On functions with conditions on the mean oscillation

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1. Introduction

Function spaces defined using the mean oscillation have been studied by several authors, e.g. John and Nirenberg [5], Campanato [2], Meyers [7], Spanne [13], Peetre [9], [10], Fefferman [3], [4] and Sarason [12].

To define these spaces, let $I(x, r)$ be the cube $\{y \in \mathbf{R}^d;\ |y_i - x_i| \leqq r/2\}$ whose edges have length r and are parallel to the coordinate axis (only such cubes will be considered in the sequel).

For a cube I, define $f(I)$ as $m(I)^{-1}\int_I f(x)\,dx$ and $\Omega(f, I)$, the mean oscillation of f on I, as $m(I)^{-1}\int_I |f(x)-f(I)|\,dx$.

We can now define

$$\mathrm{BMO}_\varphi = \left\{ f \in L^1_{\mathrm{loc}};\ \|f\|_{\mathrm{BMO}_\varphi} = \sup_{\substack{x \in \mathbf{R}^d \\ r>0}} \frac{\Omega(f, I(x, r))}{\varphi(r)} < \infty \right\}$$

and

$$\Lambda_\varphi = \left\{ f;\ \|f\|_{\Lambda_\varphi} = \operatorname*{ess\,sup}_{x, y \in \mathbf{R}^d} \frac{|f(x)-f(y)|}{\varphi(|x-y|)} < \infty \right\}$$

where φ is assumed to be a positive non-decreasing function defined on $\mathbf{R}^+$. BMO_φ and Λ_φ will be regarded as spaces of functions modulo constants and are Banach spaces. Λ_φ is evidently continuously embedded in BMO_φ. If $\varphi(r) = r^\alpha$, $0 < \alpha \leqq 1$, then BMO_φ coincide with Λ_α, the space of (possibly unbounded) Lipschitz continuous functions (Campanato [2], Meyers [7]). On the other hand, if $\int_0^1 \varphi(r) r^{-1}\,dr = \infty$, then BMO_φ contains functions that are neither continuous nor locally bounded (Spanne [13]). In the extremal case $\varphi \equiv 1$, Λ_φ is the same as L^∞ modulo constants and BMO_φ is BMO.

Fefferman [3], [4] have proved that bounded functions and Riesz transforms of bounded functions span BMO. The present paper proves the following generalization of this.

Theorem 1. *Suppose that φ satisfies the growth condition*

$$r\int_r^\infty \frac{\varphi(t)}{t^2}\,dt \leqq C\varphi(r). \tag{$*$}$$

Then $\mathrm{BMO}_\varphi = \Lambda_\varphi + \sum_1^d R_i \Lambda_\varphi$. *More precisely, if* $f_j \in \Lambda_\varphi$, *then* $\|f_0 + \sum_1^d R_j f_j\|_{\mathrm{BMO}_\varphi} \leqq C \sum \|f_j\|_{\Lambda_\varphi}$ *and if* $f \in \mathrm{BMO}_\varphi$, *then there exist* $f_j \in \Lambda_\varphi$ *such that* $f = f_0 + \sum_1^d R_j f_j$ *and* $\sum_0^d \|f_j\|_{\Lambda_\varphi} \leqq C\|f\|_{\mathrm{BMO}_\varphi}$.

C denotes always some positive constant. It is to be noted that the constants in the statement of Theorem 1 depend only on the dimension d and the constant in $(*)$.

This theorem is valid for the corresponding spaces of functions on $\mathbf{T}^d$. However, in this case we will not identify functions differing by a constant. (We can take the norm as $\sup \Omega(f, I(x,r))/\varphi(r) + \|f\|_{L^1}$ and it is sufficient to have φ defined for $0 < r < \delta$, $\delta > 0$.) This enables us to study multiplication of elements in BMO_φ by a function f. The functions f such that this is a bounded operator from the space to itself are called the pointwise multipliers.

Theorem 2. *If* $\varphi(r)/r$ *is almost decreasing, then the set of pointwise multipliers for* $\mathrm{BMO}_\varphi(\mathbf{T}^d)$ *is* $\mathrm{BMO}_\psi \cap L^\infty$ *where* $\psi(r) = \varphi(r)/\int_r^1 \varphi(t)t^{-1}\,dt$.

(A positive function f is said to be almost decreasing if $\sup_{x \geqq y} f(x)/f(y) < \infty$.)

This theorem is proved in Section 4. We then use the duality between H^1 and BMO and construct a predual to H^1. This gives the corresponding result for H^1.

Theorem 3. *The set of pointwise multipliers for* $H^1(\mathbf{T}^d)$ *is* $\mathrm{BMO}_{|\log r|^{-1}} \cap L^\infty$.

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2. Preliminary lemmas

We state some simple lemmas without proofs. (See e.g. Spanne [13].) We write $\varrho(f, r) = \sup_{x, r' \leqq r} \Omega(f, I(x, r'))$ and $\omega(f, r) = \operatorname{ess\,sup}_{|x-y| \leqq r} |f(x) - f(y)|$.

Lemma 1. $\Omega(f, I) \leqq 2 \inf_a m(I)^{-1} \int_I |f(x) - a|\,dx$.

Lemma 2. *If* $|F(x) - F(y)| \leqq C|x - y|$, *then* $\Omega(F(f), I) \leqq C\Omega(f, I)$.

Lemma 3. *Suppose that* $I(x', r') \subset I(x, r)$. *Then*

$$|f(I(x', r')) - f(I(x, r))| \leqq C \int_{r'}^{2r} \frac{\varrho(f, t)}{t}\,dt.$$

Let ψ_r denote $r^{-d}\chi_{I(0,r)}$. Thus $\psi_r * f(x) = f(I(x, r))$.

Lemma 4. $\|f - \psi_r * f\|_{\mathrm{BMO}} \leqq C\varrho(f, r)$.

The next lemma gives an example of a function in BMO_φ which is in a certain sense extremal.

Lemma 5. *If $\varphi(t)t^{-1}$ is almost decreasing, then $f(x)=\int_{|x|}^{1}\varphi(t)t^{-1}dt$ belongs to* BMO_φ.

Lemma 6. *The growth condition $(*)$ is equivalent to the existence of $\alpha<1$ such that $t^{-\alpha}\varphi(t)$ is almost decreasing. In particular $\varphi(t)t^{-1}$ will be almost decreasing and converging to zero as $t\to\infty$.*

Note that

$$r\int_r^\infty \frac{\varphi(t)}{t^2}\,dt \geqq \varphi(r).$$

The condition $(*)$ thus implies that

$$r\int_r^\infty \frac{\varphi(t)}{t^2}\,dt,$$

which is a continuous non-decreasing positive function, defines the same space BMO_φ. Consequently, φ can in this case be assumed to be continuous.

Let K be a Calderon—Zygmund kernel, $K(x)=|x|^{-d}\Omega(x/|x|)$ with $\int_{S^{d-1}}\Omega=0$ and $|\Omega(x)-\Omega(y)|\leqq C|x-y|$ ([1]). Peetre [9] has shown that convolution by K is a bounded operator in BMO_φ for every φ satisfying $(*)$. The convolution can in these spaces be defined as

$$\lim_{\varepsilon\to 0}\int (K_\varepsilon(x-y)-K_1(-y))f(y)\,dy \text{ ([3])}, \quad K_\varepsilon(x)=\begin{cases}K(x), & |x|>\varepsilon\\ 0, & |x|\leqq\varepsilon.\end{cases}$$

The Riesz transforms R_j are defined as this convolution by $C_d x_j/|x|^{d+1}$, $j=1, \ldots, d$ ([14]). R_0 is defined as the identity operator.

3. Proof of Theorem 1

In this section $(*)$ is assumed. Also φ is assumed to be continuous; this can as stated in Section 2 be done without losing generality.

Since the Riesz transforms are bounded operators in BMO_φ, they are bounded as operators from Λ_φ to BMO_φ. Thus

$$\|R_j f\|_{\mathrm{BMO}_\varphi} \leqq C\|f\|_{\Lambda_\varphi}.$$

(This corollary to Peetre's theorem can also be proved by partitioning f and directly estimating the integrals.) This gives the first half of Theorem 1.

Now assume that $f\in\mathrm{BMO}_\varphi$ and $\|f\|_{\mathrm{BMO}_\varphi}=1$. All signs of equality between functions in BMO or BMO_φ are interpreted modulo constants.

Choose r_i such that $\varphi(r_i)=2^i\varphi(r_j)$ for some r_0 and every integer i such that this is possible, say $i\in[-L, M]$ when L and M are finite or infinite.

Lemma 4 shows that $\|f-\psi_{r_i}*f\|_{\mathrm{BMO}}\leqq C\varphi(r_i)$. Thus

$$\|\psi_{r_i}*f-\psi_{r_{i+1}}*f\|_{\mathrm{BMO}} \leqq C\varphi(r_i)+C\varphi(r_{i+1}) = C\varphi(r_i) \quad (-L\leqq i<M).$$

Thus there exist functions u_j^i such that

$$\|u_j^i\|_{L^\infty}\leqq C\varphi(r_i) \quad \text{and} \quad \psi_{r_i}*f-\psi_{r_{i+1}}*f=\sum_0^d R_j u_j^i$$

(Fefferman [3]). We introduce

$$v_j^i=(\psi_{r_i}+\psi_{r_{i+1}})*u_j^i\in C(\mathbf{R}^d) \quad \text{and} \quad w_j^i=v_j^i-v_j^i(0)$$

$$\omega(w_j^i, r)=\omega(v_j^i, r)\leqq C\left(\frac{r}{r_i}+\frac{r}{r_{i+1}}\right)\|u_j^i\|_{L^\infty}\leqq Cr\frac{\varphi(r_i)}{r_i}.$$

We also have

$$\omega(w_j^i, r)=\omega(v_j^i, r)\leqq 2\|v_j^i\|_C\leqq 4\|u_j^i\|_{L^\infty}\leqq C\varphi(r_i).$$

Since $w_j^i(0)=0$, $|w_j^i(x)|\leqq\omega(w_j^i, |x|)$.

$$\sum_0^d R_j w_j^i=\sum_0^d R_j v_j^i=\sum(\psi_{r_i}+\psi_{r_{i+1}})*R_j u_j^i=(\psi_{r_i}+\psi_{r_{i+1}})*(\psi_{r_i}-\psi_{r_{i+1}})*f=$$
$$=\psi_{r_i}*\psi_{r_i}*f-\psi_{r_{i+1}}*\psi_{r_{i+1}}*f.$$

We have

$$\sum_i\omega(w_j^i, r)\leqq\sum_{r_i\leqq r}C\varphi(r_i)+\sum_{r_i>r}Cr\frac{\varphi(r_i)}{r_i}\leqq$$
$$\leqq C\varphi(r)+Cr\int_r^\infty\frac{\varphi(t)}{t^2}\,dt\leqq C\varphi(r),$$

since

$$\sum_k^m\frac{\varphi(r_i)}{r_i}=2\sum_k^m\frac{\varphi(r_i)-\varphi(r_{i-1})}{r_i}=$$
$$=2\sum_k^{m-1}\left(\frac{\varphi(r_i)}{r_i}-\frac{\varphi(r_i)}{r_{i+1}}\right)+2\frac{\varphi(r_m)}{r_m}-\frac{2\varphi(r_{k-1})}{r_k}\leqq$$
$$\leqq 2\sum_k^{m-1}\varphi(r_i)\int_{r_i}^{r_{i+1}}\frac{dt}{t^2}+2\varphi(r_m)\int_{r_m}^\infty\frac{dt}{t^2}\leqq 2\int_{r_k}^\infty\frac{\varphi(t)}{t^2}\,dt.$$

Consequently $\sum_i w_j^i$ converges absolutely to a continuous function g_j with $\omega(g_j, r)\leqq\leqq C\varphi(r)$. Thus $\|g_j\|_{\Lambda_\varphi}\leqq C$.

Let η be a function in C_0^∞ with $\int\eta=0$. Then $(\eta, R_jg_j)=(\check{R}_j\eta, g_j)=\sum_i(\check{R}_j\eta, w_j^i)==\sum_i(\eta, R_jw_j^i)$ as can be easily verified. Thus

$$(\eta, \sum_0^d R_jg_j)=\sum_i(\eta, \sum_0^d R_jw_j^i)=$$
$$=\sum_i\big((\eta, \psi_{r_i}*\psi_{r_i}*f)-(\eta, \psi_{r_{i+1}}*\psi_{r_{i+1}}*f)\big)=$$
$$=\lim_{\to -L}(\eta, \psi_{r_i}*\psi_{r_i}*f)-\lim_{i\to M}(\eta, \psi_{r_i}*\psi_r*f).$$

We have to treat this in different ways depending on the behavior of $\varphi(r)$ when $r\to 0$ and $r\to\infty$.

Case 1. If $L=\infty$, then $r_i\to 0$ when $i\to -L$. Thus

$$\lim_{i\to -L}(\eta,\psi_{r_i}*\psi_{r_i}*f)=\lim_{i\to -L}(\psi_{r_i}*\psi_{r_i}*\eta,f)=(\eta,f).$$

Take $g_j^1\equiv 0$.

Case 2. If $L<\infty$, then $\varphi(r)>\frac{1}{2}\varphi(r_{-L})$ for every r.

$$\|f-\psi_{r_{-L}}*\psi_{r_{-L}}*f\|_{\mathrm{BMO}}\leqq 2\|f-\psi_{r_{-L}}*f\|_{\mathrm{BMO}}\leqq C\varphi(r_{-L}).$$

Thus we have

$$f-\psi_{r_{-L}}*\psi_{r_{-L}}*f=\sum R_j g_j^1 \quad\text{with}\quad \|g_j^1\|_{L^\infty}\leqq C\varphi(r_{-L}).$$

$$\varphi(g_j^1,r)\leqq 2\|g_j^1\|_{L^\infty}\leqq C\varphi(r_{-L})\leqq C\varphi(r)\quad\text{and thus}\quad \|g_j^1\|_{\Lambda_\varphi}\leqq C.$$

$$\lim_{i\to -L}(\eta,\psi_{r_i}*\psi_{r_i}*f)=(\eta,f)-(\eta,\sum R_j g_j^1).$$

And similarly at infinity

Case 1. If $M=\infty$, then $r_i\to\infty$ when $i\to M$. Suppose that $\operatorname{supp}\eta\subset I(0,r_i)$. Then we have

$$(\eta,\psi_{r_i}*\psi_{r_i}*f)=(\psi_{r_i}*\psi_{r_i}*\eta,f)=$$
$$=\int_{I(0,3r_i)}(\psi_{r_i}*\psi_{r_i}*\eta(x))(f(x)-f(I(0,3r_i)))\,dx\leqq$$
$$\leqq \sup|\psi_{r_i}*\psi_{r_i}*\eta|(3r_i)^d\Omega(f,3r_i)\leqq Cr_i^{-d-1}r_i^d\varphi(r_i)\to 0,\quad i\to\infty.$$

Take $g_j^2\equiv 0$.

Case 2. If $M<\infty$, then $\varphi(r)<2\varphi(r_M)$. Thus $\|f\|_{\mathrm{BMO}}\leqq 2\varphi(r_M)$ and

$$f=\textstyle\sum_0^d R_j h_j,\quad \|h_j\|_{L^\infty}\leqq C\varphi(r_M).$$

We write $g_j^2=\psi_{r_M}*\psi_{r_M}*h_j$.

$$\omega(g_j^2,r)\leqq C\frac{r}{r_M}\|h_j\|_{L^\infty}\leqq Cr\frac{\varphi(r_M)}{r_M}.$$

For $r<r_M$ Lemma 6 gives $\omega(g_j^2,r)\leqq C\varphi(r)$. For $r>r_M$ we use

$$\omega(g_j^2,r)\leqq 2\|g_j^2\|_{L^\infty}\leqq C\varphi(r_M)\leqq C\varphi(r).$$

Thus $\|g_j^2\|_{\Lambda_\varphi}\leqq C$.

Consequently we have, for any φ,

$$(\eta,\textstyle\sum_0^d R_j g_j)=(\eta,f)-(\eta,\sum_0^d R_j g_j^1)-(\eta,\sum_0^d R_j g_j^2).$$

Thus $f=\sum_0^d R_j(g_j+g_j^1+g_j^2)$. Q.E.D.

4. Pointwise multipliers

Given two spaces B_1 and B_2 of functions on some set X, the pointwise multipliers from B_1 to B_2 are defined as the functions g such that $f\to fg$ is a continuous operator from B_1 into B_2. In many cases, e.g. for the Banach spaces treated here, it is sufficient that $f\in B_1\Rightarrow fg\in B_2$; the continuity will then follow by the closed-graph theorem. Note also that the following proof shows that the pointwise multipliers from the subspace of BMO_φ consisting of continuous functions to BMO_φ are in fact pointwise multipliers for BMO_φ (i.e. from BMO_φ to itself). This will be used in the proof of Theorem 3. $\mathbf{T}^d$ is identified with $I(0,1)\subset\mathbf{R}^d$.

5. Proof of Theorem 2

First we see that if $r<\frac{1}{2}$, then by Lemma 3

$$|f(I(x,r))| \leqq \left|f\left(I\left(x,\tfrac{1}{2}\right)\right)\right|+C\int_r^1 \frac{\varrho(f,t)}{t}\,dt \leqq$$

$$\leqq 2^d\|f\|_{L^1}+C\|f\|_{\mathrm{BMO}_\varphi}\int_r^1\frac{\varphi(t)}{t}\,dt \leqq C\|f\|_{\mathrm{BMO}_\varphi}\int_r^1\frac{\varphi(t)}{t}\,dt.$$

Now, suppose that $g\in\mathrm{BMO}_\psi\cap L^\infty$ and $f\in\mathrm{BMO}_\varphi$. Let $I=I(x_0,r)$ be any cube with $r<\frac{1}{2}$. Then

$$r^{-d}\int_I|fg(x)-f(I)g(I)|\,dx \leqq$$

$$\leqq r^{-d}\int_I|g(x)||f(x)-f(I)|\,dx+r^{-d}\int_I|f(I)||g(x)-g(I)|\,dx \leqq$$

$$\leqq \|g\|_{L^\infty}\Omega(f,I)+|f(I)|\,\Omega(g,I) \leqq$$

$$\leqq \varphi(r)\|f\|_{\mathrm{BMO}_\varphi}\|g\|_{L^\infty}+C\|f\|_{\mathrm{BMO}_\varphi}\int_r^1\frac{\varphi(t)}{t}\,dt\,\psi(r)\|g\|_{\mathrm{BMO}_\psi} \leqq$$

$$\leqq \varphi(r)\|f\|_{\mathrm{BMO}_\varphi}(\|g\|_{L^\infty}+C\|g\|_{\mathrm{BMO}_\psi}).$$

Thus, $fg\in\mathrm{BMO}_\varphi$.

Conversely, suppose that g is a pointwise multiplier from the subspace of continuous functions in BMO_φ to BMO_φ. Again, let $I=I(x_0,r)$ be any cube with $r<\frac{1}{2}$. Let f be the function defined in Lemma 5 and let $h(x)$ be $\sup\left(f(x-x_0),\ \int_r^1\varphi(t)t^{-1}dt\right)$. h will be continuous and $\|h\|_{\mathrm{BMO}_\varphi}\leqq C\|f\|_{\mathrm{BMO}_\varphi}$ by Lemma 2. Thus $gh\in\mathrm{BMO}_\varphi$ and $\|gh\|_{\mathrm{BMO}_\varphi}\leqq C$ independently of I. This gives

$$r^{-d}\int_I|gh(x)|\,dx \leqq \Omega(gh,I)+|gh(I)| \leqq$$

$$\leqq \|gh\|_{\mathrm{BMO}_\varphi}\varphi(r)+C\|gh\|_{\mathrm{BMO}_\varphi}\int_r^1\frac{\varphi(t)}{t}\,dt \leqq C\int_r^1\frac{\varphi(t)}{t}\,dt.$$

But $h(x)=\int_r^1\varphi(t)t^{-1}dt$, $x\in I$. Consequently $r^{-d}\int_I|g(x)|\,dx\leqq C$ and $g\in L^\infty$.

Take the same I and h. Then, since h is constant on I

$$\Omega(gh, I) = \int_r^1 \frac{\varphi(t)}{t}\, dt\, \Omega(g, I).$$

Thus

$$\Omega(g, I) \leqq \frac{C\varphi(r)}{\int_r^1 \frac{\varphi(t)}{t}\, dt} = C\psi(r) \quad \text{and} \quad g \in \mathrm{BMO}_\psi.$$

H^1 is the space of functions belonging to L^1 together with their Riesz transform. (Note that this is not the same $H^1(\mathbf{T})$ as the classical space of boundary values of analytic functions.) Fefferman [3] has shown that BMO is the dual space to H^1. The duality is given by $(f, g) = \int fg$, e.g. when f is a trigonometrical polynomial and g belongs to BMO. By continuity this also holds when $f \in H^1$ and $g \in L^\infty \subset \mathrm{BMO}$.

We will now show that a certain subspace of BMO is a predual of H^1.

Lemma 7. *The following two conditions on $f \in \mathrm{BMO}(\mathbf{T}^d)$ are equivalent:*

(i) $\varrho(f, r) \to 0, \quad r \to 0$

(ii) $f = f_0 + \sum_1^d R_j f_j, \quad f_j \in C(\mathbf{T}^d)$.

Proof. Suppose that $\varrho(f, r) \to 0$, $r \to 0$. We would like to use Theorem 1 with $\varphi(r) = \varrho(f, r)$, but this does not necessarily satisfy $(*)$. However, it is possible to construct a function φ_1 satisfying $(*)$ such that $\varrho(f, r) \leqq \varphi_1(r)$ and $\varphi_1(r) \to 0$, $r \to 0$. Thus, there exist $f_j \in \Lambda_{\varphi_1} \subset C$ such that $f = f_0 + \sum R_j f_j$. The converse is proved by the same method.

CMO is defined to be the set of functions in BMO satisfying the conditions in Lemma 7. The first condition shows that this is a closed subspace of BMO. Also, it shows that the Fejér sums of a function in CMO will converge in norm ([6]). Thus, trigonometrical polynomials are dense in CMO and CMO is the closure of trigonometrical polynomials (or equivalently continuous functions) in BMO.

Theorem 4. *H^1 is the dual space of* CMO. *The duality is given by $\int fg$, e.g. if $f \in C \subset \mathrm{CMO}$ and $g \in H^1$.*

Proof. If $f \in C$ and $g \in H^1$, then as stated above

$$\left|\int fg\right| \leqq C\|f\|_{\mathrm{BMO}}\|g\|_{H^1} = C\|f\|_{\mathrm{CMO}}\|g\|_{H^1}.$$

Conversely, suppose that $\chi \in \mathrm{CMO}^*$. $R_j: C \to \mathrm{CMO}$ are bounded $j = 0, \ldots, d$, and thus $R_j^* \chi \in C^* = M$. Consequently χ is a measure whose Riesz transforms are measures and by the F. and M. Riesz theorem it is a function in H^1.

Remark. Sarason [11], [12] denotes the space $\{f \in \mathrm{BMO};\ \varrho(f, r) \to 0, r \to 0\}$ by VMO. In the case of functions on $\mathbf{R}^d$, VMO is strictly larger than

$\mathrm{CMO}=\{f;\ f=f_0+\sum R_j f_j;\ f\in C_0\}$, proved to be a predual of H^1 by Neri [8]. CMO is the closure of C_0 in BMO.

Finally we give the proof of Theorem 3.

Suppose that g is a pointwise multiplier for BMO and $h\in H^1$. Let $f\in C$. Then $fg\in L^\infty$. Thus $\left|\int ghf\right|=\left|\int hfg\right|\leqq C\|h\|_{H^1}\|fg\|_{\mathrm{BMO}}\leqq C\|h\|_{H^1}\|f\|_{\mathrm{BMO}}$ and gh gives a bounded linear functional on the dense subspace C of CMO. Thus $gh\in H^1$ by Theorem 4.

Conversely, suppose that g is a pointwise multiplier for H^1 and $f\in C$. Let $h\in H^1$. Then $\left|\int fgh\right|\leqq C\|f\|_{\mathrm{BMO}}\|gh\|_{H^1}\leqq C\|f\|_{\mathrm{BMO}}\|h\|_{H^1}$. Thus $fg\in\mathrm{BMO}$.

Consequently, H^1 and BMO have the same pointwise multipliers, i.e. $\mathrm{BMO}_{|\log t|^{-1}}\cap L^\infty$.

Lemmas 5 and 2 show that $\sin\log|\log x_1|$ gives an example of a pointwise multiplier for H^1 that is not continuous.

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