Local surjectivity in C^∞ for a class of pseudo-differential operators

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Let X be a C^∞ paracompact n-dimensional manifold and P a properly supported pseudo-differential operator on X, having a principal symbol p homogeneous of degree m. P is said to be locally solvable at $x_0 \in X$ if there is a neighbourhood V of x_0 such that for all $f \in C^\infty(X)$, there exists $u \in \mathscr{D}'(X)$ satisfying $Pu=f$ in V. P is said to be of principal type if for all $(x, \xi) \in T^*X \setminus 0$, one has $\partial p/\partial \xi \neq 0$. For operators of principal type, Beals and Fefferman [1] proved that the so-called condition $(\mathscr{P})$ is sufficient for local solvability. On the other hand, Egorov [1] proved local solvability for a class of pseudo-differential operators of principal type which need not satisfy condition $(\mathscr{P})$. Egorov's result is the following. Let U be an open set of X and suppose that each point of $p^{-1}(0)$ above U has an open conic neighbourhood Γ in $T^*X \setminus 0$ in which one of the following conditions is fulfilled:

(a) There exists in Γ a C^∞ function μ, homogeneous of degree $(m-1)$, such that $i^{-1}\{\bar{p}, p\} \leqq 2\,\mathrm{Re}\,(\mu p)$ in Γ (here $\{\bar{p}, p\}$ is the Poisson bracket of $\bar{p}$ and p and its expression in local coordinates is

$$\sum_1^n \left(\frac{\partial \bar{p}}{\partial \xi_j} \frac{\partial p}{\partial x_j} - \frac{\partial \bar{p}}{\partial x_j} \frac{\partial p}{\partial \xi_j} \right).$$

(b) For some $z \in \mathbf{C} \setminus \{0\}$, $(\partial/\partial\xi)\,\mathrm{Re}\,(zp) \neq 0$ in Γ and $\mathrm{Im}\,(zp) \geqq 0$ in Γ.

(c) For some $z \in \mathbf{C} \setminus \{0\}$, $(\partial/\partial\xi)\,\mathrm{Re}\,(zp) \neq 0$ in Γ and there exists a conic submanifold Σ of Γ, of codimension 1, such that each null bicharacteristic strip of $\mathrm{Re}\,(zp)$, emanating from a point of Γ, intersects Σ exactly at one point and transversally. Furthermore one assumes that for each null bicharacteristic strip γ of $\mathrm{Re}\,(zp)$ in Γ, one has $\mathrm{Im}\,(zp) \geqq 0$ on γ^- and $\mathrm{Im}\,(zp) \leqq 0$ on γ^+. Here γ^- and γ^+ are defined as follows: Let ϱ_γ be the intersection point of $\gamma \cap \Gamma$ and Σ; if s is the parameter of $\gamma \cap \Gamma$ occuring in the Hamilton—Jacobi equations $dx/ds=$

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$=(\partial/\partial\xi)\,\mathrm{Re}\,(zp)$, $d\xi/ds=-(\partial/\partial x)\,\mathrm{Re}\,(zp)$, one defines γ^- (resp. γ^+) as the part of $\gamma\cap\Gamma$ described for $s<0$ (resp. $s>0$) if ϱ_γ corresponds to $s=0$.

Under those conditions, Egorov proved that for each point $x_0\in U$ and each $s\in\mathbf{R}$, there exists a neighbourhood $W_s(x_0)$ of x_0 such that for each $f\in H_s(X)$, one can find $u\in H_{s+m-1}(X)$ satisfying $Pu=f$ in $W_s(x_0)$.

Unfortunately, W_s could depend on s and consequently Egorov's result does not answer the question of local surjectivity of P in C^∞, i.e. the question is if for each $x_0\in U$ one can find a neighbourhood $V(x_0)$ of x_0 in U such that for all $f\in C^\infty(X)$ there exists $u\in C^\infty(X)$ satisfying $Pu=f$ in V. We will prove that the answer to this question is positive, if we make some hypotheses on the null bicharacteristic strips. More precisely we will prove:

Theorem 1. *Suppose that P is a properly supported pseudo-differential operator with homogeneous principal symbol p of degree m on a C^∞ manifold X. Let U be an open relatively compact subset of X, and suppose that for each m_0 in $T^*X\diagdown 0$ above $\overline{U}$ such that $p(m_0)=0$, one of the following conditions is satisfied:*

(a) *There exists a conic neighbourhood Γ of m_0 in $T^*X\diagdown 0$ and a C^∞ function μ in Γ, homogeneous of degree $(m-1)$, such that $(1/i)\{\bar p, p\}\leqq 2\,\mathrm{Re}\,(\mu p)$ in Γ.*

(b′) *For some $z\in\mathbf{C}\diagdown\{0\}$, the Hamiltonian field $H_{\mathrm{Re}(zp)}$ (given by*

$$\sum\left(\frac{\partial\,\mathrm{Re}\,(zp)}{\partial\xi_j}\frac{\partial}{\partial x_j}-\frac{\partial\,\mathrm{Re}\,(zp)}{\partial x_j}\frac{\partial}{\partial\xi_j}\right)$$

in local coordinates) is not parallel to the cone axis at m_0, the null bicharacteristic strip γ_{m_0} of $\mathrm{Re}\,(zp)$ through m_0 does not stay forever above $\overline{U}$ and if $m_0^\mp$ are the boundary points of the connected component of $\gamma_{m_0}\cap T^*\overline{U}$ containing m_0, one has $\mathrm{Im}\,(zp)\leqq 0$ in a neighbourhood of $[m_0, m_0^+]$ (we assume here that the positive direction goes from m_0^- to m_0^+ on the oriented curve γ_{m_0}).

(c′) *m_0 has a conic neighbourhood Γ in $T^*X\diagdown 0$ in which there exists a conic submanifold Σ codimension 1, passing through m_0, with the following property: there exists $z\in\mathbf{C}\diagdown\{0\}$ such that for all $\tilde m_0\in\Sigma\cap p^{-1}(0)$, $\gamma_{\tilde m_0}$ does not stay forever above $\overline{U}$, intersects Σ only at $\tilde m_0$ and transversally, and such that for all $\tilde m_0\in\Sigma\cap p^{-1}(0)$, $\mathrm{Im}\,(zp)\leqq 0$ in neighbourhood of $]\tilde m_0, \tilde m_0^+]$ and $\mathrm{Im}\,(zp)\geqq 0$ in a neighbourhood of $[\tilde m_0^-, \tilde m_0[$. Here $\gamma_{\tilde m_0}$ is the bicharacteristic strip of $\mathrm{Re}\,(zp)$ through $\tilde m_0$, and $\tilde m_0^\mp$ the boundary points of the connected component of $\gamma_{\tilde m_0}\cap T^*\overline{U}$ containing $\tilde m_0$ (such that the positive direction goes from $\tilde m_0^-$ to $\tilde m_0^+$ on the oriented curve $\gamma_{\tilde m_0}$).*

Then if U is sufficiently small and $f\in C^\infty(X)$, there exists $u\in C^\infty(X)$ such that $Pu=f$ in a neighbourhood of $\overline{U}$. (In fact the same result holds if we replace the complex number z occurring in (b′) or (c′) by some elliptic symbol, homogeneous of degree 0).

Remark. For operators satisfying (a) everywhere in $T^*\overline{U}$, the result of Theorem 1 has been proved by Hörmander [2]. In [3] (Theorem 3.5.1) he has also proved propagation of singularities result for pseudo-differential operators P satisfying $\operatorname{Im} p \geqq 0$ in a conic neighbourhood of an arc of bicharacteristic strip of $\operatorname{Re} p$.

The main part of the proof of Theorem 1 will be the following result of propagation of singularities.

Theorem 2. *Let P be a properly supported pseudo-differential operator with principal symbol p homogeneous of degree m. Suppose that $m_0 \in T^*X \setminus 0$, $p(m_0)=0$ and that m_0 has a conic neighbourhood Γ in $T^*X \setminus 0$ in which there exists a conic submanifold Σ of codimension 1, passing through m_0 and transversal to $H_{\operatorname{Re} p}$ at m_0. We assume that Γ is so small that for each point $\tilde{m}_0$ in Σ, the bicharacteristic strip $\gamma_{\tilde{m}_0}$ of $\operatorname{Re} p$ through $\tilde{m}_0$ intersects Σ exactly once and transversally. If s is the parameter of $\gamma_{\tilde{m}_0} \cap \Gamma$ occurring in the Hamilton—Jacobi equations*

$$\frac{dx}{ds} = \frac{\partial \operatorname{Re} p}{\partial \xi}; \quad \frac{d\xi}{ds} = -\frac{\partial \operatorname{Re} p}{\partial x},$$

define $\gamma_{\tilde{m}_0}^-$ (resp. $\gamma_{\tilde{m}_0}^+$) as the part of $\gamma_{\tilde{m}_0} \cap \Gamma$ described for $s<0$ (resp. $s>0$) if $\tilde{m}_0$ corresponds to $s=0$. Assume that for each $\tilde{m}_0 \in \Sigma$, $\operatorname{Im} p \leqq 0$ in a neighbourhood of $\gamma^-(\tilde{m}_0)$ and $\operatorname{Im} p \geqq 0$ in a neighbourhood of $\gamma_{\tilde{m}_0}^+$. Let now $m_1 \in \gamma_{m_0}^-$, $m_2 \in \gamma_{m_0}^+$. Then if $u \in \mathscr{D}'(X)$, $Pu \in H_s$ on $[m_1, m_2]$ and $u \in H_{s+m-1}$ at m_1 and m_2, one has $u \in H_{s+m-1}$ on $[m_1, m_2]$.

The proof of Theorem 2 will be a modification of the proof of Theorem 3.5.1 of Hörmander [3], which tells us already that $u \in H_{s+m-1}$ on $[m_1, m_2] \setminus \{m_0\}$.

By multiplying P by a real elliptic symbol and using Fourier integral operators associated to a suitable canonical transformation as in Duistermaat—Hörmander [1], we may as well assume that P has order 1, and that its principal symbol is

$$p(x, t, \xi, \tau) = \tau + iq(x, t, \xi, \tau) \tag{$*$}$$

in a conic neighbourhood of $m_0 = (x_0, t_0, \xi_0, \tau_0)$. Here we are working in local coordinates (x, t) of X, $x \in \mathbf{R}^{n-1}$, $t \in \mathbf{R}$, and (ξ, τ) are the dual variables of (x, t); and q is a real-valued function.

In the proof of Theorem 2, it will be convenient to use the following lemma.

Lemma 1. *Let P satisfy the hypotheses of Theorem 2 and be of the form $(*)$. Then there exists a real function ϱ, homogeneous of degree 0, and a conic neighbourhood V of m_0, such that in V we have*

(1) $\varrho^{-1}(0) = \Sigma$,

(2) $H_{\operatorname{Re} p}\varrho > 0$,

(3) $\sum_{j=1}^{n-1} \frac{\partial^2 \varrho}{\partial x_j \partial \xi_j} + \frac{\partial^2 \varrho}{\partial t \partial \tau} = 0$ *on* Σ,

(4) $q = \varrho r$ *for some function r which is $\geqq 0$ and homogeneous of degree 1 everywhere.*

Proof of Lemma 1. The existence of a function satisfying (1), (2), (4) follows from Taylor's formula. Let $\tilde{\varrho}$ satisfy (1), (2), (4). Then if f is homogeneous of degree 0, and >0 at m_0, $f\tilde{\varrho}$ will satisfy (1), (2), (4) and for (3) to be valid it is sufficient that f satisfies

$$\sum_{j=1}^{n-1}\frac{\partial\tilde{\varrho}}{\partial x_j}\frac{\partial f}{\partial \xi_j}+\sum_{j=1}^{n-1}\frac{\partial\tilde{\varrho}}{\partial \xi_j}\frac{\partial f}{\partial x_j}+\frac{\partial\tilde{\varrho}}{\partial t}\frac{\partial f}{\partial \tau}+\frac{\partial\tilde{\varrho}}{\partial \tau}\frac{\partial f}{\partial t}+f\left(\sum_{j=1}^{n-1}\frac{\partial^2\tilde{\varrho}}{\partial\xi_j\partial x_j}+\frac{\partial^2\tilde{\varrho}}{\partial\tau\,\partial t}\right)=0. \tag{5}$$

Since $\tilde{\varrho}$ satisfies (2), the vector field $(\tilde{\varrho}'_\xi, \tilde{\varrho}'_\tau, \tilde{\varrho}'_x, \tilde{\varrho}'_t)$ is not parallel to the cone axis at m_0 and we can find a solution of (5) which is homogeneous of degree 0, and >0 at m_0. Q.E.D.

Denote by ϱ a function as in Lemma 1 and define $\Gamma_\varepsilon=\{m\in\Gamma;\ |\varrho(m)|\leqq\varepsilon\}$, where Γ is the set occuring in the statement of Theorem 2, and which we suppose to be so small that ϱ satisfies conditions (1), (2), (3), (4) of Lemma 1 in some neighbourhood of it. Let $M\subset S^{s-1}(\mathbf{R}^n\times\mathbf{R}^n)$ be a set of real symbols with support in Γ_ε, bounded in the S^s-topology. The main part of the proof of Theorem 2 will be the following result, in which we assume that $\Gamma\subset T^*\mathbf{R}^n\setminus 0$.

Lemma 2. *Let $u\in\mathscr{E}'(\mathbf{R}^n)$ be such that $u\in H_{s-1/2}$ in Γ and $Pu\in H_s$ in Γ. If $c\in M$, define C as the pseudo-differential operator with symbol c and $\mathscr{L}_\eta(c)$ as the pseudo-differential operator with symbol*

$$\frac{1}{2}\varrho\frac{\partial c^2}{\partial t}+\frac{1}{2}\left(\frac{\partial\varrho}{\partial t}-6\eta\right)c^2.$$

Then for each $\eta>0$, there exists $\varepsilon>0$, independent of s, such that $\operatorname{Re}(\mathscr{L}_\eta(c)u, u)\leqq \leqq K_1(\eta)\|CPu\|_0^2+K_2(\eta)$. *Here $K_1(\eta)$ and $K_2(\eta)$ are uniform constants when c runs over M.*

Proof of Lemma 2. By K we will denote various positive constants, which may depend on u, but which are valid uniformly when c runs over M. We write $P=A+iB$, with $A=2^{-1}(P+P^*)$ and $B=(2i)^{-1}(P-P^*)$, where P^* is the adjoint of P with respect to the scalar product $(f, g)=\int f\bar{g}$. We shall denote by b the principal symbol of B, and by S a properly supported pseudo-differential operator with symbol equal to ϱ (except perhaps when $|\xi|+|\tau|$ is very small). Taking the imaginary part of both members of the identity

$(SCPu, Cu)=(ASCu, Cu)+i(BSCu, Cu)+([SC, A]u, Cu)+i([SC, B]u, Cu),$ we get

$$\operatorname{Im}(SCPu, Cu)=(1)+(2)+(3)+(4),\quad \text{where}$$

$$(1)=\frac{(A(S-S^*)Cu, Cu)}{2i}+\frac{(Cu, [S, A]Cu)}{2i}$$

$(2) = \mathrm{Re}\,(BSCu, Cu)$

$(3) = \mathrm{Im}\,([SC, A]u, Cu)$

$(4) = \mathrm{Re}\,([SC, B]u, Cu)$.

About the term (1).

If Γ is sufficiently small, then in view of Lemma 1, $(2i)^{-1}(S-S^*)$ has symbol $\sim \varrho\lambda_{-1}+l_{-2}$ in a neighbourhood of Γ (the subscripts indicate the orders of the symbols). Furthermore in a neighbourhood of Γ, $[S, A]$ has symbol $\sim -i^{-1}\partial\varrho/\partial t+ +l_{-1}$. So (1) $\geqq \mathrm{Re}\,(ASA_{-1}Cu, Cu)-\mathrm{Re}\,(Cu, 2^{-1}S_t Cu)-K$, where A_{-1} and S_t are pseudo-differential operators with principal symbols λ_{-1} and $\partial\varrho/\partial t$ respectively.

But we can find properly supported pseudo-differential operators S_j with principal symbols ϱ_j homogeneous of degree 0, $j=1, 2$ such that $S=S_1+S_2$, $|\varrho_1|\leqq 2\varepsilon$ and $\mathrm{WF}(S_2)\cap\Gamma_\varepsilon=\emptyset$. Furthermore, if $H(x, t, D_x, D_t)$ is a properly supported pseudo-differential operator with principal symbol h homogeneous of degree 0, such that $h=0$ except when (x, t) belongs to some compact set, we have:

For each compact set $T\subset\mathbf{R}^n$, there exists a constant C_T such that $\|H\varphi\|_0^2\leqq \leqq \max|h(x, t, \xi, \tau)|^2\|\varphi\|_0^2+C_T\|\varphi\|_{-1/2}^2$ when $\varphi\in L^2_{\mathrm{comp}}(T)$. Combining those 2 facts, we find that:

$$\begin{aligned}\|SD_tA_{-1}Cu\|_0^2 &\leqq 2\|S_1D_tA_{-1}Cu\|_0^2+2\|S_2D_tA_{-1}Cu\|_0^2\\ &\leqq 2(4\varepsilon^2\|D_tA_{-1}Cu\|_0^2+K\|D_tA_{-1}Cu\|_{-1/2}^2+K)\\ &\leqq 8\varepsilon^2K\|Cu\|_0^2+K\|Cu\|_{-1/2}^2+K.\end{aligned}$$

So for each $\omega>0$ we have:

$$|(SD_tA_{-1}Cu, Cu)| \leqq \frac{1}{\omega}\|SD_tA_{-1}Cu\|_0^2+\omega\|Cu\|_0^2 \leqq \frac{8\varepsilon^2K}{\omega}\|Cu\|_0^2+\frac{K}{\omega}+\omega\|Cu\|_0^2.$$

Now if we take $\omega=\varepsilon\leqq(8K+1)^{-1}\eta$, we obtain

$$\mathrm{Re}\,(SD_tA_{-1}Cu, Cu) \geqq -\eta\|Cu\|_0^2-K(\eta),$$

which implies

$$(1) \geqq -\eta\|Cu\|_0^2-\mathrm{Re}(Cu, \tfrac{1}{2}S_tCu)-K(\eta). \qquad (1')$$

About the term (2).

If Γ is small enough, B has symbol $\sim\varrho r+l_0$ in a neighbourhood of Γ, where l_0 has order 0. Denoting by R a properly supported pseudo-differential operator with principal symbol r, we find that there exists a properly supported first order pseudo-differential operator L with principal symbol vanishing in a neighbourhood of Γ such that

$$\begin{aligned}(BSCu, Cu) &= (SRSCu, Cu)+(LSCu, Cu)\\ &= ((S-S^*)RSCu, Cu)+(RSCu, SCu)+(LSCu, Cu).\end{aligned}$$

But we have

$$\left|(((S-S^*)R+L)SCu, Cu)\right| = \left|(SCu, ((S-S^*)R+L)^*Cu)\right| \leqq$$
$$\leqq \left|(S_1Cu, ((S-S^*)R+L)^*Cu)\right| + K,$$

where S_1 is the same as in (1). Now the same argument as in (1) gives:

$$\operatorname{Re}\left((S-S^*)RSCu, Cu\right) + \operatorname{Re}(LSCu, Cu) \geqq -\eta\,\|Cu\|_0^2 - K(\eta),$$

if ε is small enough.

Furthermore the sharp Gårding's inequality gives that

$$\operatorname{Re}(RSCu, SCu) \geqq -K\|SCu\|_0^2 \geqq -4\varepsilon^2 K\|Cu\|_0^2 - K.$$

So if ε is small enough we have

$$(2) \geqq -\eta\,\|Cu\|_0^2 - K(\eta). \tag{2'}$$

About the term (3).

$$\operatorname{Im}([SC, A]u, Cu) = \operatorname{Im}(C^*[SC, A]u, u) = (Wu, u)$$

if we define

$$W = \frac{C^*[SC, A] - [SC, A]^*C}{2i}.$$

The principal symbol of W is $\operatorname{Im}(ci^{-1}\{\varrho c, a\}) = -c\{\varrho c, a\} = c(\partial/\partial t)(\varrho c)$. (Here we have denoted by a the function $(x, t, \xi, \tau)\to\tau$.) When c runs over M, the full symbol of W is $c(\partial/\partial t)(\varrho c)$+an error bounded in S^{2s-1}. So

$$\operatorname{Im}([SC, A]u, Cu) \geqq \operatorname{Re}(Zu, u) - K \tag{3'}$$

where Z has principal symbol $2^{-1}(\varrho\,\partial/\partial t + 2\,\partial\varrho/\partial t)c^2$.

About the term (4).

$\operatorname{Re}([SC, B]u, Cu) = \operatorname{Re}(C^*[SC, B]u, u)$. $C^*[SC, B]$ has the purely imaginary principal symbol $i^{-1}c\{\varrho c, b\}$ and its full symbol differs from $i^{-1}c\{\varrho c, b\}$ by an error which is bounded in S^{2s-1}, when c runs over M. So

$$\operatorname{Re}([SC, B]u, Cu) \geqq -K. \tag{4'}$$

Collecting (1′), (2′), (3′), (4′) we get

$$\operatorname{Im}(SCPu, Cu) \geqq -2\eta\,\|Cu\|_0^2 - \operatorname{Re}(Cu, 2^{-1}S_t Cu) + \operatorname{Re}(Zu, u) - K(\eta).$$

But on the other hand we have

$$\operatorname{Im}(SCPu, Cu) \leqq |(SCPu, Cu)| \leqq \frac{1}{\omega}\,\|SCPu\|_0^2 + \omega\,\|Cu\|_0^2 \leqq \frac{M}{\omega}\,\|CPu\|_0^2 + \omega\,\|Cu\|_0^2$$

for all $\omega > 0$.

So if we take $\omega = \eta$, we finally get:

$$\operatorname{Re}\left(\mathcal{L}_\eta(c)u, u\right) \leqq K_1(\eta)\,\|CPu\|_0^2 + K_2(\eta),$$

and this completes the proof of Lemma 2.

Proof of Theorem 2. We may assume that $u\in H_{s-1/2}$ and $Pu\in H_s$ in Γ, and that $u\in\mathscr{E}'$. Using Lemma 2 we will now construct a pseudo-differential operator Θ of order s, elliptic at m_0, such that $\Theta u\in L_2$. Set $k_\eta=2^{-1}(\partial\varrho/\partial t-6\eta)$. If η is small enough, k_η is >0 at m_0. Choose a symbol δ homogeneous of degree s for $|\xi|+|\tau|>>2^{-1}(|\xi_0|+|\tau_0|)$, with support in a very small conic neighbourhood of m_0, such that $(2^{-1}\varrho\partial\delta^2/\partial t+k_\eta\delta^2)(m_0)>0$ and $(2^{-1}\varrho\partial\delta^2/\partial t+k_\eta\delta^2)\geqq 0$ on Σ. (We denote $\xi(m_0)$ and $\tau(m_0)$ by ξ_0 and τ_0 respectively, where (x, t, ξ, τ) are local coordinates near m_0.) All this is possible thanks to the transversality at m_0 of Σ with respect to $\partial/\partial t$. Then choose real symbols θ, g with small support in Γ, homogeneous of degree s for $|\xi|+|\tau|>2^{-1}(|\xi_0|+|\tau_0|)$, such that θ is elliptic at m_0 and $\theta^2\leqq 2^{-1}\varrho\,\partial\delta^2/\partial t+k_\eta\delta^2+g^2$.

If we now define, for $0<\alpha\leqq 1$, $\sigma_\alpha(x, t, \xi, \tau)=\big(1+\alpha^2(|\xi|^2+\tau^2)\big)^{-1/2}$ and if Θ_α and G_α denote pseudo-differential operators with symbols equal to $\sigma_\alpha\theta$, $\sigma_\alpha g$ respectively, an application of the sharp Gårding's inequality to $\mathscr{L}_\eta(\delta\sigma_\alpha)-\Theta_\alpha^2+G_\alpha^2$ combined with use of Lemma 2 gives

$$\|\Theta_\alpha u\|_{(0)}\leqq K \text{ where } K \text{ is independent of } \alpha.$$

So if $\alpha\to 0$, we get $\Theta u\in L_2$, where Θ has symbol θ. This proves the theorem.

Proof of Theorem 1. We denote by P^* the L_2-adjoint of P with respect to some smooth positive density.

If $u\in\mathscr{E}'(\overline{U})$ and $P^*u\in H_s(X)$, one has $u\in H_{s+m-1}$ in the region corresponding to condition (b′), by Theorem 3.5.1 of Hörmander [3], and also in the region corresponding to condition (c′), by Theorem 2. So $\mathrm{WF}_{s+m-1}(u)$ is contained in the region corresponding to condition (a). But now the proof of Proposition 2.2 in Duistermaat [1] shows that $u\in H_{s+m-1}(X)$ because if $\overline{U}$ is small enough, we can as in Chapter VIII of Hörmander [1], find a function φ satisfying the hypotheses of this Proposition 2.2. The conclusion of Theorem 1 now follows by standard results on surjections in Fréchet spaces (cf. e.g. Duistermaat—Hörmander [1], Theorem 6.3.1).

Remarks. (1) In some cases it is possible to prove Theorem 2 by constructing a parametrix, for example if P has the symbol

$$\tau+it|x''|^2\xi_{n-1},\quad |x''|^2=\textstyle\sum_{j=1}^k x_j^2,\quad k\leqq n-2,\quad\text{and}\quad \xi_{n-1}>0,$$

one can construct near each point where $t=0$, a microlocal left parametrix with wave front set contained in the union of the diagonal of $T^*X\setminus 0$, and of the bicharacteristic relation for τ above $x''=0$, and prove propagation for P in the same way as it is done for D_n in Theorem 6.1.1 of Duistermaat—Hörmander [1].

(2) It is easy to construct non-elliptic boundary value problems which are described on the boundary by operators satisfying the hypotheses of Theorems 1 or 2, in view of the reduction theory developed in Hörmander [2].

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