The inverse of vowel articulation

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Introduction. The input of vowel articulation consists of air pulses at the glottis, the vocal tract acts as a filter and the output from the lips is heard as a vowel. To make a simple mathematical model of this, one considers the vocal tract as a tube T specified by a single area function $x \to A(x)$ where x is the distance from the glottis along an axis of the tube and $A(x)$ is the area of a cross-section of the tube at x orthogonal to the axis. The pressure $\bar{p}$, density $\bar{\varrho}$ and velocity v of the air in the tube are assumed to be functions only of the time t and the coordinate x. Further assumptions are that $\bar{p}=p^*+p$ and $\bar{\varrho}=\varrho^*+\varrho$ deviate very little from their mean values p^* and ϱ^* and that v is small. If we introduce the volume velocity $u(x, t)=A(x)\bar{\varrho}(x, t)v(x, t)$, Newton's law gives $u_t+A\bar{p}_x=0$ and conservation of mass $u_x+(A\bar{\varrho})_t=0$ where the indices denote derivatives. Since $p=c^2\varrho$ where c is the velocity of sound, this gives us two equations for the pressure p and the volume velocity u,

$$Ap_x+u_t = 0, \quad Ap_t+u_x = 0 \tag{1}$$

provided we choose our units so that $c=1$. At the same time we can prescribe that $x=0$ at the glottis and $x=1$ at the lips. We note in passing that if $A=A_0$ is constant then the general solution p_0, u_0 of (1) is given by

$$A_0p_0 = f(x+t)+g(x-t), \quad u_0 = -f(x+t)+g(x-t)$$

where f and g are arbitrary. If f is identically zero, we have $A_0p_0=u_0$ and conversely and that characterizes a solution which is outgoing, i.e. a wave travelling from the glottis.

To account for the radiation from the mouth in the simplest possible way we assume that T connects there with a bigger tube T_0 of constant cross-section $A_0>A(1)$ and that, at the lips, $p=p_0$, $u=u_0$ where p_0, u_0 is an outgoing solution in T_0 (see Figure 1). This gives the boundary condition

$$x = 1 \Rightarrow Ap = bu \tag{2}$$

where $b=A(1)/A_0$ is a number between 0 and 1 which we shall call the loss coefficient. Another possibility, closer to physical reality, is to assume that T connects with a conical flange with a wide opening. If its mathematical vertex is at $1-x_0$ (see Figure 1) it turns out that (2) should be replaced by

$$(2') \qquad x = 1 \Rightarrow x_0 A p_t + A p = x_0 u_t.$$

This radiation condition is close to one used in phonetics and based on three-dimensional radiation from a circular disk in a wall (Flanagan [3] p. 61). In order not to complicate the mathematics too much we shall stick to the simplest case (2).

Eliminating p or u from (1) one gets the Webster horn equations

$$(3) \qquad (Ap_x)_x - Ap_{tt} = 0 \quad \text{and} \quad (A^{-1}u_x)_x - A^{-1}u_{tt} = 0.$$

For simplicity we shall call (1) the Webster system. By the elementary theory of linear hyperbolic second order equations in two variables, the Webster system has unique solutions under a variety of boundary conditions. A forward (backward) solution is one which vanishes for large negative (positive) time. We shall restrict ourselves to infinitely differentiable area functions.

Let p, u be the forward solution of (1), (2) with given $u(0, t)$, necessarily vanishing for large negative t. The linear map from the function $u(0, t)$ to the function $u(1, t)$, conveniently denoted by $u(0, t) \to u(1, t)$, is called the vowel transfer. Physically, the vowel transfer represents the action at the lips of an input volume velocity at the glottis. It turns out that the impulse response $\delta(t) \to f(t)$ of the vowel transfer has a Fourier—Laplace transform

$$F(\omega) = \int e^{-2\pi i \omega t} f(t)\, dt$$

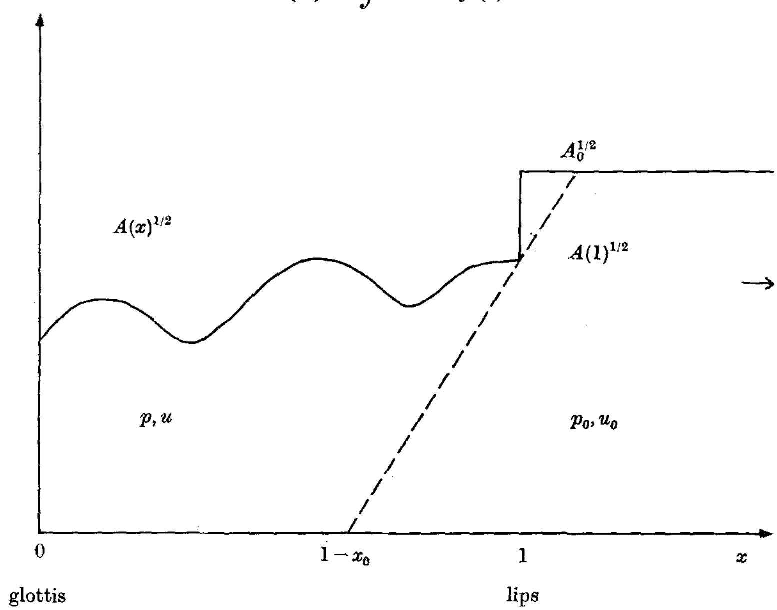

Figure 1. The area functions of the tubes T and T_0.

which is meromorphic with poles ω_n, $-\bar{\omega}_n$ in the upper complex half-plane such that $0 \leqq \operatorname{Re} \omega_1 \leqq \operatorname{Re} \omega_2 \leqq \dots$ and $\operatorname{Im} \omega_n > 0$ for all $n = 1, 2, \dots$. We shall call $\omega_1, \dots$ the vowel resonances. In general one has of course $0 < \operatorname{Re} \omega_1 < \dots$. For the straight tube with a constant area function, $\omega_n = n/2 - 1/4$. In phonetics, the numbers $\operatorname{Re} \omega_1$, $\operatorname{Re} \omega_2, \dots$ and $\operatorname{Im} \omega_1$, $\operatorname{Im} \omega_2, \dots$ are called, respectively, the formants of our idealized vowel and their bandwidths. The first three or four formants can be measured also for actual vowels and are widely used as vowel characteristics. There are also numerical schemes how to compute them from the form of the vocal tract (see Fant [2] and Flanagan [3] and the literature quoted there). One much discussed problem is the possibility of reversing the procedure, namely to compute the area function when the vowel resonances are known. At least one numerical scheme for this already exists (Wakita [8]). A main difficulty is measuring the bandwidths and taking various physical features into account, e.g. losses through vibrations of the walls of the vocal tract.

In our mathematical model, the vowel resonances turn out to be essentially arbitrary except for a regular asymptotic behaviour. We shall prove

Theorem. *The vowel resonances* $\omega_1, \omega_2, \dots$ *of a tube with an indefinitely differentiable area function* $A(x)$ *and loss coefficient* $b > 0$ *have the property that* $\operatorname{Im} \omega_n > 0$, $\operatorname{Re} \omega_n \geqq 0$ *for all* n *and if they area labelled so that* $\operatorname{Re} \omega_n$ *does not decrease with* n, *there is an asymptotic expansion*

$$\omega_n \sim 2^{-1} n - 4^{-1} + ic + c_1 n^{-1} + c_2 n^{-2} + \dots$$

for large n *where* $4\pi c = \log (1+b)/(1-b) > 0$. *Conversely, given such numbers, they are the vowel resonances of a tube with loss coefficient* $b = \tanh 2\pi c$ *and an infinitely differentiable area function* $A(x)$, *unique when normalized so that* $A(1) = 1$.

When $b = 0$, the tube is loss-free and the vowel resonances are real. As explained below, they are then not sufficient to reconstruct the tube. Generally speaking, losses move the resonances into the complex plane giving them one more degree of freedom and more informative value.

With an appropriate asymptotic expansion of the vowel resonances, the theorem is certainly true also for more sophisticated radiation conditions like (2′) but the proof will then be more delicate.

The proof of the theorem uses the fact that the vowel resonances are identical with the zeros of the function $\omega \to P(1, \omega) - bU(1, \omega)$ where

$$P(x, \omega) = \int e^{-2\pi i \omega t} p(x, t)\, dt, \quad U(x, \omega) = \int e^{-2\pi i \omega t} u(x, t)\, dt$$

are the Fourier—Laplace transforms of the glottis reflection pulse, i.e. the solution p, u of the Webster system such that $p(0, t) = \delta(t)$, $u(0, t) = 0$. The functions

P and U are entire analytic of exponential type and for $x=1$ given by canonical products

$$P(1,\omega)=\prod_1^\infty(1-\omega^2/\alpha_n^2),\quad U(1,\omega)=-\bar{A}2\pi i\omega\prod_1^\infty(1-\omega^2/\beta_n^2)$$

where $\bar{A}=\int_0^1 A(x)dx$ is the mean area and $0<\alpha_1<\beta_1<\alpha_2<\dots$ with asymptotic expansions

$$\alpha_n\sim 2^{-1}n-4^{-1}+a_1n^{-1}+a_2n^{-2}+\dots,\quad \beta_n\sim 2^{-1}n+b_1n^{-1}+b_2n^{-2}+\dots.$$

These numbers are the eigenvalues of the Webster system under the boundary conditions $u(0,t)=0$, $u_x(1,t)=0$ and $u(0,t)=0$, $u(1,t)=0$ respectively and will be called the pure resonances and antiresonances of the tube. They are determined by the vowel resonances. On the other hand, it has been known since the work of Borg [1] that the resonances and antiresonances determine the area function and explicit formulas are available in the work of Gelfand and Lewitan [4] and M. Krein [5]. These papers deal with Sturm—Liouville operators with an unknown potential but are easily adapted to the Webster system. Following Sondhi and Gopinath [7] we shall get the area function from the lip transfer $u(1,t)\to p(1,t)$ for forward solutions with $u(0,t)=0$. Its impulse response $\delta(t)\to -A(1)h(t)$ has the property that for $t<2$, $h(t)=\delta(t)+g(t)$ where g vanishes for $t<0$ and is an infinitely differentiable functions when $0\leqq t\leqq 2$. The integral equation

$$\int_{-y}^{y}(h(s-t)+h(t-s))w(y,s)\,ds=2$$

has a unique solution w defined and of class C^∞ when $|t|\leqq y\leqq 1$ and we have $A(1-y)=A(1)w(y,y)^2$.

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1. **Existence.** We shall collect existence results for some boundary problems for the Webster system

$$Ap_x+u_t=0,\quad Ap_t+u_x=0$$

in the strip $0\leqq x\leqq 1$. The area function $A(x)$ is assumed to be positive and of class C^∞. The solutions p, u are allowed to be distributions. They are then infinitely differentiable in x considered as distributions in time t. In particular, the boundary values of p and u as x tends to 0 or 1 exist as distributions. It is sometimes useful to rewrite the Webster system as

(1) $$v_x-v_t=a(v+w)/2,\quad w_x+w_t=a(v+w)/2$$

where $v=Ap-u$, $w=Ap+u$ and $a(x)=A'(x)/A(x)$.

Let T be a triangle in the strip consisting of half a square with sides parallell to the coordinate axes or to the diagonals $t\pm x=0$. Lines with this last property are called characteristics. It is a classical fact that the system (1) has a unique C^∞ solution in T with arbitrary C^∞ data u, v on the large side when it is not characteristic (a Cauchy problem) or with v given on one small side and w on the other (a characteristic problem when these sides are characteristic) or with v given on a small side and w on the large side (a mixed problem). In a characteristic problem, the solution extends to the entire square of which T is a part. In these statements the boundary data can also be independent linear combinations of v and w.

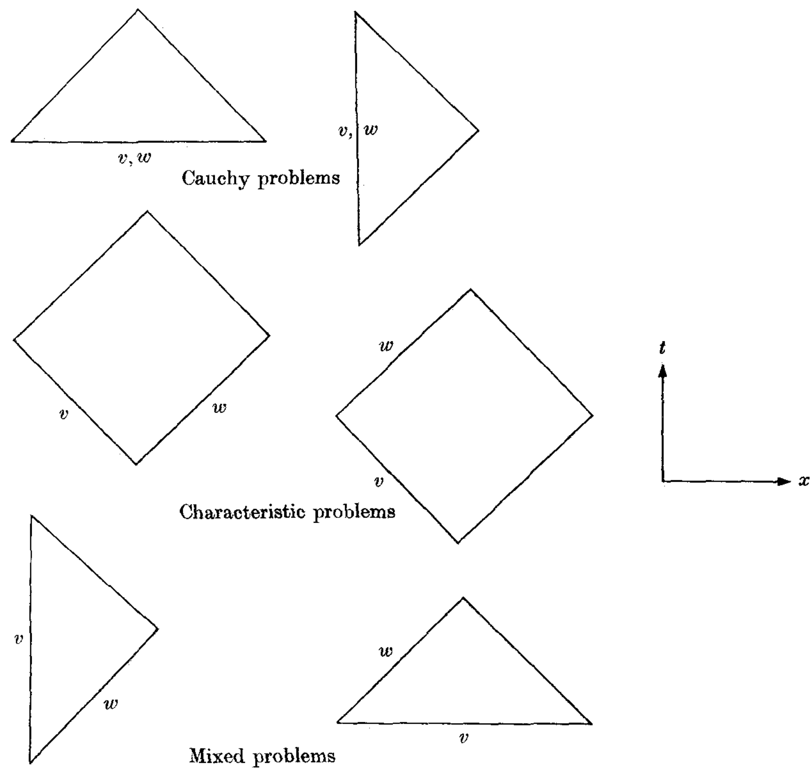


Figure 2. Boundary problems for the Webster system.

Cutting the strip $0 \leqq x \leqq 1$ into suitable triangles it follows from this that (1) has a unique C^∞ solution with C^∞ data v, w at $x=0$ or $x=1$ and that the solution vanishes at a point unless a characteristic through that point meets the support of the data. In particular, if that support is contained in an interval $x=0$, $|t|<\varepsilon$, the solution v, w vanishes when $|t|>x+\varepsilon$. It also follows that (1) has a unique forward (backward) C^∞ solution with given C^∞ data at $x=0$ and $x=1$ which are non-zero linear combinations of v and w and vanish when t is large negative (positive). In this case a forward (backward) solution vanishes at a point when the boundary data vanish below (above) the characteristics through the point. All this is also true for the inhomogeneous system (1) with C^∞ right sides. We can also allow the boundary data and the solution to be distributions in time. This is easy to prove by regularization and by estimating C^∞ solutions in terms of C^∞ boundary data in the inhomogeneous case.

We shall first consider solutions with $Ap=\delta(t), u=0$ when $x=0$ or $Ap=b\delta(t)$, $u=\delta(t)$ when $x=1$. They will be called the glottis and lip reflection pulses respectively and are fundamental for the study of the vowel transfer $u(0,t)\to u(1,t)$ of forward solutions with $Ap=bu$ for $x=1$. We let $\delta_0(t)=(1+\operatorname{sgn} t)/2$ denote the Heaviside function.

Theorem 1. *The vowel transfer. Let* $A\in C^\infty$. *Then*

(i) *there is a unique solution* p, u, *the glottis reflection pulse, such that* $Ap=\delta(t)$, $u=0$ *when* $x=0$. *It vanishes when* $|t|>x$ *and has the form*

$$(2)\quad \begin{aligned} A(x)p(x,t) &= c(x)\big(\delta(x-t)+\delta(x+t)\big)+\delta_0(x^2-t^2)A(x)\tilde{p}(x,t)\\ u(x,t) &= c(x)\big(\delta(x-t)-\delta(x+t)\big)+\delta_0(x^2-t^2)\tilde{u}(x,t)\end{aligned}$$

where $2c(x)=\big(A(x)/A(0)\big)^{1/2}$ *and* $\tilde{p}$, $\tilde{u}$ *are* C^∞ *functions when* $|t|\leqq x\leqq 1$.

(ii) *there is a unique solution* p, u, *the lip reflection pulse, such that* $Ap=b\delta(t)$, $u=\delta(t)$ *when* $x=1$. *It vanishes when* $|t|>1-x$ *and has the form*

$$(3)\quad \begin{aligned} A(x)p(x,t) &= -b_1(x)\delta(y-t)+b_2(x)\delta(y+t)+\delta_0(y^2-t^2)A(x)\tilde{p}(x,t)\\ u(x,t) &= b_1(x)\delta(y-t)+b_2(x)\delta(y+t)+\delta_0(y^2-t^2)\tilde{u}(x,t)\end{aligned}$$

where $y=1-x$, $2b_1(x)=(1-b)b(x)$, $2b_2(x)=(1+b)b(x)$, $b(x)=\big(A(x)/A(1)\big)^{1/2}$ *and* $\tilde{p}$, $\tilde{u}$ *are* C^∞ *functions when* $|t|\leqq y\leqq 1$.

(iii) *there is a unique forward solution* p, u, *the impulse response of the vowel transfer, such that* $u=\delta(t)$ *when* $x=0$, $Ap=bu$ *when* $x=1$ *and, if* $t\leqq x<1$,

$$A(x)p(x,t) = c(x)\delta(x-t)+\delta_0(x-t)A(x)p(x,t)$$

$$u(x,t) = c(x)\delta(x-t)+\delta_0(x-t)u(x,t)$$

where $c(x)=\big(A(x)/A(0)\big)^{1/2}$ *and* p, u *are* C^∞ *functions when* $t\leqq x\leqq 1$.

This theorem and its proof are illustrated by Figure 3.

Proof. (i) If p, u, c are C^∞ functions, (2) defines distributions p, u such that

$$Ap_t+u_x = (-A\tilde{p}+\tilde{u}+c')\delta(x-t)+(A\tilde{p}+\tilde{u}-c')\delta(x+t)+\delta_0(x^2-t^2)(A\tilde{p}_t+\tilde{u}_x)$$

$$Ap_x+u_t = (A\tilde{p}-\tilde{u}+c'-ac)\delta(x-t)+(A\tilde{p}+\tilde{u}+c'-ac)\delta(x+t)+\delta_0(x^2-t^2)(A\tilde{p}_x+\tilde{u}_t)$$

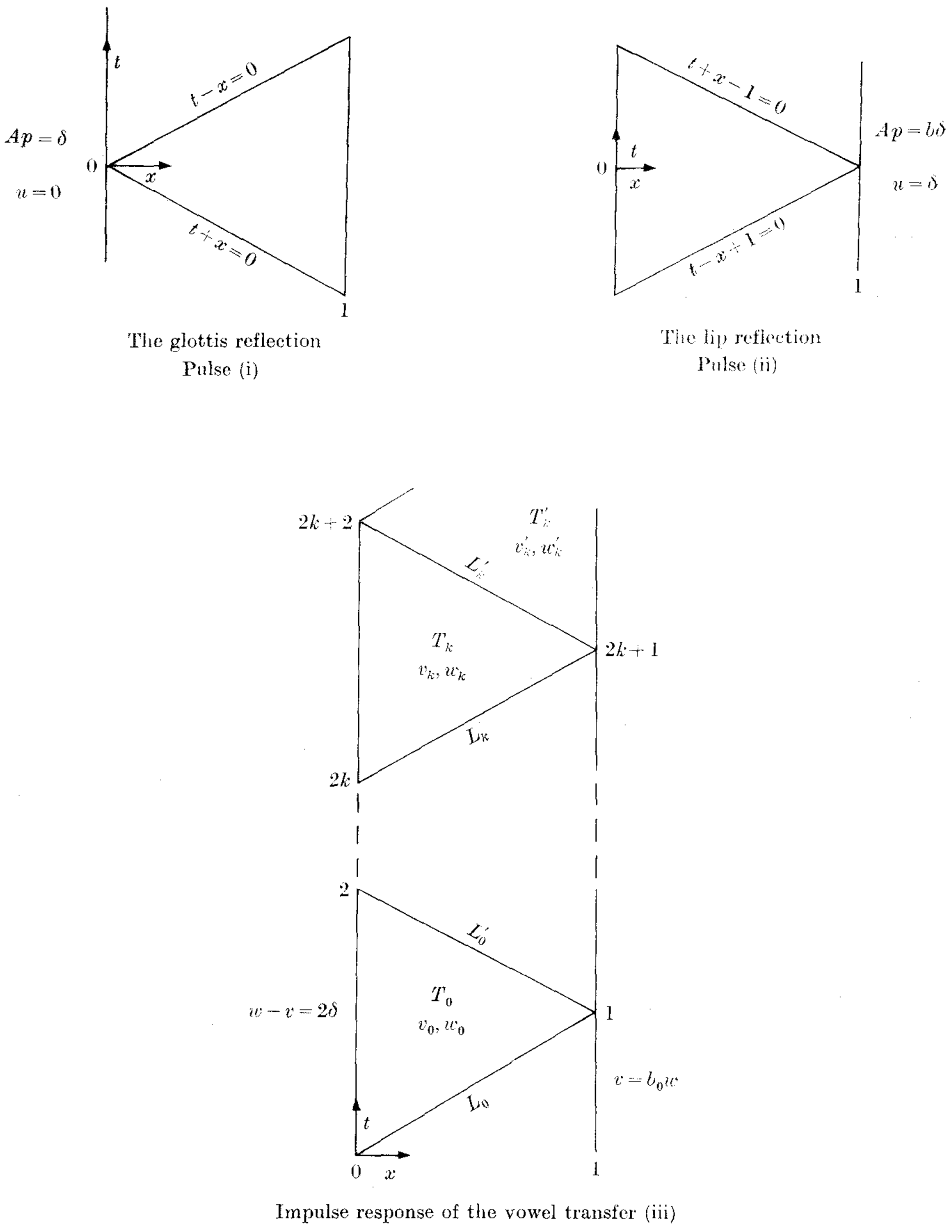


Figure 3. To Theorem 1.

where $a = A'/A$. Putting the right side equal to zero gives

$$t = x \Rightarrow A\tilde{p} - \tilde{u} - c' = 0, \quad 2c' = ac$$

$$t = -x \Rightarrow A\tilde{p} + \tilde{u} - c' = 0, \quad 2c' = ac$$

$$x^2 - t^2 \geqq 0 \Rightarrow A\tilde{p}_t + \tilde{u}_x = 0, \quad A\tilde{p}_x + \tilde{u}_t = 0.$$

This gives $c(x) = (A(x)/A(0))^{1/2}$ and a characteristic boundary problem for $v = A\tilde{p} - \tilde{u}$, $w = A\tilde{p} + \tilde{u}$ in the region $t^2 \leqq x^2$ with C^∞ data on the boundary. This proves (i) and the proof of (ii) is entirely similar.

(iii) Putting $v = Ap - u$, $w = Ap + u$, the problem is to find a forward solution of (1) such that

$$x = 0 \Rightarrow w - v = 2\delta(t), \quad x = 1 \Rightarrow v = b_0 w$$

where $b_0 = (b-1)/(b+1)$. Putting $c(x) = (A(x)/A(0))^{1/2}$ we try to find a solution of the form

$$w = 2c(x) \textstyle\sum_0^\infty b_0^k \delta(t - x - 2k) + \sum_0^\infty (\chi_k w_k(x, t) + \chi_k' w_k'(x, t))$$

(5)

$$v = 2c(x) \textstyle\sum_0^\infty b_0^k \delta(t + x - 2k - 2) + \sum_0^\infty (\chi_k v_k(x, t) + \chi_k' v_k'(x, t))$$

where the χ_k and χ_k' are the characteristic functions of the triangles T_k: $|t - 2k - 1| \leqq 1 - x$ and T_k': $|t - 2k + 1| \leqq x$ and the pairs w_k, v_k and w_k', v_k' are C^∞ solutions of (1) inside T_k and T_k' such that $w_k = v_k$ when $x = 0$ and $v_k' = b_0 w_k'$ when $x = 1$. Note that if $A(x)$ is constant, we get a solution by taking $c = 1$ and all w_k, v_k, w_k', v_k' equal to zero. Near the line L_k: $t - x - 2k = 0$ separating the triangles T_k and T_{k-1}' (see Figure 3) (5) gives

$$w = 2b_0^k c(x)\delta + \delta_0 w_k + \check{\delta}_0 w_{k-1}', \quad v = \delta_0 v_k + \check{\delta}_0 v_{k-1}'$$

where $\delta = \delta(t - x - 2k)$, $\delta_0 = \delta_0(t - x - 2k)$, $\check{\delta}_0 = \delta_0(x + 2k - t)$. Hence

$$w_t + w_x = 2b_0^k c'(x)\delta + \delta_0(w_{kt} + w_{kx}) + \check{\delta}_0(w_{k-1,t}' + w_{k-1,x}')$$

$$v_t - v_x = 2(v_k - v_{k-1}')\delta + \delta_0(v_{kt} - v_{kx}) + \check{\delta}_0(v_{k-1,t}' - v_{k-1,x}')$$

$$2^{-1}a(v + w) = b_0^k ac\delta + \delta_0 2^{-1}a(v + w) + \check{\delta}_0 2^{-1}a(v + w).$$

Since $2c' = ac$, (1) holds for (5) across L_k if and only if $2(v_k - v_{k-1}') + b_0^k ac = 0$ on that line. Similarly, (5) satisfies (1) across the line L_k': $t + x - 2k - 2 = 0$ separating T_k and T_k' if and only if $2(w_k - w_k') + b_0^k ac = 0$ on that line. Hence (5) is a solution of (1) provided w_k, v_k solve the mixed problem in T_k given by $w_k = v_k$ when $x = 0$, $v_k = v_{k-1}' + 2^{-1} b_0 ac$ on L_k and w_k', v_k' solve the mixed problem in T_k' given by

$v_k' = b_0 w_k'$ when $x=1$ and $w_k = w_k' + 2^{-1} b_0^k ac = 0$ on L_k' (see Figure 3). Since all these mixed problems have C^∞ solutions uniquely determined by C^∞ data, putting $v_{-1}'=0$ and solving them in $T_0, T_0', T_1, \dots$ gives a solution (5) of (1) with all the desired properties.

Our next theorem deals with the lip transfer, i.e. the map $u(1,t) \to p(1,t)$ for forward solutions of the Webster system such that $u=0$ when $x=0$.

Theorem 2. *The lip transfer. Let $A \in C^\infty$. Then*

(i) *the impulse response $\delta(t) \to p(1,t)$ of the lip transfer has the property that*

$$-A(1)p(1,t) = \delta(t) + g(t) \tag{6}$$

when $t<2$ where g is a C^∞ function when $0 \leqq t \leqq 2$ and vanishes when $t<0$.

(ii) *for every $0<y<1$ there is a unique solution p, u when $t \leqq 0$ vanishing for $t<1-x-y$ such that $p=-1$ when $1-y \leqq x \leqq 1$. It is of class C^∞ when $0 \geqq t \geqq 1-x-y$ and depends on the parameter y.*

(iii) *If p, u is the solution under* (ii), *the function $w(y,t) = u(1,t)/A(1)$ has the property that*

$$w(y,y) = \big(A(1-y)/A(1)\big)^{1/2}. \tag{7}$$

Extended by symmetry so that $w(y,-t) = w(y,t)$ it is of class C^∞ when $1 \geqq y \geqq t$ and it is the unique solution of the integral equation

$$2w(y,t) + \int_{-y}^{+y} \big(g(t-s) + g(s-t)\big) w(y,s)\, ds = 2$$

where g is given by (6).

Proof. (i) Put $w = Ap+u$, $v = Ap-u$. Precisely as in the proof of Theorem 1, one sees that the impulse response is unique and given by

$$w = -2b(x) \sum_0^\infty \delta(t-x-2k-1) + \sum_0^\infty \big(\chi_k w_k(x,t) + \chi_k' w_k'(x,t)\big)$$

$$v = -2b(x) \sum_0^\infty \delta(t+x-2k-1) + \sum_0^\infty \big(\chi_k v_k(x,t) + \chi_k' v_k'(x,t)\big).$$

Here χ_k and χ_k' are the characteristic functions of the triangles T_k: $|t-2k-1| \leqq x$ and T_k': $|t-2k-2| \leqq 1-x$, $b(x) = \big(A(x)/A(1)\big)^{1/2}$ solves the differential equation $2b' = ab$ and the w_k, v_k and w_k', v_k' are C^∞ functions in T_k and T_k' respectively given by mixed boundary problems with C^∞ boundary data. In particular, $-2A(1)p(1,t) = -v(1,t) - w(1,t)$ has the desired properties.

(ii) We try to construct a solution of the form $p=\delta_0\tilde{p}$, $u=\delta_0\tilde{u}$ where $\delta_0=$ $=\delta_0(t+x+y-1)$ and $\tilde{p}$ and $\tilde{u}$ are C^∞ functions when $0\geqq t\geqq 1-x-y$. We get

$$Ap_x+u_t = (A\tilde{p}+\tilde{u})\delta+(A\tilde{p}_x+\tilde{u}_t)\delta_0$$
$$Ap_t+u_x = (A\tilde{p}+\tilde{u})\delta+(A\tilde{p}_t+\tilde{u}_x)\delta_0.$$

Together with the condition that $p=-1$ when $t=0$, $1-y\leqq x\leqq 1$, this gives the system (1) with the boundary conditions

$$0=t=1-x-y \Rightarrow w=0 \quad \text{and} \quad 0=t, \quad 1-y\leqq x\leqq 1 \Rightarrow v+w=-2A(x)$$

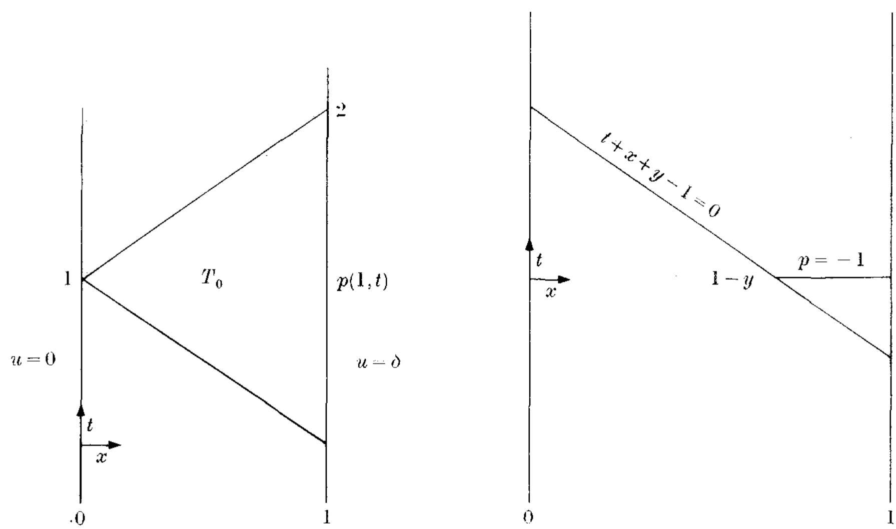
for $v=A\tilde{p}-\tilde{u}$, $w=A\tilde{p}+\tilde{u}$. It has a unique C^∞ solution in the triangle $0\geqq t\geqq$ $\geqq 1-x-y\geqq 0$ and this proves (ii). To prove (iii) note that when $t+x+y-1=0$,

$$A(\tilde{p}+A^{-1}\tilde{u})' = -A\tilde{p}_t+A\tilde{p}-\tilde{u}_t+\tilde{u}_x-a\tilde{u} = -(A\tilde{p}_t+\tilde{u}_x)+(A\tilde{p}_x+\tilde{u}_t)+2u'-au$$

where the prime denotes the derivative with respect to x. Hence

$$0=t=1-x-y \Rightarrow \tilde{u} = \big(A(1-y)A(x)\big)^{1/2}$$

so that (7) follows. Since $p=-1$ when $t=0$, $x>1-y$, all odd t-derivatives of u vanish in the same interval. Hence $u(1,t)=u(1,-t)$ defines a C^∞ extension of u from the interval $x=1$, $0\geqq t\geqq -y$ to the interval $x=1$, $|t|\leqq y$. Using this, the solution p, u in the triangle $0\geqq t\geqq 1-x-y$ extends to a C^∞ solution in the



(i)
The lip transfer.

(ii), (iii)
A forward solution with $p=-1$ on the interval $t=0, 1-y\leqslant x\leqslant 1$

Figure 4. To Theorem 2.

triangle $|t| \leqq x+y-1 \geqq 0$ with u given as above when $x=1$. Call this new solution p, u also. Then $p^*(x,t)=p(x,-t)$, $u^*(x,t)=-u(x,-t)$ is another C^∞ solution in the same triangle and hence also $P=p+p^*$, $U=u+u^*$ with the property that $P=-2$ when $t=0$ and $U=0$ when $x=1$ and also when $t=0$. But then $P=-2$, $U=0$ when $|t|=1-x$ and hence also in the rest of the triangle. In particular,

$$|t| \leqq y \Rightarrow p(1,t)+p^*(1,t) = -2.$$

On the other hand, by (i)

$$-A(1)p(1,t) = u(1,t)+\int g(t-s)u(1,s)\,ds$$

so that also

$$-A(1)p^*(1,t) = -u^*(1,t)-\int g(-t+s)u^*(1,s)\,ds$$

and hence, inserting $u(1,t)=A(1)w(y,t)$ and $u^*(1,t)=-A(1)w(y,t)$ we get

(8) $$2 = 2w(y,t)+\int_{-y}^{+y}(g(t-s)+g(s-t))w(y,s)\,ds$$

when $|t| \leqq y$. Here $t \to w(y,t)$ is a C^∞ function. Next suppose that $t \to w(y,t)$ is a C^∞ function solving the corresponding homogeneous integral equation. Construct a solution p, u of the Webster system when $t \leqq 0$ and $0 \leqq t \leqq x+y-1$ vanishing when $t<1-y-x$ such that $u(1,t)=A(1)w(y,t)$ when $|t| \leqq y$. Let $p^*(x,t)=p(x,t)$, $u^*(x,t)=-u(x,-t)$. Then $P=p+p^*$, $U=u+u^*$ is a solution in the triangle $|t| \leqq x+y-1$ such that $P=U=0$ when $t=0$ and $U=0$ when $x=1$. But then $P=U=0$ in the entire triangle so that $p=p^*=0$ when $t=0$ and, by the computations under (i), $u=p=0$ when $t=1-x-y<0$. Hence p and also p^* vanish when $x=1$, $|t| \leqq y$. But then u and u^* vanish there also so that $w(y,t)=0$ when $|t| \leqq y$. This finishes the proof of (iii) since differentiations of (8) with respect to y show that a solution $t \to w(y,t)$ which is bounded is of class C^∞ in both variables when $|t| \leqq y$.

2. **Fourier—Laplace transforms.** Our various solutions p, u of the Webster system have been C^∞ functions of x considered as tempered distributions in t, i.e. the functions $x \to \int p(x,t)f(t)\,dt$ and $x \to \int u(x,t)f(t)\,dt$ where the integrals are taken in the distribution sense are infinitely differentiable when f belongs to the Schwartz class S of C^∞ functions such that $t^q f^{(r)}(t)$ is bounded for all nonnegative integers q and r. Hence, by the theory of distributions, the Fourier—Laplace transforms

$$P(x,\omega) = \int e^{-2\pi i\omega t}p(x,t)\,dt, \quad U(x,\omega) = \int e^{-2\pi i\omega t}u(x,t)\,dt$$

are also C^∞ functions of x considered as tempered distributions in ω and they satisfy the system

(1) $$P_x+2\pi i\omega A^{-1}U = 0, \quad U_x+2\pi i\omega AP = 0.$$

Conversely, if P and U have these properties, the inverse transforms

$$p(x, t) = \int e^{2\pi i\omega t} P(x, \omega)\, d\omega, \quad u(x, t) = \int e^{2\pi i\omega t} U(x, \omega)\, d\omega$$

are solutions of the Webster system and C^∞ functions of x considered as tempered distributions in t. Finally, by the Paley—Wiener theorem for distributions, $p(x, t)$ and $u(x, t)$ vanish for $t < t(x)$ if and only if the distributions $\omega \to P(x, \omega)$ and $\omega \to U(x, \omega)$ have analytic extensions to the lower half-plane which are $O(e^{2\pi \operatorname{Im}(t(x)-\varepsilon)\omega})$ for every $\varepsilon > 0$.

We shall now analyze our transforms in detail.

Theorem 3. *Let* $A \in C^\infty$. *Then*

(i) *the Fourier—Laplace transforms* P, U *and* P_b, U_b *of the glottis reflection pulse and the lip reflection pulse of Theorem* 1 *are* C^∞ *functions of* x, ω *whose derivatives are of at most polynomial growth for real values of* ω. *They are also entire analytic functions of* ω *with unique asymptotic expansions*

$$AP \sim e^{-2\pi i\omega x}\big(c(x)+\dots\big)+e^{2\pi i\omega x}\big(c(x)+\dots\big), \tag{2}$$

$$U \sim e^{-2\pi i\omega x}\big(c(x)+\dots\big)-e^{2\pi i\omega x}\big(c(x)+\dots\big), \tag{3}$$

$$AP_b \sim e^{-2\pi i\omega(1-x)}\big(-b_1(x)+\dots\big)+e^{2\pi i\omega(1-x)}\big(b_2(x)+\dots\big), \tag{4}$$

$$U_b \sim e^{-2\pi i\omega(1-x)}\big(b_1(x)+\dots\big)+e^{2\pi i\omega(1-x)}\big(b_2(x)+\dots\big) \tag{5}$$

where the dots stand for asymptotic series of the form $f_1(x)\omega^{-1}+f_2(x)\omega^{-2}+\dots$ *with* C^∞ *coefficients which are continuous with respect to* A.

(ii) *if* $0 \leqq b < 1$ *there are unique factorizations*

$$P(1, \omega) = \prod_1^\infty (1-\omega^2/\alpha_n^2), \tag{6}$$

$$U(1, \omega) = -\bar{A}2\pi i\omega \prod_1^\infty (1-\omega^2/\beta_n^2), \quad \bar{A} = \int_0^1 A(x)\, dx, \tag{7}$$

$$A(1)P(1, \omega)-bU(1, \omega) = A(1) \prod_1^\infty (1-\omega/\omega_n)(1+\omega/\bar{\omega}_n) \tag{8}$$

where the zeros are such that

$$0 < \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \dots \tag{9}$$

and, for all n,

$$0 \leqq \operatorname{Re}\omega_1 \leqq \operatorname{Re}\omega_2 \leqq \dots, \quad b > 0 \Rightarrow \operatorname{Im}\omega_n > 0. \tag{10}$$

They have asymptotic expansions

$$\alpha_n \sim 2^{-1}n-4^{-1}+a_1 n^{-1}+a_2 n^{-2}+\dots, \tag{11}$$

$$\beta_n \sim 2^{-1}n+b_1 n^{-1}+b_2 n^{-2}+\dots, \tag{12}$$

$$\omega_n \sim 2^{-1}n-4^{-1}+ic+c_1 n^{-1}+c_2 n^{-2}+\dots, \quad 4\pi c = \log(1+b)/(1-b). \tag{13}$$

where the coefficients are continuous functions of A.

(iii) *one has*

(14) $$A(1)U_b(0,\omega) = A(1)P(1,\omega) - bU(1,\omega)$$

and $1/U_b(0,\omega)$ *is the Fourier—Laplace transform of the impulse response of the vowel transfer* $u(0,t)\to u(1,t)$ *under the condition that* $Ap=bu$ *for* $x=1$.

Note. The expansions under (i) are taken in the sense that the differences between the left sides and partial sums with n terms of the right sides are $O(e^{2\pi|\mathrm{Im}\,\omega|x}|\omega|^{-n})$ as $\omega\to\infty$, uniformly in x. Continuity with respect to A is taken in the C^∞ topology.

Proof. (i) follows from (i) of Theorem 1, differentiations under the integral sign and integrations by parts. To prove (ii) note that it follows from (1) that the functions $x\to P(x,\omega)$ and $x\to U(x,\omega)$ cannot both vanish for any x with $0\leqq x\leqq 1$. In fact, if they do, they vanish for all such x. We also get the identity

$$P_x\bar{U}+\bar{P}U_x+2\pi i\omega(A^{-1}|U|^2+A|P|^2) = 0.$$

Taking the real part and integrating over the tube gives

$$\mathrm{Re}\,P(1,\omega)\overline{U(1,\omega)} - 2\pi\,\mathrm{Im}\,\omega\int_0^1 (A^{-1}|U|^2+A|P|^2)\,dx = 0.$$

When the first term vanishes, $\mathrm{Im}\,\omega$ must also vanish. Hence the functions $\omega\to P(1,\omega)$ and $\omega\to U(1,\omega)$ have only real zeros and since the two functions cannot vanish simultaneously, the identity shows that their zeros are simple. When $P(1,\omega)=$ $=bU(1,\omega)$ and $b>0$, the first term is positive and hence $\mathrm{Im}\,\omega>0$. Hence the zeros of the function $\omega\to P(1,\omega)-bU(1,\omega)$ lie in the upper half-plane when $b>0$. As solutions of (1), P and U are uniquely determined by the condition that $P=1$, $U=0$ when $x=0$. This proves that $\overline{P(x,\omega)}=P(x,-\bar{\omega})$, $\overline{U(x,\omega)}=U(x,-\bar{\omega})$ and hence if γ is a zero of one of the three functions above, so is $-\bar{\gamma}$. When $A(x)$ is constant, then $AP=\cos 2\pi\omega x$, $U=-i\sin 2\pi\omega x$ and hence

$$A(1)P(1,\omega) = \cos 2\pi\omega,\quad U(1,\omega) = -i\sin 2\pi\omega,$$

$$A(1)P(1,\omega)-bU(1,\omega) = (1-b^2)^{1/2}\cos 2\pi(\omega-ic)$$

with c as in (13). In this case, the formulas (2), (3), (4), (5) and (11), (12), (13) are exact if we delete the negative powers of ω and n respectively. It follows from (2) that

$$A(1)P(1,\omega) \sim (1+u_2\omega^{-2}+\ldots)\cos 2\pi\omega+(v_1\omega^{-1}+v_3\omega^{-3}+\ldots)\sin 2\pi\omega$$

with real coefficients in the asymptotic series. In fact, $P(1,\omega)$ is real when ω is real. Hence, for large n, $P(1,\omega)$ has precisely one real zero α_n close to $\alpha_{0n}=n/2-1/4$ with an asymptotic development (11) in terms of negative powers of α_{0n} and hence also of n. That this zero has number n in the sequence of positive zeros of $P(1,\omega)$ follows by a homotopy through tubes with area functions $x\to 1-s+sA(x)$ where

$0 \leqq s \leqq 1$. The formulas (12) and (13) are proved in precisely the same way. Since $P(1, \omega)$ and $U(1, \omega)$ have no common zeros, the homotopy also shows that the separation (9), true when $A(x)$ is constant, holds in the general case. The ordering of (10) is just a convention.

Next, let F be any left side of (6), (7) or (8). By (2) and (3),

$$\log |F(\omega)| = 2\pi\, |\mathrm{Im}\, \omega| \big(1+o(1)\big), \quad \omega \to \infty \quad \mathrm{Re}\, \omega = 0. \tag{15}$$

On the other hand, if G is the corresponding right side, a comparison with, e.g., the product

$$\cosh 2\pi |\omega| = \prod_1^\infty (1+|\omega|^2/\alpha_{0n}^2), \quad \alpha_{0n} = 2^{-1} n - 4^{-1},$$

shows that $\log |G(\omega)| = O(2\pi |\omega|)$ and that (15) holds for G when $\mathrm{Re}\, \omega = 0$. In particular, F and G are entire analytic functions of exponential type. Since they have the same zeros, Hadamard's factorization theorem shows that $F(\omega) = e^{B\omega + C} G(\omega)$. Here, since $\overline{F(\omega)} = F(-\bar{\omega})$ and the same for G, B is purely imaginary. Letting $\mathrm{Re}\, \omega = 0$ $\mathrm{Im}\, \omega \to \infty$ and using that (15) holds for both F and G it follows that $B = 0$. Hence (6), (7), (8) hold modulo constants on the right sides. They are determined by putting $\omega = 0$ and noting that then $P = 1$ and $dU/d\omega = -2\pi i \bar{A}$.

To prove (iii) note that, by virtue of (i), the two sides of (14) are entire analytic in ω of exponential type, that they share the symmetry property $\overline{F(\omega)} = F(-\bar{\omega})$ and that their two quotients are bounded far away on the imaginary axis. Hence, as before, the two sides are equal modulo a constant factor which must be one since the two sides are equal when $\omega = 0$. According to (ii), all the zeros of $U_b(0, \omega)$ lie in the upper half-plane and, by (5), $U_b(0, \omega) = \big(A(0)/A(1)\big)^{1/2} \cos 2\pi(\omega - ic) + O(\omega^{-1})$ for large real ω with c as in (13). Hence $P^*(x, \omega) = P_b(x, \omega)/U_b(0, \omega)$ and $U^*(x, \omega) = U_b(x, \omega)/U_b(0, \omega)$ are C^∞ functions, solutions of (1) with the boundary condition $AP^* = bU^*$ when $x = 1$, whose derivatives are of at most polynomial growth in ω. Hence their inverse Fourier—Laplace transforms, p^* and u^*, solve Webster's system with $Ap^* = bu^*$ when $x = 1$ and $u^* = \delta$ when $x = 0$. In fact, $U^* = 1$ when $x = 0$. This identifies $U^*(1, \omega) = 1/U_b(0, \omega)$ with the impulse response of the vowel transfer provided p^*, u^* is a forward solution. But this is so since P^* and U^* are analytic in the lower half-plane and, by (4) and (5), equal to $O(e^{2\pi\, \mathrm{Im}\, \omega x})$ there. Hence p^* and u^* vanish when $t < x$.

3. **Canonical products.** To prepare for the reconstruction of the area function from the vowel resonances we shall now study canonical products of the type employed in (8) of the preceding section whose zeros have asymptotic expansions. First an auxiliary result that has to do with analytic functions having asymptotic expansions

$$g(\omega) \sim \sum_0^\infty h_k(\omega)\, \omega^{-k} \tag{1}$$

where the h_k are analytic of period $c>0$ and the expansion is such that

(2) $$g(\omega)-\sum_0^k h_j(\omega)\omega^{-j} = o(|\omega|^{-k})$$

for all k.

Lemma (i) *Suppose that* (2) *holds on the real axis as* $\omega\to\infty$ *and that* $h_0(0)=0$, $h_0'(0)\neq 0$. *Then, for large integral* n, *the equation* $g(\omega-\tau n)=0$ *has precisely one small zero* $\omega=\omega(n)$. *This zero has an asymptotic expansion in terms of negative powers of* n.

(ii) *Suppose that the* h_k *are meromorphic with poles at the integral multiples of* τ *and that* (2) *holds on all circles of some fixed radius* $0<\varepsilon<\tau$ *around these multiples. Then the numbers*

$$\int_{|\omega|=\varepsilon} g(\omega-\tau n)\,d\omega/\omega$$

have an asymptotic expansion in terms of negative powers of n.

Proof. We may assume that $\tau=1$. (i) By assumption,

$$g(\omega-n) = \sum_0^k h_j(\omega)(\omega-n)^{-j}+g_k(\omega-n)$$

where $g_k(\omega-n)=o(|\omega-n|^{-k})$. Putting $y=n^{-1}$ and writing $(\omega-n)^{-k}=$ $=(-y)^k(1-\omega y)^{-k}$ as a power series in y and ω we have

$$g(\omega-n) = P_k(\omega, y)+y^kQ_k(\omega, y)$$

where $P_k(\omega, y)=h_0'(0)\omega'+\dots$ is a polynomial of degree $\leqq k$ in ω and y and Q_k is analytic in ω and continuous at the origin and vanishes there. Consider now the function $F_k=P_k(\omega, y)+y^kz$ of the small complex variables ω, y, z. By the Weierstrass preparation theorem, $F_k(\omega, y, z)=0$ if and only if $\omega=f_k(y, z)$ where f_k is analytic at the origin. By the form of F_k, $f_k(y, z)=a_1y+\dots+a_{k-1}y^{k-1}+$ $+g_k(y, z)y^k$ where $a_1, \dots$ are constants and $g_k(y, z)$ analytic at the origin. Hence $g(\omega-n)=0$ with small ω if and only if $\omega=f_k(n^{-1}, Q_k(\omega, n^{-1}))$ and this completes the proof.

(ii) By assumption,

$$g(\omega-n) = \sum_0^k h_j(\omega)(\omega-n)^{-j}+o(|n|^{-k})$$

when $|\omega-n|=\varepsilon$. The assertion follows by a series expansion of $(\omega-n)^{-k}=$ $=(-n)^{-k}(1-\omega/n)^{-k}$.

We can now prove

Theorem 4. (i) *Let* $\omega_1, \omega_2, \dots$ *be complex numbers with asymptotic expansion*

$$\omega_n \sim \omega_{0n}+c_1n^{-1}+c_2n^{-2}+\dots, \quad \omega_{0n} = n/2-1/4+ic$$

where $c>0$ *and assume that* $0\leqq \operatorname{Re}\omega_1\leqq \operatorname{Re}\omega_2\leqq\ldots$ *and that* $\operatorname{Im}\omega_n>0$ *for all* n. *Then*

$$F(\omega)=\prod_1^\infty (1-\omega/\omega_n)(1+\omega/\bar\omega_n)$$

is entire analytic and

$$F(\omega)+F(-\omega)=2\prod_1^\infty (1-\omega^2/\alpha_n^2),$$

$$F(\omega)-F(-\omega)=2F'(0)\omega\prod_1^\infty (1-\omega^2/\beta_n^2)$$

where $0<\alpha_1<\beta_1<\alpha_2<\ldots$ *with asymptotic expansions*

$$\alpha_n\sim\alpha_{0n}+a_1n^{-1}+\ldots,\quad \alpha_{0n}=n/2-1/4,$$

$$\beta_n\sim\beta_{0n}+b_1n^{-1}+\ldots,\quad \beta_{0n}=n/2.$$

(ii) *Suppose that* $\alpha_1,\ldots$ *and* $\beta_1,\ldots$ *have the properties above and let* $\sigma_n/2\pi i$ *be the residue at* $\omega=\beta_n$ *of the quotient*

$$H(\omega)=\prod_1^\infty (1-\omega^2/\alpha_n^2)/2\pi i\omega\prod_1^\infty (1-\omega^2/\beta_n^2).$$

Then there is an asymptotic expansion

$$\sigma_n\sim\sigma_0+s_1n^{-1}+\ldots$$

where

$$\sigma_0=\prod_1^\infty (\beta_n/\beta_{0n})^2/\prod_1^\infty (\alpha_n/\alpha_{0n})^2=\lim H(-i\omega)\quad \textit{as}\quad \omega\to+\infty.$$

Proof. (i) Put

$$F_n(\omega)=(1-\omega/\omega_n)(1+\omega/\bar\omega_n)$$

so that $F=F_1F_2\ldots$ and consider

$$F_n(-\omega)/F_n(\omega)=(\omega+\omega_n)(\omega+\bar\omega)/(\omega-\omega_n)(\omega-\bar\omega_n).$$

The absolute value of the right side equals 1 precisely when ω is real. Hence $F(\omega)\pm\pm F(-\omega)\neq 0$ unless ω is real and $f(\omega)=\arg G(\omega)$ with $G(\omega)=F(\omega)/F(-\omega)$ equals π or 0 respectively modulo 2π. Since

$$f'(\omega)=G'(\omega)/iG(\omega)=i^{-1}\sum_1^\infty ((\omega-\omega_n)^{-1}-(\omega-\bar\omega_n)^{-1}+(\omega+\bar\omega_n)^{-1}-(\omega+\omega_n)^{-1})=$$

$$=2\sum_1^\infty (|\omega-\omega_n|^{-2}+|\omega+\omega_n|^{-2})\operatorname{Im}\omega_n>0$$

when ω is real, the zeros $\pm\alpha_n$, $\pm\beta_n$ of functions $F(\omega)\pm F(-\omega)$ are real and simple and separate each other. To investigate their asymptotic properties we shall compare F to the product

$$F_0(\omega)=\cos 2\pi(\omega-ic)/\cos ic=\prod_1^\infty (1-\omega/\omega_{0n})(1+\omega/\bar\omega_{0n})$$

where $\omega_{0n}=n/2-1/4+ic$ and compare G to the quotient

$$G_0(\omega)=F_0(-\omega)/F_0(\omega)=\cos 2\pi(\omega+ic)/\cos 2\pi(\omega-ic).$$

It has the property that

$$G_0(\omega) = -1 \Leftrightarrow \omega = \pm\alpha_{0n} = \pm(n/2-1/4),$$
$$G_0(\omega) = 1 \Leftrightarrow \omega = \pm\beta_{0n} = \pm n/2.$$

We shall see that, in the sense of the first part of the lemma,

(3) $$G(\omega)\pm 1 \sim h_0(\omega)\pm 1+h_1(\omega)\omega^{-1}+\dots$$

where $h_0=G_0$ and all coefficients have the period $1/2$. Since $h_0'(\omega)\neq 0$ when $\omega=\alpha_{0n}$ or β_{0n}, an appeal to the lemma then finishes the proof. In fact, the only remaining part, i.e. that the equations $F(\omega)\pm F(-\omega)=0$ have, respectively, precisely $2n$ zeros when, e.g., $|\omega|<\alpha_{0n}+8^{-1}$ and precisely $2n+1$ zeros when, e.g., $|\omega|<\beta_{0n}+8^{-1}$, is taken care of by a homotopy from the case when $\omega_n=\omega_{0n}$ for all n.

Our asymptotic series will be linear combinations of the functions

$$h_{jk}(\omega) = \sum_{-\infty}^{+\infty} (\omega-\omega_{0n})^{-k}\omega_{0n}^{-j}$$

where $j, k\geqq 0$ and $j+k>1$. Here, by definition, $\omega_{0n}=n/2-1/4+ic$ for all n. In particular,

$$h_{11}(\omega) = \sum_1^{\infty} \left((\omega-\omega_{0n})^{-1}\omega_{0n}^{-1}+(\omega+\bar{\omega}_{0n})\bar{\omega}_{0n}^{-1}\right) = -2\pi\tan 2\pi(\omega-ic).$$

Using the fact that $\omega h_{jk}=h_{j-1,k}+h_{j,k-1}$ an easy argument shows that

(4) $$h_{jk}(\omega) = h_{jk,0}(\omega)+h_{jk,1}(\omega)\omega^{-1}+\dots+h_{jk,p}(\omega)\omega^{-p}$$

where $p=\min(j,k)$ and the coefficients are periodic with period $1/2$. In the same way,

(5) $$\sum_{-\infty}^{+\infty} |\omega-\omega_{0n}|^{-j}|\omega_{0n}|^{-k} = o(|\omega|^{1-\min(j,k)}).$$

Next, consider

$$\log F(\omega)/F_0(\omega) = \sum_{-\infty}^{+\infty} \log(1-\omega/\omega_n)/(1-\omega/\omega_{0n})$$

where, by definition, $\omega_{-n}=-\bar{\omega}_n$. Assuming for the moment that

(6) $$|\omega_n-\omega_{0n}| < c/2$$

for all n, this can be rewritten as

$$\log F(\omega)/F_0(\omega) = c_0+\sum_{-\infty}^{+\infty} \log\left(1-(\omega_n-\omega_{0n})/(\omega-\omega_{0n})\right)$$

where $c_0=\sum\log\omega_{0n}/\omega_n$ and the logarithms can be expanded in power series. The result is that

$$\log F(\omega)/F_0(\omega) = c_0+\sum_1^{\infty}\sum_{-\infty}^{+\infty} k^{-1}(\omega_n-\omega_{0n})^k(\omega-\omega_{0n})^{-k}$$

where the series converges absolutely. In view of the asymptotic expansion of $\omega_n-\omega_{0n}$, which we can write in terms of negative powers of ω_{0n}, this together with

(4) and (5) shows that

(7) $$\log F(\omega)/F_0(\omega) \sim c_0+h_1(\omega)\omega^{-1}+h_2(\omega)\omega^{-2}+\dots$$

where the $h_k(\omega)$ are analytic and periodic with period 1/2. Since this statement holds true also when $F(\omega)/F_0(\omega)$ acquires a rational factor which is regular at $\omega=\infty$, the assumption (6) is now superfluous. In fact, changing a factor F_n with parameter ω_n to another such factor with a different parameter amounts to multiplying F by a rational function regular at ∞. Since

$$G(\omega)/G_0(\omega) = e^{\log (F(\omega)F_0(-\omega)/F(-\omega)F_0(\omega))}$$

the desired expansion (3) now follows from (7).

(ii) Put

$$H_0(\omega) = \Pi_1^\infty (1-\omega^2/\omega_{0n}^2)/2\pi i\omega \, \Pi_1^\infty (1-\omega^2/\omega_{0n}^2) = i^{-1}\cot 2\pi\omega.$$

It suffices to prove that

(8) $$H(\omega) \sim \sigma_0 H_0(\omega)+h_1(\omega)\omega^{-1}+\dots$$

in the sense of the second part of the lemma with coefficients $h_1, \dots$ of period 1/2. In fact, the residue of $H_0(\omega)$ at $\omega=\beta_{0n}$ equals $1/2\pi i$. Consider the quotient

$$H(\omega)/H_0(\omega) = \Pi_{-\infty}^{+\infty} (1-\omega/\alpha_n)(1-\omega/\alpha_{0n})^{-1}(1-\omega/\beta_{0n})(1-\omega/\beta_n)^{-1}$$

where the product runs over all integers $\neq 0$. With σ_0 as in the theorem we can write this as

$$\sigma_0 \Pi_{-\infty}^{+\infty} (1-(\alpha_n-\alpha_{0n})/(\omega-\alpha_{0n}))(1-(\beta_n-\beta_{0n})/(\omega-\beta_{0n}))^{-1}.$$

The limit of the product as $\omega\to\infty$ with $\operatorname{Re}\omega$ bounded is 1. Since $H_0(-i\infty)=1$, this shows that $\sigma_0=H(-i\infty)$. If we restrict ω to small circles $|\omega-\beta_{0n}|=\varepsilon$ we can take the logarithm and make series expansions provided

(9) $$|\alpha_n-\alpha_{0n}| < \varepsilon/2, \quad |\beta_n-\beta_{0n}| < \varepsilon/2$$

for all n. The result is that

$$\log H(\omega)/H_0(\omega) = \sum_1^\infty k^{-1} \sum_{-\infty}^{+\infty} (-(\alpha_n-\alpha_{0n})^k(\omega-\alpha_{0n})^{-k}+(\beta_n-\beta_{0n})^k(\omega-\beta_{0n})^{-k}).$$

Inserting asymptotic expansions of α_n and β_n in terms of, respectively, negative powers of α_{0n} and β_{0n}, (8) follows as before. Since changing the parameters of a finite number of factors of H does not affect the form of (8), (9) can now be disregarded and this finishes the proof.

4. **The lip response and the integral equation for the area function.** Let $u(1, t)= =\delta(t)\to -A(1)h(t)$ be the impulse response of the lip transfer $u(1, t)\to p(1, t)$ obtained from forward solutions of the Webster system such that $u=0$ when $x=0$.

By Theorem 2, if $t<2$, $h(t)=\delta(t)+g(t)$ where $g(t)=0$ when $t<0$ and g is a C^∞ function when $0\leqq t\leqq 2$ such that the integral equation

(1) $$2w(y, t)+\int_{-y}^{+y}(g(t-s)+g(s-t))w(y, s)\,ds = 2$$

has a unique C^∞ solution $w(y, t)=w(y, -t)$ when $|t|\leqq y\leqq 1$. We shall now call any such function a lip response even if we do not know that it comes from the impulse response of a tube with a C^∞ area function. More precisely, we require that $g(t)=0$ when $t<0$, that g is infinitely differentiable when $0\leqq t\leqq 2$ and that $w(y, t)=0$ is the only continuous solution of the homogeneous equation (1) when $y\leqq 1$.

We can now prove a result due to Sondhi and Gopinath [7]. A tube whose area function is such that $A(1)=1$ is said to be normalized.

Theorem 5. *Let g be a lip response. Then*

(i) *the integral equation* (1) *has a unique C^∞ solution $w(y, t)=w(y, -t)$ defined when $|t|\leqq y\leqq 1$ such that*

(2) $$w_{yy}-w_{tt} = 2w_y f'(y)/f(y)$$

where $f(y)=w(y, y)>0$.

(ii) *the function g is the lip response of a tube, unique when normalized, whose area function is given by*

(3) $$A(1-y) = A(1)w(y, y)^2.$$

Proof. (i) By Fredholm theory, the equation (1), considered for fixed $y\leqq 1$, has a unique continuous solution $t\to w(y, t)$. Since $t\to w(y, -t)$ solves the same equation, w is an even function of t. By the properties of the integral, a continuous solution $w(y, t)$ of (1) is successively seen to be bounded, continuous and of class C^∞ in both variables when $|t|\leqq y\leqq 1$. Putting

$$Zw(y, t) = 2w(y, t)+\int_0^{y+t} w(y, t-u)g(u)\,du+\int_0^{y-t} w(y, t+u)g(u)\,du$$

we can write (1) as $Zw=2$. Differentiating the equality $Zv(y, t)+h(y, t)=0$ we get

(4) $$Zv_y(y, t)+v(y, -y)g(y+t)+v(y, y)g(y-t)+h_y(y, t) = 0$$

(5) $$Zv_t(y, t)+v(y, -y)g(y+t)-v(y, y)g(y-t)+h_t(y, t) = 0$$

so that

(6) $$Z(v_y\pm v_t)(y, t)+2v(y, \mp y)g(y\pm t)+(h_y\pm h_t)(y, t) = 0$$

with corresponding signs. In particular, if $v=w$ and $h=2$,

$$Z(w_y+w_t)(y, t)+2f(y)g(y+t) = 0$$

where $f(y)=w(y,y)$. Hence, applying (6) with the lower signs and with

$$v = w_y+w_t, \quad h = 2f(y)g(y+t), \quad \text{noting that} \quad (w_y+w_t)(y,y) = f'(y) \quad \text{we get}$$

$$Z(w_{yy}-w_{tt})(y,t)+2f'(y)\big(g(y-t)+g(y+t)\big) = 0.$$

Comparing this to (4) with $v=w$ and $h=2$ shows that Z annihilates $w_{yy}-w_{tt}-2w_y f'/f$ so that (2) follows. If $f(y)=0$ for some y, then, by (5), $w_t(y,t)=0$ when $|t|\leqq y$ so that also $w(y,t)=0$ when $|t|\leqq y$. But then, by (1), $w(y,0)=1$ and this contradiction shows that $w(y,t)>0$ when $|t|\leqq y$. By Theorem 2, if g is the lip transfer of a tube with area function $A(x)$, then $A(1-y)=A(1)w(y,y)^2$.

(ii) If g is just a lip response, let $w(y,t)$ be the solution of (2) and define $A(1-y)$ by (3) with $A(1)=1$ (note that $w(0,0)=1$). Let $g^*(y)$ be the lip response of a tube with area function $A(x)$ and let w^* solve the corresponding integral equation $Zw^*(y,t)=2$. Then $A(1-y)=w^*(y,y)^2$ so that $w(y,y)=w^*(y,y)$. On the other hand, both w and w^* satisfy the differential equation (2) with the same function $f=f^*$ on the right and the same value $f(y)=f^*(y)$ when $|t|=y$. Since this characteristic boundary problem has a unique solution, it follows that $w^*(y,t)=w(y,t)$ for all $|t|\leqq y\leqq 1$. Now the equation (1) holds for both g and g^*. Hence their difference $h=g-g^*$ satisfies the equation

$$\int_{-y}^{y}\big(h(s-t)+h(t-s)\big)w_y(y,s)\,ds+f(y)\big(h(y+t)+h(y-t)\big) = 0$$

when $|t|\leqq y\leqq 1$. Putting $t=0$ this gives a homogeneous Volterra equation for $h(y)$ so that $h(y)=0$ when $0\leqq y\leqq 1$. But then the equation reduces to

$$\int_{-y}^{0} h(t-s)w_y(y,s)\,ds+f(y)h(y+t) = 0$$

when $y\geqq t\geqq 0$. But then again, $h(y+t)=0$ when $y+t\geqq 0$ so that $h(t)=0$ when $0\leqq t\leqq 2$. This finishes the proof.

5. **The resonaces and antiresonances and the area function.** We shall now prove that there is a unique normalized tube with given resonances and antiresonances having a proper asymptotic behavior.

Theorem 6. (i) *Let* $\delta(t)\to -h(t)/A(1)$ *be the impulse response of the lip transfer of a tube with a* C^∞ *area function* $A(x)$. *Then* $h(t)$ *has the Fourier—Laplace transform* $H(\omega)=-A(1)P(\omega)/U(\omega)$ *where* P, U *is the Fourier—Laplace transform at* $x=1$ *of the glottis reflection pulse of Theorem* 1. *More precisely,*

$$\text{(1)} \qquad \operatorname{Im}\omega < 0 \Rightarrow H(\omega) = \int e^{-2\pi i\omega t}h(t)\,dt$$

with the integral taken in the distribution sense.

(ii) *Conversely, let*

$$0 < \alpha_1 < \beta_1 < \alpha_2 < \dots$$

and suppose that α_n *and* β_n *have asymptotic expansions* (11), (12) *of Theorem* 3, *define* P *and* U *by* (6), (7) *of the same theorem with arbitrary* $\bar{A}>0$ *and put*

$$H(\omega) = -A(1)P(\omega)/U(\omega)$$

where $A(1)$ *is chosen so that* $H(-i\infty)=1$. *Then* (1) *defines a tempered distribution* $h(t)$ *such that*

$$t < 2 \Rightarrow h(t) = \delta(t)+g(t)$$

defines a lip response g. *The corresponding tube is unique when normalized and has the resonances* $\alpha_1, \alpha_2, \dots$ *and antiresonances* $\beta_1, \beta_2, \dots$. *The pair* P, U *is the Fourier—Laplace transform of its glottis reflection pulse.*

Proof. (i) According to Theorem 3(i), $P^*(x, \omega)=P(x, \omega)/U(1, \omega)$ and $U^*(x, \omega)=U(x, \omega)/U(1, \omega)$ are C^∞ functions of x and ω when $\operatorname{Im}\omega\neq 0$, they are $O(e^{2\pi(1-x)\operatorname{Im}\omega})$ when $\operatorname{Im}\omega \leqq \text{const} < 0$ and their derivatives are of at most polynomial growth on lines $\operatorname{Im}\omega=\text{const}\neq 0$. They are also solutions of the transformed Webster system (1) of Section 2 and $U^*=0$ when $x=0$ and $U^*=1$ when $x=1$. Hence their inverse Fourier—Laplace transforms

$$p^*(x, t) = \int P^*(x, \omega)e^{2\pi i\omega t}\,d\omega, \quad u^*(x, t) = \int U^*(x, \omega)e^{2\pi i\omega t}\,d\omega$$

with integration over $\operatorname{Im}\omega=\text{const}<0$ are solutions of the Webster system, vanishing when $t<1-x$, such that $u^*=0$ when $x=0$ and $u^*=\delta$ when $x=1$. This identifies $\delta(t)\to p^*(1, t)$ with the impulse response of the lip transfer so that (1) follows.

(ii) In view of the fact that $\alpha_n=\alpha_{0n}+O(n^{-1})$, $\beta_n=\beta_{0n}+O(n^{-1})$ a comparison factor by factor of $H(\omega)=-A(1)P(\omega)/U(\omega)$ with $H_0(\omega)=-P_0(\omega)/U_0(\omega)$ corresponding to the tube with area function equal to 1 shows that $H(\omega)$ is bounded away from the real axis. Hence, if $c>0$,

$$h(t) = \int_{\operatorname{Im}\omega=-c} e^{2\pi i\omega t}H(\omega)\,d\omega \tag{2}$$

defines a distribution independent of c which vanishes when $t<0$. Also, (2) and (1) are then equivalent. For the residues of H we have the formula

$$\varrho_{\pm n} = 2\pi i \operatorname{Res}(H, \pm\beta_n) = A(1)P(\pm\beta_n)/\bar{A}V'(\pm\beta_n)$$

where $V(\omega)=\omega\Pi(1-\omega^2/\beta_n^2)$. Since the numbers α_n separate the numbers β_n, all $\varrho_n=\varrho_{-n}$ and $\varrho_0=A(1)/\bar{A}$ are positive. Moreover, by the second part of Theorem 4 they have asymptotic expansions

$$\varrho_n \sim 1+c_1\beta_{0n}^{-1}+\dots. \tag{3}$$

Hence, taking residues,

$$t>0 \Rightarrow h(t)=\sum_{-\infty}^{+\infty} \varrho_n e^{2\pi i\beta_n t}$$

with the sum taken in the distribution sense. This shows that h is a tempered distribution and (2) shows that

$$h(t)+h(-t)=\sum \varrho_n e^{2\pi i\beta_n t} \tag{4}$$

for all t. Inserting (3) we can write

$$h(t)+h(-t)=\sum_0^n \sum_0^m c_{jk} Q_j(t) t^k + f_{nm}(t)$$

where $n \geqq 0$, $m \geqq 0$, $f_{nm}(t)$ is of class C^{n+m-2} and

$$Q_j(t)=\sum_{n\neq 0} \beta_{0n}^{-j} e^{2\pi i\beta_{0n} t}, \quad j>0,$$

are the Bernoulli "polynomials", successive integrals of

$$Q_0(t)=\sum e^{2\pi i\beta_{0n} t}=2\sum \delta(t-2n).$$

This means that

$$h(t)=\delta(t)+g(t)$$

where $g(t)=0$ for $t<0$ and g is a C^∞ function when $0\leqq t\leqq 2$. We shall now see that g is a lip response.

Consider the integral equation

$$\int_{-y}^{y} \big(h(s-t)+h(t-s)\big) w(s)\, ds = 0$$

where w is a real C^∞ function and $|t|\leqq y$. Multiplying by $w(t)$, integrating and inserting (4) gives

$$2\sum \varrho_n \left|\int_{-y}^{y} e^{-2\pi i\beta_n s} w(s)\, ds\right|^2 = 0.$$

Hence g is a lip response provided the functions $e^{2\pi i\beta_n t}$ form a complete set in $L^2(-1, 1)$. But this follows from the asymptotics of β_n and a result by N. Levinson ([6] p. 6). In fact, the number $f(x)$ of integers n such that $|\beta_n|\leqq x$ is at least $4x-O(x^{-1})$ for large x and Levinson's criterion requires only that $\int_1^x f(y)dy/y \geqq$ $\geqq 4x-\log x-\text{const.}$

Since g is a lip response, we know from Theorem 5 that there is a unique normalized tube with a C^∞ area function whose lip response is $g(t)$. Let p^*, u^* be its glottis reflection pulse and h^* the impulse response of its lip transfer. With capital letters denoting Fourier—Laplace transforms we then have $H^*(\omega)=$ $=-P^*(\omega)/U^*(\omega)$. Also, $h(t)=h^*(t)$ when $t<2$. Since $h(t)-h^*(-t)$ is a tempered distribution, it follows from this that for some $N>0$, $H(\omega)-H^*(\omega)=$ $=O\big(e^{-4\pi|\mathrm{Im}\,\omega|}(1+|\omega|)^N\big)$ when $\mathrm{Im}\,\omega<-1$. Hence $F(\omega)=A(1)P(\omega)U^*(\omega)-$ $-P^*(\omega)U(\omega)=O\big((1+|\omega|)^N\big)$ in the same region. In fact, a comparison factor by factor of P, P^*, U, U^* with P_0, U_0 show all of them to be $O(e^{2\pi|\mathrm{Im}\,\omega|})$ in the whole complex plane. Since they are bounded close to the real axis, F has the same property. But F is also an odd function and hence $F=O((1+|\omega|)^N)$ everywhere. Hence F is an odd polynomial bounded close to the real axis and has to vanish. Hence $H=H^*$ and $h=h^*$ so that the tube has the resonances α_n and the antiresonances β_n. The proof is finished.

6. **The vowel resonances and the area function.** Finally, we shall show that there exists a tube with given vowel resonance having appropriate asymptotic properties. We shall prove

Theorem 7. *Let* $\omega_1, \omega_2, \ldots$ *be complex numbers and assume that* $\operatorname{Im}\omega_n>0$ *for all* n, *that* $0\leqq \operatorname{Re}\omega_1 \leqq \operatorname{Re}\omega_2 \leqq \ldots$ *and that there is an asymptotic expansion*

$$\omega_n \sim 2^{-1}n-4^{-1}+ic+c_1n^{-1}+c_2n^{-2}+\ldots$$

for large n *where* $c>0$. *Then* $\omega_1, \omega_2, \ldots$ *are the vowel resonances of a unique normalized* C^∞ *tube closed at the glottis and with loss coefficient* $b=\cosh 2\pi c$ *at the lips.*

Proof. Put

$$F(\omega) = \prod_1^\infty (1-\omega/\omega_n)(1+\omega/\bar{\omega}_n)$$

and

$$F(\omega)+F(-\omega) = 2P(\omega), \quad F(\omega)-F(-\omega) = 2F'(0)Q(\omega).$$

By Theorem 4, there are numbers $0<\alpha_1<\beta_1<\ldots$ satisfying the requirements of Theorem 6 such that

$$P(\omega) = \prod_1^\infty (1-\omega^2/\alpha_n^2), \quad Q(\omega) = \omega \prod_1^\infty (1-\omega^2/\beta_n^2).$$

Note that

$$F'(0) = \sum (\omega_n-\bar{\omega}_n)/|\omega_n|^2$$

is purely imaginary with positive imaginary part. Now put

$$U(\omega) = -\bar{A}2\pi i \prod_1^\infty (1-\omega^2/\beta_n^2)$$

with $\bar{A}>0$ chosen so that $P(\omega)/U(\omega)$ tends to -1 as ω tends to $-i\infty$. Then, by virtue of Theorem 6, there is a unique normalized tube with resonances $\alpha_1, \alpha_2, \ldots$ and antiresonances $\beta_1, \beta_2, \ldots$ for which P and U are the Fourier—Laplace transforms at $x=1$ of its glottis reflection pulse. Putting $F'(0)=2\pi i\bar{A}b$ we have $b>0$ and

$$F(\omega) = P(\omega)-bU(\omega).$$

Hence, by Theorem 3(iii), $\omega_1, \omega_2, \ldots$ are the vowel resonances of the tube and Theorem 3(ii) shows that $b=\cosh 2\pi c$. This finishes the proof.

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