Injectors in certain classes of locally π-soluble groups

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0. Introduction

In [2], Fischer, Gaschütz, and Hartley showed that any finite soluble group possesses a unique conjugacy class of $\mathfrak{F}$-injectors for any Fitting class $\mathfrak{F}$ of finite soluble groups. The results of [2] have been extended to certain classes of periodic locally soluble groups by Tomkinson [5] and the author [4].

The purpose of this note is to show that the restriction of local solubility, imposed in the above papers, is not necessary in order to obtain well-behaved classes of $\mathfrak{F}$-injectors. In section 2 we consider finite π-soluble groups and prove that such a group possesses a unique conjugacy class of $\mathfrak{F}$-injectors, where $\mathfrak{F}$ is any Fitting class of finite π-soluble groups such that $O^{\pi'}(G)\in\mathfrak{F}$ implies that $G\in\mathfrak{F}$. Section 3 indicates an extension of the results of section 2 to a class of locally finite locally π-soluble groups; the injectors obtained here form a unique local conjugacy class of subgroups.

1. Notation and Terminology

Throughout this paper π will denote a fixed but arbitrary set of primes. If $\mathfrak{X}$ is a class of groups we denote by $\mathfrak{X}^*$ the class of finite $\mathfrak{X}$-groups and by $\mathfrak{X}_\pi$ the class of π-groups in $\mathfrak{X}$. $\mathfrak{N}$ denotes the class of finite nilpotent groups. A finite group G is said to be *π-soluble* if every chief factor of G is either a π-group or a π'-group, and those which are π-groups are soluble.

Let H be a subgroup of an arbitrary group G, and let Ω be a totally ordered set. By a *series of type Ω from H to G* we mean a set $\{U_\sigma, V_\sigma; \sigma\in\Omega\}$ of pairs of subgroups of G containing H and satisfying

(i) $V_\sigma \trianglelefteq U_\sigma$, for all $\sigma\in\Omega$,

(ii) $U_\alpha \leqq V_\sigma$, if $\alpha<\sigma$,

(iii) $G-H=\bigcup_{\sigma\in\Omega}(U_\sigma-V_\sigma)$, where $G-H$ denotes the set of elements of G which do not belong to H. In such a situation we say that H is *serial* in G, and write H ser G.

We define a group $G\in\mathfrak{M}$ if and only if G possesses a local system $\Sigma=\{F_\lambda: \lambda\in\Lambda\}$ consisting of finite π-soluble groups such that $R\leqq N_G(\Sigma)=\bigcap_{F\in\Sigma} N_G(F)$, where R is the locally nilpotent residual of G, i.e. R is the unique normal subgroup of G minimal with respect to the factor group G/R being locally nilpotent. Clearly R is an FC-group and F ser G, for all $F\in\Sigma$. Throughout when considering an $\mathfrak{M}$-group G we shall assume that Σ denotes a local system which defines G as an $\mathfrak{M}$-group, i.e. satisfies the above definition. $\mathfrak{M}^*$ is then the class of finite π-soluble groups.

An automorphism α of a group G is called a *locally inner automorphism* of G if, given any finite set of elements $g_1, \ldots, g_n$ of G, there exists $g\in G$, depending in general on the set in question, such that $g_i^\alpha=g^{-1}g_ig$, for $i=1, \ldots, n$. Two subgroups X and Y of G are said to be *locally conjugate* in G if there exists a locally inner automorphism of G mapping X onto Y.

We shall use the notation of group classes and closure operations developed by P. Hall and set out for example in [3].

2. Finite π-soluble groups

A *Fitting class of finite π-soluble groups* is a subclass $\mathfrak{F}$ of $\mathfrak{M}^*$ such that

(i) if $G\in\mathfrak{F}$ and $N\trianglelefteq G$, then $N\in\mathfrak{F}$,

(ii) if $N_1, N_2\trianglelefteq G\in\mathfrak{M}^*$ and $N_1, N_2\in\mathfrak{F}$, then $N_1N_2\in\mathfrak{F}$.

Lemma 2.1. *Let $\mathfrak{F}$ be a Fitting class of finite π-soluble groups, then the following are equivalent:*

(i) if $O^{\pi'}(G)\in\mathfrak{F}$, then $G\in\mathfrak{F}$,

(ii) $\mathfrak{F}=\mathfrak{F}\mathfrak{M}^*_{\pi'}$.

Proof. Suppose that $\mathfrak{F}$ satisfies (i). Clearly $\mathfrak{F}\leqq\mathfrak{F}\mathfrak{M}^*_{\pi'}$, so assume that $G\in\mathfrak{F}\mathfrak{M}^*_{\pi'}$. Then there exists a normal subgroup N of G such that $N\in\mathfrak{F}$ and $G/N\in\mathfrak{M}^*_{\pi'}$. But $O^{\pi'}(G)\leqq N$ and, since $\mathfrak{F}$ is a Fitting class, $O^{\pi'}(G)\in\mathfrak{F}$. Hence $G\in\mathfrak{F}$.

On the other hand, if $\mathfrak{F}$ satisfies (ii) and $O^{\pi'}(G)\in\mathfrak{F}$, then $G\in\mathfrak{F}\mathfrak{M}^*_{\pi'}=\mathfrak{F}$.

We therefore define a Fitting class $\mathfrak{F}$ of finite π-soluble groups to be *extendable* if $\mathfrak{F}$ satisfies the conditions of Lemma 2.1.

Our next result shows that extendable Fitting classes are plentiful.

Lemma 2.2. *If $\mathfrak{F}$ is a Fitting class of finite π-soluble groups, then $\mathfrak{F}\mathfrak{M}^*_{\pi'}$ is an extendable Fitting class of finite π-soluble groups.*

Proof. (i) Let N be a normal subgroup of $G \in \mathfrak{F}\mathfrak{M}^*_{\pi'}$, and let $K \trianglelefteq G$ such that $K \in \mathfrak{F}$ and $G/K \in \mathfrak{M}^*_{\pi'}$. Then $N/N \cap K \cong NK/K \in \mathfrak{M}^*_{\pi'}$ and $N \cap K \trianglelefteq K \in \mathfrak{F}$. Hence $N \cap K \in \mathfrak{F}$.

(ii) Let N_1, N_2 be normal subgroups of a finite π-soluble group G such that $N_1, N_2 \in \mathfrak{F}\mathfrak{M}^*_{\pi'}$. Let K_i be the $\mathfrak{F}$-radical of N_i, $i=1, 2$. Then $K_i \trianglelefteq G$ and $N_i/K_i \in \mathfrak{M}^*_{\pi'}$, $i=1, 2$. Hence $K_1K_2 \in \mathfrak{F}$ and $N_1N_2/K_1K_2 = N_1K_2/K_1K_2 \cdot K_1N_2/K_1K_2 \in \mathfrak{M}^*_{\pi'}$.

Therefore $\mathfrak{F}\mathfrak{M}^*_{\pi'}$ is a Fitting class of finite π-soluble groups. Clearly $\mathfrak{F}\mathfrak{M}^*_{\pi'}$ is extendable.

Lemma 2.3. *Let N be a normal subgroup of a finite π-soluble group G such that either $G/N \in \mathfrak{M}^*_{\pi'}$ or $G/N \in \mathfrak{N}_\pi$. Let $\mathfrak{F}$ be an extendable Fitting class of finite π-soluble groups, let W be an $\mathfrak{F}$-maximal subgroup of N, and let V_1, V_2 be $\mathfrak{F}$-maximal subgroups of G with $W \leq V_1 \cap V_2$. Then there exists an element g belonging to the nilpotent residual of G such that $V_1^g = V_2$.*

Proof. Clearly we may assume that $W = V_1 \cap N = V_2 \cap N$ is a normal subgroup of G. Let R be the nilpotent residual of G.

Case (a). $G/N \in \mathfrak{M}^*_{\pi'}$. If X is any Sylow π'-subgroup of G, then, since $\mathfrak{F}$ is extendable, WX is an $\mathfrak{F}$-subgroup of G. Let S, T_1 be respectively Sylow π-, π'-subgroups of G such that $S \cap V_1$, $T_1 \cap V_1$ are respectively Sylow π-, π'-subgroups of V_1. Then $V_1 = (S \cap V_1)(T_1 \cap V_1)$. But $S \leq N$ and so $S \cap V_1 \leq N \cap V_1 = W$. Therefore $V_1 \leq WT_1$ which, by the $\mathfrak{F}$-maximality of V_1, implies that $V_1 = WT_1$. Similarly there exists a Sylow π'-subgroup T_2 of G such that $V_2 = WT_2$. Since, by [1], the Sylow π'-subgroups of G are all conjugate and $T_1R = T_2R$, we have that there exists $g \in R$ such that $T_1^g = T_2$. Hence $V_1^g = V_2$.

Case (b). $G/N \in \mathfrak{N}_\pi$. If X is a Sylow π'-subgroup of G, then $X \leq N$ and so, by the $\mathfrak{F}$-maximality of W in N, $X \leq W$. Therefore $G/W \in \mathfrak{M}^*_\pi$ and hence G/W is a finite soluble group. Let E_i/W be a Carter subgroup of $M_i/W = N_{G/W}(V_i/W)$. Then $M_i/M_i \cap N$ is nilpotent and so, for large enough r, we have that

$$[V, \underbrace{M_i, \dots, M_i}_{r}] \leq V_i \cap N = W.$$

Hence V_i/W is hypercentral in M_i/W. Therefore V_i/W is a normal subgroup of E_i/W, for $i=1, 2$.

Claim. E_i/W is a Carter subgroup of G/W. For, suppose that for some $x \in G$, $E_i^x = E_i$. Then $V_i^x \trianglelefteq E_i$ and so $V_iV_i^x \in \mathfrak{F}$. Therefore, by the $\mathfrak{F}$-maximality of V_i, $V_i = V_i^x$ i.e. $x \in M_i$. Hence $x \in E_i$ and so E_i/W is a Carter subgroup of G/W.

Therefore, since $E_1R = E_2R$, there exists $g \in R$ such that $E_1^g = E_2$. Therefore V_1^g, V_2 are normal $\mathfrak{F}$-subgroups of E_2. Hence $V_1^g = V_2$.

If $\mathfrak{F}$ is a Fitting class of finite π-soluble groups and $G \in \mathfrak{M}^*$, then an $\mathfrak{F}$-*injector* of G is a subgroup V of G such that

(i) $V \in \mathfrak{F}$,

(ii) $V \cap N$ is a maximal $\mathfrak{F}$-subgroup of N, for all normal subgroups N of G.

Theorem 2.4. *Let G be a finite π-soluble group, and let $\mathfrak{F}$ be an extendable Fitting class of finite π-soluble groups. Then G possesses $\mathfrak{F}$-injectors, and if V_1, V_2 are $\mathfrak{F}$-injectors of G then there exists an element g of the nilpotent residual of G such that $V_1^g = V_2$.*

Proof. We prove the existence of $\mathfrak{F}$-injectors by induction on $|G|$, the case $|G|=1$ being trivial. So, suppose that $|G| \neq 1$ and that the theorem has been proved for all groups of order less than $|G|$. Then there exists a proper normal subgroup K of G such that either $G/K \in \mathfrak{M}_\pi^*$ or $G/K \in \mathfrak{N}_\pi$. By induction K possesses an $\mathfrak{F}$-injector U, say. Let V be an $\mathfrak{F}$-maximal subgroup of G such that $U \leqq V$.

Claim. V is an $\mathfrak{F}$-injector of G. For, let M be a maximal normal subgroup of G, then we shall show that $M \cap V$ is an $\mathfrak{F}$-injector of M. Let $N = M \cap K$, then either $M/N \in \mathfrak{M}_\pi^*$ or $M/N \in \mathfrak{N}_\pi$. By induction M possesses an $\mathfrak{F}$-injector V^*, say. Then $W = N \cap V = N \cap U$ and $N \cap V^*$ are $\mathfrak{F}$-injectors of N. But, by induction, there exists an element g of the nilpotent residual of N such that $W^g = N \cap V^*$. Clearly $g \in R$, the nilpotent residual of G, and so we may assume that $W = N \cap V^*$. Now, let $\bar{V}$ be an $\mathfrak{F}$-maximal subgroup of G such that $V^* \leqq \bar{V}$. Then, by Lemma 2.3, there exists $x \in R$ such that $V^x = \bar{V}$. Then we have

$$(V \cap M)^x = V^x \cap M = \bar{V} \cap M = V^*.$$

Hence $V \cap M$ is an $\mathfrak{F}$-injector of M. Since this is true for all maximal normal subgroups M of G, V is an $\mathfrak{F}$-injector of G.

To prove conjugacy let M again be a maximal normal subgroup of G. Then either $G/M \in \mathfrak{M}_\pi^*$ or $G/M \in \mathfrak{N}_\pi$. By induction we may assume that $M \cap V_1 = M \cap V_2$. Then the result follows immediately by Lemma 2.3.

Corollary 2.5. *Let $1 = G_0 \trianglelefteq G_1 \trianglelefteq \dots \trianglelefteq G_n = G$ be a series of a finite π-soluble group G such that either $G_{i+1}/G_i \in \mathfrak{M}_\pi^*$ or $G_{i+1}/G_i \in \mathfrak{N}_\pi$, for all $1 \leqq i \leqq n-1$, and let $\mathfrak{F}$ be an extendable Fitting class of finite π-soluble groups. Then V is an $\mathfrak{F}$-injector of G if and only if $V \cap G_i$ is an $\mathfrak{F}$-maximal subgroup of G_i, for all $1 \leqq i \leqq n$.*

Proof. Follows from the proof of Theorem 2.4.

Corollary 2.6. *Let $\mathfrak{F}$ be an extendable Fitting class of finite π-soluble groups, and let V be an $\mathfrak{F}$-injector of a finite π-soluble group G such that $V \leqq H \leqq G$. Then V is an $\mathfrak{F}$-injector of H.*

Proof. Let $1=G_0 \trianglelefteq G_1 \trianglelefteq \ldots \trianglelefteq G_n=G$ be a series of G such that either $G_{i+1}/G_i \in \mathfrak{M}^*_{\pi'}$ or $G_{i+1}/G_i \in \mathfrak{N}_\pi$, for all $1 \leqq i \leqq n-1$. Then, if $H_i=H \cap G_i$ for all $1 \leqq i \leqq n$, $1=H_0 \trianglelefteq H_1 \trianglelefteq \ldots \trianglelefteq H_n=H$ is a series of H of the same type. Then $H_i \cap V=G_i \cap H \cap V=G_i \cap V$ is an $\mathfrak{F}$-maximal subgroup of H_i, for all $1 \leqq i \leqq n$. The result now follows by Corollary 2.5.

3. A class of locally π-soluble groups

A *Fitting class* of $\mathfrak{M}$-groups is a subclass $\mathfrak{F}$ of $\mathfrak{M}$ such that

(i) if $G \in \mathfrak{F}$ and H ser G, then $H \in \mathfrak{F}$,

(ii) if $\{S_\lambda; \lambda \in \Lambda\}$ is a family of serial $\mathfrak{F}$-subgroups of an $\mathfrak{M}$-group G such that $G=\langle S_\lambda; \lambda \in \Lambda\rangle$, then $G \in \mathfrak{F}$.

A Fitting class $\mathfrak{F}$ of $\mathfrak{M}$-groups is said to be *extendable* if $\mathfrak{F}=\mathfrak{F}\mathfrak{M}_{\pi'} \cap M$.

The following is immediate from the definition of $\mathfrak{F}$.

Lemma 3.1. *If $\mathfrak{F}$ is an extendable Fitting class of $\mathfrak{M}$-groups, then $\mathfrak{F}^*$ is an extendable Fitting class of finite π-soluble groups.*

Theorem 3.2. *If $\mathfrak{G}$ is an extendable Fitting class of finite π-soluble groups, then $\mathfrak{G} \cap \mathfrak{M}$ is an extendable Fitting class of $\mathfrak{M}$-groups.*

Proof. (i) Let H ser $G \in \mathrm{L}\mathfrak{G} \cap \mathfrak{M}$, and let $h_1, \ldots, h_n \in H$. Then there exists a subgroup E of G such that $h_1, \ldots, h_n \in E \in \mathfrak{G}$. Now $H \cap E$ ser E which implies that $H \cap E$ is subnormal in E. Therefore $h_1, \ldots, h_n \in H \cap E \in \mathfrak{G}$, since $\mathfrak{G}$ is closed under taking subnormal subgroups. Hence $H \in \mathrm{L}\mathfrak{G} \cap \mathfrak{M}$.

(ii) Suppose that the $\mathfrak{M}$-group G is generated by $\{S_\lambda; \lambda \in \Lambda\}$ a family of serial $\mathfrak{G}$-subgroups of G, and let $g_1, \ldots, g_n \in G$. Then there exists $F \in \Sigma$ such that $g_1, \ldots, g_n \in F$, and there exists $\lambda_1, \ldots, \lambda_r \in \Lambda$ such that $F \leqq \langle S_{\lambda_1}, \ldots, S_{\lambda_r}\rangle$.

We now need only show that if an $\mathfrak{M}$-group G is generated by a finite number of serial $\mathfrak{G}$-subgroups then it is itself a $\mathfrak{G}$-group. So, let $H_1, \ldots, H_m$ be serial $\mathrm{L}\mathfrak{G}$-subgroups of an $\mathfrak{M}$-group G such that $G=\langle H_1, \ldots, H_m\rangle$, and let $g_1, \ldots, g_n \in G$. Then there exists $F \in \Sigma$ such that $g_1, \ldots, g_n \in \langle H_1 \cap F, \ldots, H_m \cap F\rangle$ and $H_i \cap F$ is a serial subgroup of H_i, and so is a $\mathfrak{G}$-group, $1 \leqq i \leqq n$. Therefore $\langle H_1 \cap F, \ldots, H_m \cap F\rangle \in \mathfrak{G}$, since it is generated by subnormal $\mathfrak{G}$-subgroups. Hence $G \in \mathrm{L}\mathfrak{G} \cap \mathfrak{M}$, as required.

Now let $\mathfrak{F}=\mathrm{L}\mathfrak{G} \cap \mathfrak{M}$, and suppose that $G \in \mathfrak{F}\mathfrak{M}_{\pi'} \cap \mathfrak{M}$. Then, if $F \in \Sigma$, $F \in \mathfrak{G}\mathfrak{M}^*_{\pi'}=\mathfrak{G}$. Hence $G \in \mathrm{L}\mathfrak{G} \cap \mathfrak{M}=\mathfrak{F}$ and $\mathfrak{F}$ is extendable.

Corollary 3.3. *The extendable Fitting classes of $\mathfrak{M}$-groups are precisely the $\mathrm{L}\mathfrak{G} \cap \mathfrak{M}$, where $\mathfrak{G}$ is an extendable Fitting class of finite π-soluble groups.*

Proof. Let $\mathfrak{F}$ be an extendable Fitting class of $\mathfrak{M}$-groups and let $g_1, \ldots, g_n \in G \in \mathfrak{F}$. Then there exists $F \in \Sigma$ such that $g_1, \ldots, g_n \in F$. But F ser G, and so $F \in \mathfrak{F}^*$. Therefore $G \in \mathrm{L}\mathfrak{F}^* \cap \mathfrak{M}$. Now, suppose that G is an $\mathrm{L}\mathfrak{F}^* \cap \mathfrak{M}$-group. Then G possesses a local system Σ of finite serial $\mathfrak{F}$-subgroups and $G=\langle F; F\in\Sigma\rangle$. Hence $G\in\mathfrak{F}$. Therefore $\mathfrak{F}=\mathfrak{F}^*\cap\mathfrak{M}$.

The converse follows immediately from the previous theorem.

The proof of the following is straightforward and is therefore omitted.

Lemma 3.4. *If $\mathfrak{F}$ is an extendable Fitting class of $\mathfrak{M}$-groups, then* $\mathfrak{F}=\mathrm{L}\mathfrak{F}\cap\mathfrak{M}$.

If $\mathfrak{F}$ is an extendable Fitting class of $\mathfrak{M}$-groups and $G\in\mathfrak{M}$, then an *$\mathfrak{F}$-injector* of G is a subgroup V of G such that

(i) $V\in\mathfrak{F}$,

(ii) $V\cap S$ is a maximal $\mathfrak{F}$-subgroup of S, for all serial subgroups S of G.

We are now in a position to show, using inverse limits and the results obtained in section 2, that any $\mathfrak{M}$-group possesses a local conjugacy class of $\mathfrak{F}$-injectors. The proofs of the following results are similar to those used in the analogous results obtained in [4]. Consequently, proofs will not be given here.

Theorem 3.5. *Let $\mathfrak{F}$ be an extendable Fitting class of $\mathfrak{M}$-groups and suppose that $G\in\mathfrak{M}$. Then G possesses $\mathfrak{F}$-injectors and any two are locally conjugate in G.* (Cf. [4] Theorems 2.6 and 2.9.)

Theorem 3.6. *Let $\mathfrak{F}$ be an extendable Fitting class of $\mathfrak{M}$-groups, and let V be an $\mathfrak{F}$-injector of an $\mathfrak{M}$-group G. Then, if $V\leqq H\leqq G$, V is an $\mathfrak{F}$-injector of H.* (Cf. [4] Theorem 2.7.)

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