigcap_{q\in K} \mathscr{E}^{w}_{q}(t).$$

Once Theorem 1.5. is proved, it remains to construct, for a given $q^0 \in M_{sd}$, a subset $K \subset M_d$ such that K has the properties (iv), (v), (vi), and such that the weak A.U. spaces associated with elements from K are nonquasianalytic. This gives then Theorem 1.3.

The author is deeply indebted to Mats Neymark, who pointed out several inconsistencies in the original manuscript and made valuable remarks.

§ 2. $\bar{\partial}$-cohomology for Lipschitz continuous majorant functions

1. **Proposition 2.1.** *There are constants c and γ, such that, if $p\colon \mathbf{R}^n \to \mathbf{R}^+$ satisfies $|p(\xi^1)-p(\xi^2)| \leqq |\xi^1-\xi^2|$, then there is a plurisubharmonic function $q\colon \mathbf{C}^n \to \mathbf{R}$ such that*

$$p(\operatorname{Re}\zeta) \leqq q(\zeta)+\gamma|\operatorname{Im}\zeta| \quad \textit{and} \quad q(\zeta) \leqq 2p(\operatorname{Re}\zeta)+\gamma|\operatorname{Im}\zeta|+c.$$

The proof of this proposition is based on the following lemma:

Lemma 2.2. *There exists a continuous function $h(\zeta, t)\colon \mathbf{C}^n \times \mathbf{R}^+ \to \mathbf{R}^+$, which is plurisubharmonic in ζ for every fixed t, such that for some constants C, A the following inequalities are satisfied:*

(a) $h(\zeta, t) \leqq C(1+A|\operatorname{Im}\zeta|)$,

(b) $h(\zeta, t) \leqq -|\zeta|+A|\operatorname{Im}\zeta|$ *for* $(1/2)t \leqq |\zeta| \leqq t$,

(c) $h(\zeta, t) \leqq -t+A|\operatorname{Im}\zeta|$ *for* $|\zeta| \geqq t$,

(d) $h(i\xi, t) \geqq -A|\xi|$ *for* $\xi \in \mathbf{R}^n$.

Moreover, for each t, $h(\zeta, t) = \ln|f(\zeta)|$, $f \in \mathscr{A}(\mathbf{C}^n)$.

This lemma is standard, and its proof is very easy. We consider $\varphi \in C_0^\infty(\mathbf{R}^n)$, with $\int \varphi\, dx = 1$, $\varphi \geqq 0$ and try to find α, β such that the inequalities above are satisfied for the function $h(\zeta, t) = \alpha \ln|\hat{\varphi}(\beta\zeta/t)|$. Then (a) is a consequence of the Paley—Wiener theorem, and (d) follows from $\int \varphi\, dx = 1$ and $\varphi \geqq 0$. It then remains to find α, β for which we also have (b) and (c) (cf. almost every work concerning localizations in quasianalytic classes).

Proof of proposition 2.1. Consider $\xi_0 \in \mathbf{R}^n$ and define a plurisubharmonic function q_{ξ_0} by $q_{\xi_0}(\zeta) = p(\xi_0) + 2h(\zeta-\xi_0, p(\xi_0))$, where h is the function from the preceding lemma. We then have $q_{\xi_0}(\zeta) \leqq 2(p(\operatorname{Re}\zeta)+C(1+A|\operatorname{Im}\zeta|))$. Indeed, for $|\zeta-\xi_0| \leqq p(\xi_0)/2$ we have $|p(\operatorname{Re}\zeta)-p(\xi_0)| \leqq p(\xi_0)/2$ (in view of the Lipschitzness of p), and for $|\zeta-\xi_0| \geqq p(\xi_0)/2$ we use (b) or (c).

We now define $q(\zeta)=\sup_{\xi_0\in\mathbf{R}^n} q_{\xi_0}(\zeta)$. Then $q(\zeta)\leqq 2(p(\operatorname{Re}\zeta)+C(1+A|\operatorname{Im}\zeta|))$ (in particular q is finite in every point). Finally it follows from (d), that $p(\operatorname{Re}\zeta)\leqq \leqq q(\zeta)+A|\operatorname{Im}\zeta|$.

2. From Proposition 2.1. we obtain the following lemma concerning the solvability of the system $\bar\partial v=u$.

Lemma 2.3. *Consider $p(\xi)\colon \mathbf{R}^n\to\mathbf{R}^+$ a function which satisfies $|p(\xi^1)-p(\xi^2)|\leqq \leqq\alpha|\xi^1-\xi^2|$, $\forall\xi^1, \xi^2\in\mathbf{R}^n$, $\alpha\leqq 1$. Then there exist constants C, γ such that the following is true: if $u\in C^\infty(\mathbf{C}^n)$ satisfies*

$$|\bar\partial u(\zeta)| \leqq \exp\big(p(\operatorname{Re}\zeta)/2+B|\operatorname{Im}\zeta|+b\ln(1+|\zeta|)\big),$$

then there exists $v\in C^\infty(\mathbf{C}^n)$ such that $\bar\partial v=\bar\partial u$ and such that

$$|v(\zeta)| \leqq C\exp\big(p(\operatorname{Re}\zeta)+(B+\alpha\gamma)|\operatorname{Im}\zeta|+(b+n+2)\ln(1+|\zeta|)\big).$$

Here C does not depend on p, and γ is the constant from Proposition 2.1.

Proof. We apply Proposition 2.1. for the function $(1/\alpha)p$, and obtain a plurisubharmonic function q which satisfies the estimates $p(\operatorname{Re}\zeta)\leqq\alpha q(\zeta)+\alpha\gamma|\operatorname{Im}\zeta|$ and $\alpha q(\zeta)\leqq 2p(\operatorname{Re}\zeta)+\alpha\gamma|\operatorname{Im}\zeta|+c\alpha$.

From the estimate for $\bar\partial u$ in the hypothesis of Lemma 2.3., we now obtain that $\int|\bar\partial u(\zeta)|^2\exp\big(-\alpha q(\zeta)-(2B+\alpha\gamma)|\operatorname{Im}\zeta|-(2b+2n+1)\ln(1+|\zeta|)\big)d\zeta\wedge d\bar\zeta\leqq C'$ for some C'. Since αq is plurisubharmonic, we can apply Theorem 4.4.2 in [7], and conclude that there exists v such that $\bar\partial v=\bar\partial u$ and such that

$$\int|v(\zeta)|^2\exp\big(-\alpha q(\zeta)-(2B+\alpha\gamma)|\operatorname{Im}\zeta|-(2b+2n+3)\ln(1+|\zeta|)\big)d\zeta\wedge d\bar\zeta \leqq C'',$$

for some C'', which depends only on C'. This is already an estimate of the desired type (if we use the estimate for αq from above), only that it involves L_2-norms instead of sup-norms. The passage to sup-norms for Lipschitz majorant functions is however, by now standard; one may, e.g., use the following inequality, valid for C^1 functions (a proof can be found in [9]):

$$|w(\zeta)| \leqq C\sup_{|\theta|\leqq 1}|\bar\partial w(\zeta+\theta)|+\Big(\int_{|\theta|\leqq 1}|w(\zeta+\theta|^2\,d\theta\wedge d\bar\theta\Big)^{1/2}.$$

§ 3. C^∞ functions as functionals on spaces of entire functions

1. In the proof of Theorem 1.5 we regard elements from $\mathscr{E}_q^w(t)$-spaces essentially as functionals on some spaces of entire functions. Special care is required, however, to overcome the nonuniqueness in the representation (1). We start the section by introducing a (more or less) convenient terminology.

Consider a function $p\colon \mathbf{C}^n \to \mathbf{R}^+$ such that $p(\zeta)\leqq\alpha|\zeta|+\alpha'$ for some constants α, α'. For $B>0$, $b\geqq 0$, we define a quasinorm $|\ |_{p,B,b}$ on $\mathscr{A}(\mathbf{C}^n)$ in the following way:

$$|h|_{p,B,b} = \sup_{\zeta\in\mathbf{C}^n} |h(\zeta)|/\exp\big(p(\zeta)+B|\operatorname{Im}\zeta|+b\ln(1+|\zeta|)\big)$$

and denote

$$\mathscr{A}_{p,B,b} = \{h\in\mathscr{A}(\mathbf{C}^n);\ |h|_{p,B,b}<\infty\}.$$

On $\mathscr{A}_{p,B,b}$, $|\ |_{p,B,b}$ is now a norm, and we endow $\mathscr{A}_{p,B,b}$ with the corresponding norm-topology.

Definition 3.1. Let $t>3\alpha+B$ and consider $f\in\mathscr{E}(t)$. Then f defines a functional $\tilde{f}$ on $\mathscr{F}(\mathscr{E}'(t))$ by $\tilde{f}(\hat{u})=u(f)$. We say that f is *naturally defined* on $\mathscr{A}_{p,B,b}$ if the functional $\tilde{f}$ is continuous on $\mathscr{F}(\mathscr{E}'[3\alpha+B])\cap\mathscr{A}_{p,3\alpha+B,b+n+3}$, the intersection being considered with the topology induced from $\mathscr{A}_{p,3\alpha+B,b+n+3}$.

The choice of the constants in this definition is justified by the following result:

Proposition 3.2. *Let p be a function as the above, consider $h\in\mathscr{A}(\mathbf{C}^n)$ such that $|h|_{p,B,b}<\infty$, and suppose that there is given a sequence of positive numbers $\varepsilon_k\to 0$. Then there exists a sequence of distributions u_k, with supports concentrated in $|x|\leqq 3\alpha+B$ and such that $|h-\hat{u}_k|_{p,3\alpha+B,b+n+3}<\varepsilon_k$.*

In fact, this proposition shows that the Hahn—Banach extension of $\tilde{f}$ to $\mathscr{A}_{p,3\alpha+B,b+n+3}$ is uniquely defined on $\mathscr{A}_{p,B,b,}$.

Proposition 3.2 results from Theorem 3.1 in [8]. Since we shall use similar arguments later, we shall indicate the proof briefly, for the convenience of the reader.

We first need an elementary lemma.

Lemma 3.3. *Let $\varepsilon_k\to 0$ be a sequence of strictly positive numbers. Then there exist functions $\varphi_k\in C^\infty(\mathbf{C}^n)$ such that the following inequalities hold uniformly for α, $0\leqq\alpha\leqq 1$:*

(2) $$|\varphi_k(\zeta)| \leqq \varepsilon_k(1+|\zeta|)\exp(-|\zeta|),$$

(3) $$|\bar\partial\varphi_k(\zeta)| \leqq \varepsilon_k(1+|\zeta|)\exp(-\alpha|\zeta|+3\alpha|\operatorname{Im}\zeta|),$$

(4) $$|1-\varphi_k(\zeta)| \leqq \varepsilon_k(1+|\zeta|).$$

Proof. We denote for $\delta>0$:

$v_\delta(\zeta) = \exp(-\delta\langle\zeta,\bar\zeta\rangle) = \exp\big(-\delta\big(\sum(\operatorname{Re}\zeta_j)^2-\sum(\operatorname{Im}\zeta_j)^2+2i\sum\operatorname{Re}\zeta_j\operatorname{Im}\zeta_j\big)\big)$.

$U_\delta\ = \{\zeta\in\mathbf{C}^n;\ |\operatorname{Im}\zeta|^2 \leqq (1/2)|\operatorname{Re}\zeta|^2+1/\delta\}$.

$U_{\delta,1} = \{\zeta\in\mathbf{C}^n;$ distance from ζ to U_δ is less then $1\}$.

w_δ: a function in $C^\infty(\mathbf{C}^n)$ such that $w_\delta(\zeta)=1$ for $\zeta\in U$, $w_\delta(\zeta)=0$ for $\zeta\notin U_{\delta,1}$ and such that $|w_\delta(\zeta)|+|\bar\partial w_\delta(\zeta)|\leqq C$, for some constant C.

It is then easy to verify that, for a convenient sequence δ_k, we may set $\varphi_k(\zeta)=$ $=w_{\delta_k}(\zeta)v_{\delta_k}(\zeta)$.

Proof of proposition 3.2. Consider first the functions $h_k=\varphi_k h$, where the φ_k are the functions constructed in Lemma 3.3. From (2) it follows that $|h_k(\zeta)|\leqq$ $\leqq|h|_{p,B,b}\cdot\varepsilon_k(1+|\zeta|)^{b+1}\exp\left(p(\zeta)+B|\mathrm{Im}\,\zeta|-|\zeta|\right)\leqq C\varepsilon_k(1+|\zeta|)^{b+1}\exp B|\mathrm{Im}\,\zeta|$, and from (3) that $|\bar\partial h_k(\zeta)|\leqq|h|_{p,B,b}\cdot\varepsilon_k(1+|\zeta|)^{b+1}\exp\left(p(\zeta)+B|\mathrm{Im}\,\zeta|-\alpha|\zeta|+3\alpha|\mathrm{Im}\,\zeta|\right)\leqq$ $\leqq C\varepsilon_k(1+|\zeta|)^{b+1}\exp\left((B+3\alpha)|\mathrm{Im}\,\zeta|\right)$. Then solve $\bar\partial g_k=\bar\partial h_k$, with some g_k which satisfies the estimate $|g_k(\zeta)|\leqq C'\varepsilon_k(1+|\zeta|)^{b+1+n+2}\exp(3\alpha+B)|\mathrm{Im}\,\zeta|$ (the existence of such a g_k follows, e.g., from Lemma 2.3.). It follows that h_k-g_k is the Fourier—Borel transform of some distribution with compact support, u_k, and we have, in view of (4) and the inequality on g_k, that

$$|h-\hat u_k|\leqq|1-\varphi_k|\,|h|+|g_k|\leqq$$

$$\leqq C\varepsilon_k\left((1+|\zeta|)^{b+1}\exp\left(p(\zeta)+B|\mathrm{Im}\,\zeta|\right)+C'(1+|\zeta|)^{b+n+3}\exp\left((3\alpha+B)|\mathrm{Im}\,\zeta|\right)\right)\leqq$$

$$\leqq C''\varepsilon_k(1+|\zeta|)^{b+n+3}\exp\left(p(\zeta)+(3\alpha+B)|\mathrm{Im}\,\zeta|\right).$$

The proposition now follows by passing to a convenient subsequence.

2. The main result of this section is, that in sufficiently general situations, a function which is naturally defined on a family of spaces, is also defined on their "union" (precise statements are given below).

Let $p_\iota\colon \mathbf{C}^n\to\mathbf{R}^+$, $\iota\in I$ be a family of majorant functions (the index set I may, or may not, be countable), which satisfy for the same α, $0\leqq\alpha\leqq1$, the conditions $|p_\iota(\zeta)-p_\iota(\zeta')|\leqq\alpha|\zeta-\zeta'|$, and denote $p(\zeta)=\sup_\iota p_\iota(\zeta)$.

Proposition 3.4. *There exist constants $\tau(=2^{-n})$, $\sigma\geqq3$ and $\delta>0$ with the following property:*

Suppose that $t>B+\sigma\alpha$ and that $f\in\mathscr{E}(t)$ is such that for all $u\in\mathscr{E}'(B+\sigma\alpha)$ and all $\iota\in I$, $|u(f)|\leqq C|\hat u|_{p_\iota,B+\sigma\alpha,b+\delta}$ with C independent of u and ι. Then f is naturally defined on $\mathscr{A}_{\tau p,B,b}$, i.e. for all $v\in\mathscr{E}'(B+3\alpha)$, $|v(f)|\leqq C'|\hat v|_{\tau p,B+3\alpha,b+n+3}$.

The first remark which simplifies the expressions we have to estimate is, that we may suppose that the functions p_ι depend only on $\mathrm{Re}\,\zeta$. This follows from the inequalities $p_\iota(\mathrm{Re}\,\zeta)\leqq p_\iota(\zeta)+\alpha|\mathrm{Im}\,\zeta|$, $p(\zeta)\leqq p(\mathrm{Re}\,\zeta)+\alpha|\mathrm{Im}\,\zeta|$ which are obvious consequences of the Lipschitzness of the p_ι.

To ease the notations further, we will change the index set I to $\mathbf{Z}^n$ in the following way: for every $\lambda\in\mathbf{Z}^n$ there exists $\iota(\lambda)$ such that $p(\lambda)\leqq p_{\iota(\lambda)}(\lambda)+1$, and we define $p_\lambda(\zeta)=p_{\iota(\lambda)}(\zeta)$. In the sequel, we will work with the family p_λ, $\lambda\in\mathbf{Z}^n$. This will not change the result, in view of the obvious inequality $\sup_{\iota\in I}p_\iota(\mathrm{Re}\,\zeta)\leqq$ $\leqq\sup_{\lambda\in\mathbf{Z}^n}p_\lambda(\mathrm{Re}\,\zeta)+c$, for some c.

For the moment we will also suppose that $\alpha=1$.

Before embarking on the proof, we introduce some notations and make some constructions.

Let us consider $(A_\lambda)_{\lambda\in \mathbf{Z}^n}$, the covering of $\mathbf{R}^n$ with the cubes $A_\lambda=\{\xi;\ |\xi_i-\lambda_i|\leqq 1,\ i=1, \dots, n\}$. Further we denote for $\lambda_n \in \mathbf{Z}$

$$B_{\lambda_n} = \bigcup_{\lambda' \in \mathbf{Z}^{n-1}} A_{(\lambda', \lambda_n)}.$$

Suppose now that we are given a function $\chi\colon \mathbf{R}^n \to \mathbf{R}^+$ such that $|\chi(\xi^1)-\chi(\xi^2)|\leqq \leqq |\xi^1-\xi^2|$. Starting from the function χ, we define functions $\chi_{\lambda_n}\colon \mathbf{R}^n\to\mathbf{R}^+$, associated with χ, in the following way:

$$\begin{aligned}
\chi_{\lambda_n}(\xi) &= 0 && \text{for}\quad \xi_n \geqq \lambda_n+1+\chi(\xi', \lambda_n+1),\\
\chi_{\lambda_n}(\xi) &= \chi(\xi', \lambda_n+1)+\lambda_n+1-\xi_n && \text{for}\quad \lambda_n+1 \leqq \xi_n \leqq \lambda_n+1+\chi(\xi', \lambda_n+1),\\
\chi_{\lambda_n}(\xi) &= \chi(\xi) && \text{for}\quad \xi \in B_{\lambda_n},\\
\chi_{\lambda_n}(\xi) &= \chi(\xi', \lambda_n-1)+\xi_n+1-\lambda_n && \text{for}\quad \lambda_n-1-\chi(\xi', \lambda_n-1) \leqq \xi_n \leqq \lambda_n-1,\\
\chi_{\lambda_n}(\xi) &= 0 && \text{for}\quad \xi_n \leqq \lambda_n-1-\chi(\xi', \lambda_n-1).
\end{aligned}$$

Here we have used the notation $\xi'=(\xi_1, \dots, \xi_{n-1})$ if $\xi=(\xi_1, \dots, \xi_{n-1}, \xi_n)$.

The obtained functions are obviously Lipschitz continuous, with Lipschitz constant 1. Their main property is that, on a slab they are as great as χ, but outside that slab they decrease so rapidly that every Lipschitz continuous function, with Lipschitz constant 1, which is greater then zero, and which coincides with χ on B_{λ_n}, is greater then χ_{λ_n}.

The construction above can be effectuated in any other variable, and in doing this in the variable ξ_{n-1}, starting from the functions χ_{λ_n} we obtain functions $(\chi_{\lambda_n})_{\lambda_{n-1}}$. Continuing this procedure, we obtain inductively systems of functions $(\dots(\chi_{\lambda_n})_{\lambda_{n-1}}\dots)_{\lambda_{i+1}}$, such that the following properties are satisfied:

If g is in a system, then $|g(\xi^1)-g(\xi^2)|\leqq|\xi^1-\xi^2|$.

$$\sup_{\lambda_i\in\mathbf{Z}}\big((\dots(\chi_{\lambda_n})_{\lambda_{n-1}}\dots)_{\lambda_{i+1}}\big)_{\lambda_i} = (\dots(\chi_{\lambda_n})_{\lambda_{n-1}}\dots)_{\lambda_{i+1}}.$$

If $\mu\colon \mathbf{R}^n \to \mathbf{R}^+$ is a function such that $|\mu(\xi^1)-\mu(\xi^2)|\leqq|\xi^1-\xi^2|$, and if

$$\mu \geqq (\dots(\chi_{\lambda_n})_{\lambda_{n-1}}\dots)_{\lambda_i}$$

on

$$B_{\lambda_i, \lambda_{i+1}, \dots, \lambda_n} = \bigcup_{\lambda_1, \dots, \lambda_{i-1}} A_{(\lambda_1, \dots, \lambda_{i-1}, \lambda_i, \dots, \lambda_n)},\ \text{ then }\ (\dots(\chi_{\lambda_n})_{\lambda_{n-1}}\dots)_{\lambda_i} \leqq \mu+C \ \text{ on }\ \mathbf{R}^n.$$

The proof of Proposition 3.4 now follows in a finite number of steps, from the following proposition, for convenient choices of B' and b' (and $\chi=p$).

Proposition 3.5. *Suppose* $1 \leqq i \leqq n$, $\lambda_{i+1} \in \mathbf{Z}, \ldots, \lambda_n \in \mathbf{Z}$ *and consider constants* B', b', $t > 2^{-n+i}(3+\gamma)+B'$ *(γ is the constant from Lemma 2.3). Suppose further that for all* $\lambda_i \in \mathbf{Z}$ *and all* $u \in \mathscr{E}'(B'+[3+\gamma]2^{-n+i})$ *we have*

$$|u(f)| \leqq C|\hat{u}|_{2^{-n+i-1}(\ldots(\chi_{\lambda_n}(\mathrm{Re}\,\zeta))_{\lambda_{n-1}}\ldots)_{\lambda_i},\, B'+2^{-n+i}(3+\gamma),\, b'+2n+7}$$

where C does not depend on u and $\lambda_i, \lambda_{i+1}, \ldots, \lambda_n$.

Then we have for all $v \in \mathscr{E}'[B']$:

$$|v(f)| \leqq C'|v|_{2^{-n+i}(\ldots(\chi_{\lambda_n}(\mathrm{Re}\,\zeta))_{\lambda_{n-1}}\ldots)_{\lambda_{i+1}},\, B',\, b'}$$

for some constant C', which does not depend on v and $\lambda_{i+1}, \ldots, \lambda_n$.

It is enough to prove Proposition 3.5 for $i=n$, since the argument is similar in the general case. For $i=n$ we obtain it from the following technical lemma.

Lemma 3.6. a) *There exists a constant C such that if* $h \in \mathscr{A}(\mathbf{C}^n)$ *satisfies the inequality*

$$|h(\zeta)| \leqq \exp\left(\chi(\mathrm{Re}\,\zeta)/2 + B'|\mathrm{Im}\,\zeta| + b'\ln(1+|\zeta|)\right), \tag{5}$$

then there exist functions $h_{\lambda_n} \in \mathscr{A}(\mathbf{C}^n)$ *such that* $h = \sum_{\lambda_n \in \mathbf{Z}} h_{\lambda_n}$ *and such that*

$$(1+|\lambda_n|)^2 |h_{\lambda_n}(\zeta)| \leqq C\exp\left(\chi_{\lambda_n}(\mathrm{Re}\,\zeta) + (\gamma+B')|\mathrm{Im}\,\zeta| + (b'+n+4)\ln(1+|\zeta|)\right). \tag{6}$$

b) *Suppose that we are given entire functions h_{λ_n} which satisfy the inequalities* (6), *that* $\sum h_{\lambda_n} = 0$ *and suppose that* $\varepsilon_k \to 0$. *Then there exists a sequence of systems of distributions* $u_{\lambda_n, k} \in \mathscr{E}'[B'+3+\gamma]$ *such that*

$$\sum_{\lambda_n} u_{\lambda_n, k} = 0, \tag{7}$$

$$|h_{\lambda_n} - \hat{u}_{\lambda_n, k}| \leqq \tag{8}$$

$$\leqq \varepsilon_k (1+|\lambda_n|)^{-2} \exp\left(\chi_{\lambda_n}(\mathrm{Re}\,\zeta) + (3+\gamma+B')|\mathrm{Im}\,\zeta| + (b'+2n+7)\ln(1+|\zeta|)\right),$$

$$\sum_{|\lambda_n| \leqq r} u_{\lambda_n, k}, \quad r \in \mathbf{N}, \tag{9}$$

is a bounded set in $\mathscr{E}'$ for every k.

We will admit the lemma for a moment and prove Proposition 3.5 (for $i=n$).

To do so, it is sufficient to take $v \in \mathscr{E}'[B']$ of order not greater than b' which satisfies $|\hat{v}(\zeta)| \leqq \exp\left(1/2\chi(\mathrm{Re}\,\zeta) + B'|\mathrm{Im}\,\zeta| + b'\ln(1+|\zeta|)\right)$ and to prove that $|v(f)| \leqq C'$ (in fact, we can regularize a general v). In view of Lemma 3.6 we may write $\hat{v}$ in the form

$$\hat{v}(\zeta) = \sum_{t \in \mathbf{Z}} h_t,$$

where

$$|h_t(\zeta)| \leqq C(1+|t|)^{-2} \exp\left(\chi_t(\mathrm{Re}\,\zeta) + (\gamma+B')|\mathrm{Im}\,\zeta| + (b'+n+4)\ln(1+|\zeta|)\right)$$

with h_t entire functions. In view of the assumption on f, the expression $\tilde{f}(h_t)$ has, for every $t\in\mathbf{Z}$, a welldefined meaning, and we have $|\tilde{f}(h_t)|\leqq\tilde{C}(1+|t|)^{-2}$. It follows that $|\sum_t\tilde{f}(h_t)|\leqq C'$, and the proposition is proved, if we can show that $v(f)=\sum_t\tilde{f}(h_t)$. This assertion now follows from Lemma 3.6.b): v satisfies for some constant D the estimate $|\hat{v}(\zeta)|\leqq D(1+|\zeta|)^{b'}\exp B'|\mathrm{Im}\,\zeta|$. Let us then apply Lemma 3.6. b), in which we take as h_t those from before, with the exception of h_0, which we change to $h_0-\hat{u}$. We can then approximate these functions, in the indicated way, with Fourier—Borel transforms of distributions with compact support, $u_{t,k}$. For every fixed k we now have $\sum_t u_{t,k}(f)=0$. Indeed, we may change the order of summation in $(\sum_t u_{t,k})(f)$ in view of (9), and we obtain 0 in view of (7).

Proof of Lemma 3.6. a) Let us denote by ψ_s a family of $C^\infty(\mathbf{C}^n)$ functions with the following properties

$$\operatorname{supp}\psi_s\subset B_s\times\{i\mathbf{R}^n\},$$

$$\psi_s=1 \quad \text{for} \quad |\mathrm{Re}\,\zeta_n-s|\leqq 1/3,$$

$$|\psi_s(\zeta)|+|\bar{\partial}\psi_s(\zeta)|\leqq c, \quad \text{for a constant } c>0,$$

$$\textstyle\sum_s\psi_s=1.$$

It is then clear that the functions $f_{\lambda_n}=\psi_{\lambda_n}h$ could play the role of functions h_{λ_n} if only they were entire. We will obtain from them suitable entire functions, by adding small corrections, with the aid of some $\bar{\partial}$-argument.

To do so, consider the functions

$$\varrho_s=\begin{cases}\psi_s & \text{for} \quad \mathrm{Re}\,\zeta_n\geqq s\\ 1 & \text{for} \quad \mathrm{Re}\,\zeta_n<s\end{cases}$$

and $g_{\lambda_n}=\varrho_{\lambda_n}h$.

The functions $\bar{\partial}g_{\lambda_n}$ then have their support in $\lambda_n+1/3\leqq\mathrm{Re}\,\zeta_n\leqq\lambda_n+1$, and therefore we have

$$|\bar{\partial}g_{\lambda_n}(\zeta)|\leqq(1+|\lambda_n|)^{-2}\exp\left(\chi_{\lambda_n}(\mathrm{Re}\,\zeta)/2+B'|\mathrm{Im}\,\zeta|+(b+2)\ln(1+|\zeta|)\right).$$

It is therefore possible to find w_{λ_n} with $\bar{\partial}w_{\lambda_n}=\bar{\partial}g_{\lambda_n}$, such that

$$|w_{\lambda_n}(\zeta)|\leqq C(1+|\lambda_n|)^{-2}\exp\left(\chi_{\lambda_n}(\mathrm{Re}\,\zeta)+(\gamma+B')|\mathrm{Im}\,\zeta|+(b'+n+4)\ln(1+|\zeta|)\right).$$

We now denote $h_{\lambda_n}=f_{\lambda_n}+w_{\lambda_{n-1}}-w_{\lambda_n}$, and it is easy to see that these functions h_t satisfy all the requirements in Lemma 3.6 a).

b) The second part of the lemma is proved by repeating the arguments from the proof of Proposition 3.2, in which we now perform the arguments simultaneously for all the functions h.

In fact we denote $h_{\lambda_n,k}=\varphi_k h_{\lambda_n}$, φ_k those from Lemma 3.3, solve, with Paley—Wiener estimates, the systems $\bar\partial w_{\lambda_n,k}=\bar\partial\varphi_k h_{\lambda_n}$ for $\lambda_n\neq 0$, and set, in order to maintain $\sum u_{\lambda_n,k}=0$, $w_{0,k}=-\sum_{\lambda_n\neq 0} w_{\lambda_n,k}$. Finally we define $\hat u_{\lambda_n,k}=h_{\lambda_n,k}-w_{\lambda_n,k}$.

We have now proved Proposition 3.4 for the case $\alpha=1$, and it remains to see, what changes must be performed, in order to obtain it in the general case. Apart from the "loss" of a term $\exp(2\alpha|\operatorname{Im}\zeta|)$ in estimates which occurs when we change from the functions $p_\iota(\zeta)$ to the functions $p_\iota(\operatorname{Re}\zeta)$, we see that we only have a loss of terms $\exp|\operatorname{Im}\zeta|$ when we apply Lemma 2.3. The amount of terms $|\operatorname{Im}\zeta|$ which we loose in the exponent hereby depends linearly on the constant of Lipschitzness of the respective majorant functions. It is therefore sufficient to construct a system $(\dots(p_{\lambda_n})_{\lambda_{n-1}}\dots)_{\lambda_l}$ consisting of functions which are Lipschitz with Lipschitz constant α. This is possible by an obvious modification of the construction preceding Proposition 3.5.

§ 4. Proof of Theorem 1.5

1. **Lemma 4.1.** *Suppose $q\in M_a$ and $f\in\mathscr{E}(t)$. The following two assertions are then equivalent:*

(I) $f\in\mathscr{E}_q^w(t)$,

(II) *for all B, $0<B<t$, and all $b>0$, there exists j, $B+3/j<t$, such that f is naturally defined on $\mathscr{A}_{q_j,B,b}$.*

Proof. a) Suppose first $f\in\mathscr{E}_q^w(t)$ and choose B, B', B'', $0<B<B'<B''<t$, $b>0$, $b'=b+n+3$, $b''=b+1$. By definition there exist a Radon measure μ and a j_0, $\int d|\mu|<\infty$ such that

$$f(x)=\int_{\mathbf{C}^n}\exp i\langle x,\zeta\rangle\,d\mu(\zeta)/\exp\big(q_{j_0}(\zeta)+B''|\operatorname{Im}\zeta|+b''\ln(1+|\zeta|)\big),\quad\text{for}\quad |x|\leqq B'',$$

and from this it follows that

$$v(f)=\int\hat v(\zeta)\,d\mu(\zeta)/\exp\big(q_{j_0}(\zeta)+B''|\operatorname{Im}\zeta|+b''\ln(1+|\zeta|)\big)$$

for

$$v\in\mathscr{E}'[B'],\quad |\hat v|_{q_{j_0},B',b'}<\infty.$$

(Choose $g_\varepsilon\in C_0^\infty(B''-B')$ such that $|1-\hat g_\varepsilon(\zeta)|\leqq\varepsilon(1+|\zeta|)\exp(B''-B)|\operatorname{Im}\zeta|$. Then

$$(g_\varepsilon * v)(f)=\int g_\varepsilon(\zeta)\hat v(\zeta)\,d\mu/\exp\big(q_{j_0}(\zeta)+B''|\operatorname{Im}\zeta|+b''\ln(1+|\zeta|)\big)$$

and

$$\Big|\int(1-g_\varepsilon(\zeta))\hat v(\zeta)\,d\mu/\exp\big(q_{j_0}(\zeta)+B''|\operatorname{Im}\zeta|+b''\ln(1+|\zeta|)\big)\Big|\leqq$$

$$\leqq C\varepsilon\int\frac{(1+|\zeta|)\exp((B''-B')|\operatorname{Im}\zeta|)\exp\big(q_{j_0}(\zeta)+B'|\operatorname{Im}\zeta|+b'\ln(1+|\zeta|)\big)}{\exp\big(q_{j_0}(\zeta)+B''|\operatorname{Im}\zeta|+b''\ln(1+|\zeta|)\big)}\,d\mu\to 0$$

when $\varepsilon\to 0$.)

We now choose j such that

(10) $$j \geqq j_0, \quad B+3/j < B',$$

and prove that f is naturally defined on $\mathcal{A}_{q_j, B, b}$.

Since $|q_j(\zeta^1)-q_j(\zeta^2)| \leqq (1/j)|\zeta^1-\zeta^2|$, this amounts to

$$|v(f)| \leqq C|\hat{v}|_{q_j, B+3/j, b+n+3}, \quad \hat{v} \in \mathcal{A}_{q_j, B+3/j, b+n+3} \cap \mathcal{F}(\mathcal{E}'[B+3/j]).$$

To prove this we have only to observe that

$$|v(f)| \leqq |\hat{v}|_{q_j, B+3/j, b+n+3} \int \frac{\exp(q_j(\zeta)+(B+3/j)|\operatorname{Im}\zeta|+b'\ln(1+|\zeta|))}{\exp(q_{j_0}(\zeta)+B''|\operatorname{Im}\zeta|+b''\ln(1+|\zeta|))} d|\mu(\zeta)| \leqq$$
$$\leqq |\hat{v}|_{q_j, B+3/j, b+n+3} \int d|\mu(\zeta)|,$$

in view of the inequalities $q_j(\zeta) \leqq q_{j_0}(\zeta)$ and $(B+3/j)|\operatorname{Im}\zeta| \leqq B'|\operatorname{Im}\zeta|$ which are consequences of (10).

b) Let us, conversely, suppose that for all B, b, there is some j, $B+3/j<t$ such that f is naturally defined on $\mathcal{A}_{q_j, B, b}$. In view of the Hahn—Banach theorem, there exists a Radon measure ν such that $\int d|\nu(\zeta)| < \infty$ and such that

$$v(f) = \int \hat{v}(\zeta)\, d\nu(\zeta)/\exp(q_j(\zeta)+(B+3/j)|\operatorname{Im}\zeta|+(b+n+3)\ln(1+|\zeta|)),$$

for

$$v \in \mathcal{E}'[B+3/j], \quad |\hat{v}|_{q_j, B+3/j, b+n+3} < \infty.$$

In particular

$$f(x) = \delta_x(f) =$$
$$= (2\pi)^{-n} \int \exp i\langle x, \zeta\rangle\, d\nu(\zeta)/\exp(q_j(\zeta)+(B+3/j)|\operatorname{Im}\zeta|+(b+n+3)\ln(1+|\zeta|)),$$
$$|x| \leqq B+3/j,$$

and this gives immediately, $f \in \mathcal{E}_q^w(t)$.

2. The proof of the theorem is now very short, since we have put enough conditions on K in order to assure an easy reduction to Proposition 3.4.

Since the inclusion $\mathcal{E}_q^w(t) \supset \mathcal{E}_{q^0}^w(t)$ is clear, we have only to prove that

$$\bigcap_{q \in K} \mathcal{E}_q^w(t) \subset \mathcal{E}_{q^0}^w(t).$$

Let us then suppose $f \in \bigcap_{q\in K} \mathcal{E}_q^w(t)$ and fix $B<t$ and b. We choose j_0 such that, for $B' = B+\sigma/j_0$, $B'+3/j_0 < t$ and set $b' = b+\delta$, σ, δ the constants from Proposition 3.4. From the assumption $f \in \bigcap_{q\in K} \mathcal{E}_q^w(t)$ we obtain (in view of Lemma 4.1.) a function $I: K \to \mathbf{N}$ such that f is naturally defined on the spaces $\mathcal{A}_{I(q), B', b'}$. We may of course suppose that for all q, $I(q) \geqq j_0$. In particular

$$|v(f)| \leqq C(I(q))|\hat{v}|_{I(q), B', b'} \quad \text{for} \quad v \in \mathcal{E}'[B'], |\hat{v}|_{I(q), B', b'} < \infty.$$

If we could prove that, for a convenient choice of I, the preceding inequalities are satisfied for some C which does not depend on q, then the theorem would follow. Indeed, we could apply Proposition 3.4 and obtain that f were naturally defined on

$\mathscr{A}_{\tau\bar{q},B,b}$, with τ from Proposition 3.4. and $\bar{q}(\zeta)=\sup_{q\in K} q_{I(q)}(\zeta)$. Since $\bar{q}(\zeta)\geqq q_r^0(\zeta)$ for some r, in view of condition (vi) on K, and since $\tau q_r^0\geqq q_{r'}^0-c$, in view of (iii), f were then naturally defined on $\mathscr{A}_{q_{r'}^0,B,b}$ and it remains to apply Lemma 4.1 again.

The last thing to do is therefore to prove the existence of a function $I: K\to\mathbf{N}$ such that $|v(f)|\leqq C$ if $|\hat{v}|_{I(q),B',b'}\leqq 1$ for some $q\in K$, $v\in\mathscr{E}'[B']$. If there is no such function I then the following is true:

for every $k\in\mathbf{N}$, there exists $q^k\in K$ such that the norms of f as functionals on $\mathscr{A}_{q_j^k,B',b'}\cap\mathscr{F}(\mathscr{E}'[B'])$, $j\in\mathbf{N}$ are all greater then k (for a finite number of j it may happen that $f\notin(\mathscr{A}_{q_j^k,B',b'}\cap\mathscr{F}(\mathscr{E}'[B']))'$). We now apply (v) for the sequence q^k and denote by q the element associated with q^k by this condition. It follows that the norm of f in any space $(\mathscr{A}_{q_j,B',b'}\cap\mathscr{F}(\mathscr{E}'[B']))'$ is greater then any k. This contradicts the assumption $f\in\mathscr{E}_q^w(t)$ and this contradiction proves the theorem.

§ 5. Nonquasianalytic Denjoy—Carleman classes

1. The nonquasianalytic weak A.U. spaces which we construct in order to prove Theorem 1.3 are associated with nonquasianalytic Denjoy—Carleman classes of functions. We first introduce and study the sequences of integers for which we will consider the corresponding Denjoy—Carleman class.

Definition 5.1. We denote by $\mathscr{N}_c$, $c>3$, the set of sequences $d=\{d_j\}$ of numbers, which satisfy the following conditions:

$$d_j \geqq j, \tag{11}$$

$$d_{j+1} \geqq d_j, \tag{12}$$

$$d_{j+1} \leqq c d_j, \tag{13}$$

$$\sum 1/d_j < \infty. \tag{14}$$

In $\mathscr{N}_c$ we have a natural order relation: we say that $d^1 \preceq d^2$, if there exists a constant C such that $d_j^1\leqq C d_j^2$, $\forall j$.

For $d\in\mathscr{N}_c$ we consider the following two, essentially equivalent, "associated" functions:

$$k(\tau) = \ln \textstyle\sum_j (\tau/d_j)^j, \quad k'(\tau) = \sup_k \ln(\tau/d_k)^k, \quad \tau\in\mathbf{R}^+.$$

Note that these expressions make sense, due to (i).

Lemma 5.2. $k'(\tau)\leqq k(\tau)\leqq k'(2\tau)$.

Indeed

$$\sup_k (2\tau/d_k)^k = \sum_j (1/2^j)\sup_k (2\tau/d_k)^k \geqq \sum_j (1/2^j)(2\tau/d_j)^j = \sum_j (\tau/d_j)^j.$$

In view of this lemma, the functions k and k' play the same role in most of the problems concerning Denjoy—Carleman classes. In the sequel, we will prefer to work with k.

Proposition 5.3. *$k(\tau):\mathbf{R}^+\to\mathbf{R}^+$ satisfies $|k(\tau)-k(\tau')|\leqq|\tau-\tau'|$ and is increasing.*

Proof. For $\tau>0$ we may differentiate and obtain

$$dk(\tau)/d\tau = \left(\sum_j j\tau^{j-1}/(d_j)^j\right)/\sum (\tau/d_j)^j \leqq 1,$$

since

$$\sum j\tau^{j-1}/(d_j)^j \leqq \sum (\tau/d_j)^{j-1} \leqq \sum (\tau/d_{j-1})^{j-1} = \sum (\tau/d_j)^j,$$

in view of (11) and (12).

2. The following two propositions essentially correspond to properties (v) and (vi).

Proposition 5.4. *Suppose $r\to d(r)$ is a sequence of elements in $\mathcal{N}_c$. Then there exists $d\in\mathcal{N}_c$ such that $d\preceq d(r)$, $\forall r$.*

Proof. By hypothesis $\sum_j 1/d_j(r)\leqq c_r$ for some constants c_r, which we may suppose $\geqq 1$. Define $d_j=\inf_r r^2c_rd_j(r)$. It is then easy to see that d_j satisfies (11), (12), (13) and

$$\sum 1/d_j = \sum_r (1/c_r r^2) \sum_{j\in J_r} (1/d_j(r)) \leqq 1,$$

where

$$J_r = \{j;\ d_j = r^2 c_r d_j(r)\},$$

since

$$\sum_{j\in J_r} (1/d_j(r)) \leqq \sum_j (1/d_j(r)) \leqq c_r.$$

The order relations $d\preceq d(r)$ are trivial.

Proposition 5.5. *Suppose $q\in M_{sd}$ and let $I\colon \mathcal{N}_c\to\mathbf{N}$ be a function. Then there exists $r\in\mathbf{N}$ and a constant C such that*

$$\sup_{d\in\mathcal{N}_c} \ln \sum_j \left(q_{I(d)}(\zeta)/d_j\right)^j + C \geqq q_r(\zeta). \tag{15}$$

Proof. We will reason by contradiction. The first thing to note is, that if the proposition is not true, then there exists a sequence of points ζ_k such that

$$\sup_{d\in\mathcal{N}_c} \ln \sum \left(q_{I(d)}(\zeta_k)/d_j\right)^j + k \leqq (1/k^2)q_k(\zeta_k). \tag{16}$$

Indeed, if (15) were false for any choice of C and r, then we first obtain a sequence of points ζ_k such that

$$\sup_{d\in\mathcal{N}_c} \ln \sum \left(q_{I(q)}(\zeta_k)/d_j\right)^j + c_k + k \leqq q_{j(k)}(\zeta_k)$$

where $j(k)$ and c_k are chosen such that $q_{j(k)}\leqq(1/k^2)q_k+c_k$, and (16) then follows immediately.

(16) implies in particular that $q_k(\zeta_k) \geqq k^3$.

We want to show that this remark, together with (16) leads to a contradiction. In fact we will construct $d \in \mathcal{N}_c$ such that, at least for k in a subsequence $\{k_s\}$,

$$\ln \sum (q_k(\zeta_k)/d_j)^j \geqq q_k(\zeta_k)/k\sqrt{k}, \tag{17}$$

k in the subsequence, and this is a contradiction, since then also (in view of the fact that the sequence q_k is decreasing and that the function ℓ associated with d is increasing)

$$\ln \sum (q_{l(d)}(\zeta_k)/d_j)^j \geqq q_k(\zeta_k)/k\sqrt{k}$$

for large values of k in the subsequence.

Therefore the proposition follows, if we can prove the following assertion:

Suppose τ_k is a sequence of points in $\mathbf{R}^+$ such that $\tau_k \geqq k^3$. Then there exists $d \in \mathcal{N}_c$ and a subsequence $\{k_s\}$ for which

$$\ln \sum_j (\tau_k/d_j)^j \geqq \tau_k/k\sqrt{k} \quad \text{for} \quad k \in \{k_s\}. \tag{18}$$

Now (18) is equivalent with

$$\sum_j (\tau_k/d_j)^j \geqq \exp(\tau_k/k\sqrt{k}),$$

and to have this, it is sufficient to have $d_j \leqq \tau_k/e$ for some $j \geqq \tau_k/k\sqrt{k}$. (In this case $(\tau_k/d_j)^j \geqq (\tau_k/d_j)^{\tau_k/k\sqrt{k}} = \exp \tau_k/k\sqrt{k}$).

We would therefore like to find $\{d_j\}$ such that, at least on some subsequence, $d_{[\tau_k/k\sqrt{k}]} \leqq \tau_k/e$ ($[A]$ is the integral part of A). Denoting $[\tau_k/k\sqrt{k}] = a_k$ it is therefore sufficient to have $d_{a_k} \leqq k a_k$, and we can apply the following

Lemma 5.6. *Given an increasing sequence a_k there exists $d \in \mathcal{N}_c$ such that $d_{a_{s^2}} = s^2 a_{s^2}$.*

Proof. Define $d_{a_{s^2}} = s^2 a_{s^2}$ for s natural and

$$d_{a_{s^2}+i} = \min(d_{a_{(s+1)^2}}, e d_{a_{s^2}+i-1}) \quad \text{for} \quad 1 \leqq i < a_{(s+1)^2} - a_{s^2}.$$

It is then clear that d satisfies (11) and (12). To see that (13) is satisfied, we have only to observe that

$$(s+1)^2 a_{(s+1)^2} \leqq e^{a_{(s+1)^2} - a_{s^2}} s^2 a_{s^2},$$

for $s \geqq 2$ in view of the fact that the function e^λ/λ^2 is increasing for $\lambda \geqq 2$. It remains to check (14). This follows from

$$\sum_{0 \leqq i < a_{(s+1)^2} - a_{s^2}} \frac{1}{d_{a_{s^2}+i}} \frac{a_{(s+1)^2} - a_{s^2}}{d_{a_{(s+1)^2}}} + \frac{1}{d_{a_{s^2}}} \sum_{i=0}^{\infty} e^{-i} \leqq \frac{C}{s^2},$$

whence $\sum 1/d_j \leqq C \sum 1/s^2 < \infty$.

3. For every $d\in\mathcal{N}_c$ we now consider the associated Denjoy—Carleman class, which is defined in the following way: we first introduce quasinorms on $C^\infty(t)$, setting, for $d\in\mathcal{N}_c$, $h>0$ and $B<t$

$$|f|^{d,h,B} = \sup_{\substack{|x|\leqq B \\ \alpha}} \frac{|D^\alpha f(x)|}{h^{|\alpha|}(d_{|\alpha|})^{|\alpha|}}$$

and then we define

$$\mathcal{G}_d^w(t) = \{f\in C^\infty(t);\ \forall B<t,\ \exists h>0 \text{ such that } |f|^{d,h,B}<\infty\}.$$

The properties of the elements from $\mathcal{N}_c$ then correspond to the following properties of $\mathcal{G}_d^w(t)$:

(11) implies: $\mathcal{G}_d^w(t)$ contains the real analytic functions from $C^\infty(t)$.

(12) implies that $\mathcal{G}_d^w(t)$ is stable under multiplication.

(13) implies that $\mathcal{G}_d^w(t)$ is stable under derivation.

According to the famous Denjoy—Carleman theorem (cf. e.g. [12]) (14) implies (and is in fact equivalent to) the fact that $\mathcal{G}_d^w(t)$ is nonquasianalytic.

The relevance of the function k associated with d to the corresponding Denjoy—Carleman class $\mathcal{G}_d^w(t)$ stems from the following trivial

Proposition 5.7. *Consider $\varphi\in C_0^\infty(t)$. Then the following two assertions are equivalent:*

(a) $\varphi\in\mathcal{G}_d^w(t)$,

(b) *there exist $B<t$ and $\alpha>0$ such that for every b*

$$\exp k(\alpha|\zeta|)\,|\hat\varphi(\zeta)| \leqq C_b \exp(B|\mathrm{Im}\,\zeta| - b\ln(1+|\zeta|)).$$

(Since we do not allow the d_j to be $+\infty$, the term $-b\ln(1+|\zeta|)$ in the exponent is not relevant; k dominates $b\ln(1+|\zeta|)$ at infinity for any b. We have inserted it here for later convenience.)

Let us note in conclusion of the section that we also have the following result:

Proposition 5.8. *Consider $d\in\mathcal{N}_c$ and denote by k the associated function. Then $\{k_j\}$, $k_j(\zeta)=k(|\zeta|/j)$ defines an element in M_d and $\mathcal{G}_d^w(t)$ is the weak A.U. space associated with this element.*

The first assertion is an easy consequence of Proposition 5.3. In view of Proposition 5.7, the Fourier inversion formula gives for elements in $\mathcal{G}_d^w(t)\cap C_0^\infty(t)$ a representation of type (1). This is already enough, in view of the fact that $\mathcal{G}_d^w(t)$ is nonquasianalytic.

§ 6. Proof of Theorem 1.3.

1. Consider $q\in M_d$ and $d\in\mathcal{N}_c$. We define $(q(d))_j=\ln\sum_k (q_j(\zeta)/d_k)^k=k(q_j)$. It is immediately seen from Proposition 5.3 that $q(d)\in M_d$. This gives sense to the following proposition:

Proposition 6.1. *Suppose $q\in M_{sd}$ and $d\in\mathcal{N}_c$. Then $\mathcal{E}^w_{q(d)}(t)$ is quasianalytic.*

In some way $\mathcal{E}^w_{q(d)}(t)$ is as close to $\mathcal{E}^w_q(t)$ as is the corresponding Denjoy—Carleman class to the real analytic functions.

Proof of Proposition 6.1 (standard). It is clear that $\mathcal{E}^w_{q(d)}(t)\supset\mathcal{G}^w_d(t)\cap C^\infty_0(t)$, which gives the first two properties in Definition 1.2. It is therefore enough to prove that the elements from $\mathcal{G}^w_d(t)\cap C^\infty_0(t)$ are multipliers for $\mathcal{E}^w_{q(d)}(t)$. Suppose then that $\varphi\in\mathcal{G}^w_d(t)\cap C^\infty_0(t)$, and $f\in\mathcal{E}^w_{q(d)}(t)$. We want to prove that $\varphi f\in\mathcal{E}^w_{q(d)}(t)$.

From $\varphi\in\mathcal{G}^w_d(t)\cap C^\infty_0(t)$ it follows, for some small δ, which we may suppose smaller than 1/2, that for all b

$$|\hat{\varphi}(\eta)|\exp k(2\delta|\eta|)\leqq C\exp\left(B|\mathrm{Im}\,\eta|-b\ln(1+|\eta|)\right)\quad\text{for some}\quad B<t, \tag{19}$$

and from $f\in\mathcal{E}^w_{q(d)}(t)$ it results that f can be written in the form $f(x)=\int\exp i\langle x,\zeta\rangle\,dv(\zeta)$, $|x|\leqq B$, for some Radon measure v such that

$$v=\mu/\exp\left(k(q_r(\zeta))+B|\mathrm{Im}\,\zeta|+b\ln(1+|\zeta|)\right),\quad \int d|\mu|<\infty,$$

for some (great) r.

We now define $\psi(\zeta)=(2\pi)^{-n}\int\hat{\varphi}(\eta)\,dv(\zeta-\eta)$ and we want to prove that

(A) $|\psi(\xi)|\leqq C\exp\left(-k(\delta q_r(\xi))-b\ln(1+|\xi|)\right)$,

(B) $\widehat{\varphi f}(\xi)=\psi(\xi)$.

This would bring the proof of the proposition to an end, in view of the fact that $k(\delta q_r(\xi))\geqq k(q_{r'}(\xi)-c_r)\geqq k(q_{r'}(\xi))-c_r$ for some r' and c_r (which come out from (iii)). Indeed, we could write $(\varphi f)(x)=(2\pi)^{-n}\int\exp i\langle x,\xi\rangle\psi(\xi)\,d\xi$.

a) To prove (A) we first note that

$$k(\delta q_r(\xi+\eta))\leqq k(q_r(\xi))+k(2\delta|\eta|). \tag{20}$$

Indeed

$$k(\delta q_r(\xi+\eta))\leqq k(\delta q_r(\xi)+\delta|\eta|)\leqq k(2\delta q_r(\xi))+k(2\delta|\eta|)\leqq k(q_r(\xi))+k(2\delta|\eta|).$$

Here we have used Proposition 5.3.

We can now estimate $\psi(\zeta)$.

$$\exp\big(k(\delta q_r(\xi))+b\ln(1+|\xi|)\big)\left|\int \hat{\varphi}(\eta)\,d\nu(\xi-\eta)\right| \leqq$$
$$\leqq \exp\big(k(\delta q_r(\xi))+b\ln(1+|\xi|)\big)\cdot$$
$$\cdot\int |\hat{\varphi}(\eta)|\exp\big(-(k(q_r(\xi-\eta))+B|\operatorname{Im}\eta|+b\ln(1+|\xi-\eta|))\big)\,d|\mu(\xi-\eta)| \leqq$$
$$\leqq \sup_\eta |\hat{\varphi}(\eta)|\exp l(\xi,\eta)\int d|\mu|,$$

with l denoting some obvious exponent. Using (19), it is sufficient to prove that $l(\xi,\eta)-k(2\delta|\eta|)+B|\operatorname{Im}\eta|-b\ln(1+|\eta|)$ is bounded, and this results from (20) and the subadditivity of the function ln.

b) It is easy to see that ψ is in fact an entire function. Moreover, estimates similar to those above show also that $\psi(\xi)$ satisfies inequalities which are sufficient to write $\psi(\xi)$ as the Fourier—Borel transform of a C_0^∞ function, concentrated in $|x|\leqq B$. To prove (B) it is therefore sufficient to show that for $|x|\leqq B$, $\mathscr{F}^{-1}\psi(x)=$ $=\varphi(x)f(x)$.

Now

$$\int \exp i\langle x,\zeta\rangle\int\hat{\varphi}(\eta)\,d\nu(\xi-\eta)\,d\xi = \iint \exp i\langle x,\xi\rangle\hat{\varphi}(\eta)\,d\nu(\xi-\eta)\,d\xi =$$
$$= \iint \exp i\langle x,\xi-\eta+\eta\rangle\hat{\varphi}(\xi-\eta)\,d\xi\,d\nu(\eta) =$$
$$= \iint \exp i\langle x,\eta\rangle\big(\exp i\langle x,\xi-\eta\rangle\hat{\varphi}(\xi-\eta)\,d\xi\big)\,d\nu = (2\pi)^n f(x)\varphi(x) \quad \text{for} \quad |x|\leqq B,$$

by the definition of ν.

2. It is now clear that Theorem 1.3 follows from Theorem 1.5 in view of the following two lemmas:

Lemma 6.2. *Suppose that $q\in M_{sd}$ satisfies $q_j(\zeta)\geqq 2q_{j+1}(\zeta)$. Then $K_q=\{q(d);\,d\in\mathscr{N}_c\}$ satisfies* (v).

Lemma 6.3. *For every $q\in M_{sd}$, there exists $q'\in M_{sd}$ which satisfies*

$q'_j(\zeta)\geqq 2q'_{j+1}(\zeta)$.

$\forall r,\ \forall C,\ \exists r'$ *such that* $\max\big(q'_r(\zeta)-C,0\big)\geqq q'_{r'}(\zeta)$.

$\mathscr{E}_q^w(t)=\mathscr{E}_{q'}^w(t)$.

In fact, given $q\in M_{sd}$, we first consider q' given in Lemma 6.3 and then we define $K_{q'}$ for this q'. $K_{q'}$ then satisfies (v) in view of Lemma 6.2 and (vi) for q' in view of Proposition 5.5 and the second property of q'.

The first lemma is an easy consequence of Proposition 5.4.

Proof of Lemma 6.3. Consider $q\in M_{sd}$ and let $i(j)$ be an increasing sequence of integers. Define $\bar{q}=\{\bar{q}_j\}$, $\bar{q}_j=q_{i(j)}$. It is then clear that $\mathscr{E}^w_q(t)=\mathscr{E}^w_{\bar{q}}(t)$. Choosing $i(j)$ suitably, it is easy to see that we may suppose that $\bar{q}_j(\zeta)\geqq 2\bar{q}_{j+1}(\zeta)-c_j$, for some c_j. We now define

$$q'_j = \max\left(\bar{q}_j - \textstyle\sum_{k<j} c_k - j, 0\right).$$

Obviously $q'\in M_{sd}$, and it is immediately seen that q' has the stated properties.

§ 7. Comments and remarks

Remark 7.1. The results from this paper should be compared with the results from [3] and [2, Theorem 1.5.12]. It is not possible to derive the results from [3] as corollaries of Theorem 1.5. One reason is, that if d_j is an increasing sequence of integers which satisfy $d_j\geqq j$, then $q_k=\ln(|\zeta|/kd_j)^j$ is not necessarily in M_{sd} (take, e.g., $d_j=\exp j$). The main difference with respect to [3] is that we obtain intersection theorems for spaces far from Denjoy—Carleman classes.

Remark 7.2. Proposition 5.7. remains valid also for quasianalytic Denjoy—Carleman classes. This was pointed out to the author by Mats Neymark. In fact, one uses arguments from the proof of Lemma 4.1 and the characterization of the dual of $\mathscr{G}^w_d(t)$ spaces given in [10], [11].

Remark 7.3. Weak A.U. spaces are closely related to A.U. spaces (cf. [1], [4], [5]) and the two classes of spaces have many properties in common. One main difference is that whereas weak A.U. spaces are local (this is elementary, and follows also from Theorem 1.3.), A.U. spaces are not (nonlocal A.U. spaces appear in the study of the Cauchy problem for P.D.O.).

Remark 7.4. Being local, weak A.U. spaces make sense also for nonconvex domains, and we immediately obtain the analogues of Theorems 1.3 and 1.5. Moreover, these theorems remain true even for germs.

We have used here the following terminology: A function space $X\subset\mathscr{E}(t)$ is called *local,* if all the functions which belong locally to X are in X. A function $f\in\mathscr{E}(t)$ is said to *belong locally* to X, if for every $|x_0|<t$, there exists a neighborhood V of x_0 and $g\in X$ such that $f=g$ in V.

Remark 7.5. The arguments in this paper are essentially microlocal.

References

1. Berenstein, C. A., & Dostal, M. A., *Analytically uniform spaces and their application to convolution equations,* Lecture Notes in Mathematics, 256, Springer-Verlag, 1972.
2. Björck, G., Linear partial differential operators and generalized distributions, *Ark. Mat.* **6** (1966), 351—407.
3. Boman, J., On the intersection of classes of infinitely differentiable functions, *Ark. Mat.* **5** (1964), 301—309.
4. Chou, C. C., *La transformation de Fourier complexe et l'équation de convolution,* Lecture Notes in Mathematics, 325, Springer Verlag, 1973.
5. Ehrenpreis, L., *Fourier analysis in several complex variables,* Wiley-Interscience Publishers, New York, 1970.
6. Hörmander, L., *Linear partial differential operators,* Springer-Verlag, 1963.
7. Hörmander, L., *An introduction to complex analysis in several variables,* D. Van Nostrand Company, New York, 1966.
8. Hörmander, L., Convolution equations in convex domains, *Invent. Math.,* **4** (1968), 306—318.
9. Kiselman, C. O., On unique supports of analytic functionals, *Ark. Mat.* **6** (1965), 307—318.
10. Neymark, M., On the Laplace transform of functionals on classes of infinitely differentiable functions, *Ark. Mat.* **7** (1968), 577—594.
11. Nielssen, B. D., *On the existence of solutions to P. D. E. with constant coefficients in D—C classes of functions.* Preprint, Lund 1974.
12. Rudin, W., *Real and complex analysis,* McGraw-Hill Book Comp. New York, 1966.

Received October 30, 1974, in revised form June 10, 1975

Otto Liess
Institute of Mathematics
Strada Academiei 14
Bucarest, Romania