Weighted norm inequalities for functions of exponential type

Björn Jawerth

0. Introduction

A result by Plancherel—Polya [12] (cf. Boas [2], p. 101) says that if $f \in R_r$, i.e. if its Fourier Transform is supported by $\{\xi : |\xi| = (\xi_1^2 + \ldots + \xi_n^2)^{1/2} \leqq r\}$, and if $f \in L^p$ for some $p > 0$ then

$$\sum_{k \in Z^n} |f(k)|^p \leqq C \int |f(x)|^p dx. \tag{0.1}$$

It has been recently remarked (cf. e.g. [1]) that a proof of this and other classical inequalities can be based on the ideas of Fefferman—Stein [6]. In fact, let us put

$$n_{r\lambda} f(x) = \sup_{y \in \mathbf{R}^n} |f(x-y)|/(1+r|y|)^\lambda.$$

In [8] we proved (under the said conditions on f) that

$$n_{r\lambda} f \in L^p \tag{0.2}$$

provided $\lambda > n/p$. It is clear that (0.2) implies (0.1).

The purpose of the present paper is to extend the result (0.2) in the respect that we replace the Lebesgue measure by a general positive measure μ on $\mathbf{R}^n$. That is, assuming $f \in \mathbf{E}_r \cap L^p(d\mu)$ we want to prove

$$n_{r\lambda} f \in L^p(d\mu). \tag{0.2'}$$

It turns out that the relevant condition on μ is that

$$\int \frac{d\mu(x)}{(1+r|x-a|)^{ns(1+\varepsilon)}} \leqq C\mu(Q_R(a)) \tag{0.3}$$

for all $\varepsilon > 0$ and $a \in \mathbf{R}^n$. Here $Q_R(a)$ denotes the cube with center a and side R, and $\lambda > ns/p$, $R \approx 1/r$. (For the precise statement see Main Theorem in Sec. 1.)

In a forthcoming publication we intend to use the present results in connection with Besov spaces (cf. [10]).

Finally I would like to express my gratitude to Prof. Peetre for his advice and interest.

1. The main result

The problem considered in this paper is to determine positive measures μ such that, for some fixed r and p with $0<r,\ p<\infty$:

(i) There is a function $f\in\mathbf{E}_r\cap L^p(d\mu),\ f\not\equiv 0$.

(ii) If $f\in\mathbf{E}_r$ then $\|n_{r\lambda}f\|_{L^p(d\mu)}\leqq C\|f\|_{L^p(d\mu)}$ for $\lambda>ns/p$, where s is a given positive number.

Definition 1.1. $M_s(r,p)$ denotes all positive measures μ on $\mathbf{R}^n$ satisfying (i) and (ii) above.

Let $|Q|$ and $\mu(Q)$ denote the Lebesgue and the μ-measure respectively of a cube Q on $\mathbf{R}^n$. We shall prove that $M_s(r,p)$ can be described by the following family of measures:

Definition 1.2. $A_s(R)$, $s>0$, denotes all positive measures μ on $\mathbf{R}^n$ satisfying

$$\frac{\mu(Q)}{\mu(E)}\leqq C|Q|^s$$

for some constant $C=C(\mu)$ and all pairs of cubes E, Q such that $E\subset Q$ and the side of E equals R.

Now the main result reads

Main Theorem. Let $0<r,p<\infty$ be given. Then there exist positive constants R_0 and R_1 such that

$$A_{s+\varepsilon}(R_0)\subset M_s(r,p)\subset A_{s+\varepsilon}(R_1)$$

for every $\varepsilon>0$.

In applications we need a similar result which is independent of r. We therefore introduce another family of measures:

Definition 1.3. B_s, $s>0$, denotes all positive measures μ on $\mathbf{R}^n$ satisfying

$$\frac{\mu(Q)}{\mu(E)}\leqq C\left(\frac{|Q|}{|E|}\right)^s$$

for some constant $C=C(\mu)$ and all pairs of cubes E, Q such that $E\subset Q$.

The subsequent proof of the main theorem also gives with minor modifications the following

Corollary. Let μ be a positive measure on $\mathbf{R}^n$. The following two conditions are equivalent:

(1) $\mu \in B_{s+\varepsilon}$ for each $\varepsilon > 0$.

(2) For each r and p with $0 < r, p < \infty$

(i) There is a function $f \in \mathbf{E}_r \cap L^p(d\mu)$, $f \not\equiv 0$.

(ii) If $f \in \mathbf{E}_r$ there is a constant C, independent of r, such that $\|n_{n\lambda} f\|_{L^p(d\mu)} \leqq C\|f\|_{L^p(d\mu)}$ for all $\lambda > ns/p$.

2. Preliminaries

Later it will be convenient to have another characterization of $A_{s+\varepsilon}(R)$ ($B_{s+\varepsilon}$ can be described in a similar way): μ satisfies $A_{s+\varepsilon}(R)$ for each $\varepsilon > 0$ if and only if there is a constant C_ε such that

$$\int \frac{d\mu(x)}{(1+|x-a|/R)^{ns(1+\varepsilon)}} \leqq C_\varepsilon \mu(E) \tag{2.1}$$

for any cube E with center a and side R.

Let us also compare A_s and B_s with two other conditions:

If $d\mu(x) = w(x)dx$ and $w(x)$ is a temperate weight function of order N in the sense of Hörmander (cf. [7], p. 34), then $\mu \in A_{1+N/n}(R)$ for each $R > 0$.

If $d\mu(x) = w(x)dx$ and w satisfies the so called A_∞-condition, appearing in connection with for example weighted inequalities for the Hardy—Littlewood maximal operator and singular integrals (cf. [4], [9]), then $\mu \in B_s$ for some $s > 0$. The converse is not true: It is easy to see that $\mu \in B_s$ for some $s > 0$ if and only if there is a constant $C = C(\mu)$ such that

$$\mu(2Q) \leqq C\mu(Q) \tag{2.2}$$

for any cube Q. (Here $2Q$ is the cube with the same center as Q but with sides twice as long.) That (2.2) is not equivalent to A_∞ was shown in [5].

We also need the following fact. Let

$$m_\mu f(x) = \sup_{Q(x)} 1/\mu(Q(x)) \int_{Q(x)} |f| \, d\mu$$

where the supremum is taken over all cubes $Q(x)$ with center x. From [3] we have

Hardy—Littlewood maximal theorem. Let μ be a positive measure on $\mathbf{R}^n$. Then there is a constant $C = C(n, p)$ such that

$$\|m_\mu f\|_{L^p(d\mu)} \leqq C\|f\|_{L^p(d\mu)}$$

for every $p > 1$.

3. Proof of the main theorem

Let us first prove that there exists some $R_0>0$ such that

$$A_{s+\varepsilon}(R_0) \subset M_s(r, p)$$

for every $\varepsilon>0$. The inclusion will be an easy consequence of two lemmata based on ideas in [11].

Lemma 3.1. Suppose that f is a C^1 function on $\mathbf{R}^n$ and that $\mu \in A_{s+\varepsilon}(R_0)$ for each $\varepsilon>0$. Then there is a constant $C=C(\varepsilon, \lambda, n)$ such that

$$n_{r\lambda} f \leqq C\{r^{-\lambda}(m_\mu |f|^a)^{1/a} + R_0 n_{r\lambda}(\nabla f)\}$$

if $R_0 \leqq 1/r$ and $a=(1+\varepsilon)ns/\lambda$.

Proof: By the mean value theorem

$$|f(x-y)| \leqq C\left\{\left(\frac{1}{\mu(Q)}\int_Q |f|^a d\mu\right)^{1/a} + R_0 \sup_Q |\nabla f|\right\}$$

$$\leqq C\left\{\left(\frac{\mu(Q^*)}{\mu(Q)} m_\mu |f|^a(x)\right)^{1/a} + R_0\left(1+r(R_0+|y|)\right)^\lambda n_{r\lambda}(\nabla f)(x)\right\}$$

where $Q=Q_{R_0}(x-y)$ and $Q^*=Q_{R_0+2|y|}(x)$. Since $\mu \in A_{s+\varepsilon}(R_0)$ and if $R_0 \leqq 1/r$ we get

$$|f(x-y)| \leqq C\{r^{-\lambda}(m_\mu |f|^a(x))^{1/a} + R_0 n_{r\lambda}(\nabla f)(x)\} \cdot (1+r|y|)^\lambda.$$

This clearly gives the desired inequality.

Lemma 3.2. Let $f \in \mathbf{E}_r$. Then

$$n_{r\lambda}(\partial_i f) \leqq Crn_{r\lambda} f$$

where $C=C(\lambda, n)$.

Proof: Choose $v \in \mathcal{S}$ with $\hat{v}$ equal to 1 on $|\xi| \leqq 1$. Then $f=v_r * f$ and $\partial_i f = r(\partial_i v)_r * f$ with $v_r(x)=r^n v(rx)$. Hence

$$|\partial_i f(x-y)| \leqq r^{n+1}\int |\partial_i v(rz)|(1+r|y+z|)^\lambda dz\, n_{r\lambda} f(x) \leqq$$

$$\leqq r\int |\partial_i v(z)|(1+|z|)^\lambda dz (1+r|y|)^\lambda n_{r\lambda} f(x)$$

where we have used the inequality

$$(1+u+v) \leqq (1+u)(1+v), \quad u, v \geqq 0.$$

With $C=\max_i \int |\partial_i v(z)|\,(1+|z|)^\lambda dz$ our assertion follows.

Using (2.1) we see that any function $f \in \mathbf{E}_r \cap \mathcal{S}$, $f \not\equiv 0$, fulfils the first requirement of $M_s(r, p)$.

When proving the second we may assume $\lambda \leqq 2ns/p$. Suppose $n_{r\lambda}f<\infty$. Then by Lemma 3.1 and 3.2

$$n_{r\lambda}f(x) \leqq C(m_\mu |f|^a(x))^{1/a}.$$

Therefore, with $\varepsilon>0$ so small that $p/a>1$, the Hardy—Littlewood maximal theorem (Sec. 2) gives

$$\|n_{r\lambda}f\|_{L^p(d\mu)} \leqq C\|f\|_{L^p(d\mu)}.$$

This is at least true if $n_{r\lambda}f<\infty$. Since $f\in \mathbf{E}_r$ does not grow faster than $|x|^q$ as $|x|\to\infty$ for some q, the general case follows if we study $f(x)\prod_{i=1}^n(\sin \gamma x_i/\gamma x_i)^j$ for large j and let $\gamma\to 0$.

We shall also prove that

$$M_s(r, p) \subset A_{s+\varepsilon}(R_1)$$

for each $\varepsilon>0$. A simple lemma will be needed:

Lemma 3.3. If μ belongs to $M_s(r,p)$ then $(1+|x-h|)^{-j}\in L_1(d\mu)$ for $j>ns$ and for any $h\in\mathbf{R}^n$.

Proof: Let f denote the function given by $M_s(r,p)$. As $|f(x-h)|\leqq \leqq(1+r|h|)^\lambda n_{r\lambda}f(x)$ we can assume that $f(h)=1$. The lemma is a consequence of the estimates

$$\int \frac{d\mu(x)}{(1+r|x-h|)^{\lambda p}} \leqq \|n_{r\lambda}f\|^p_{L^p(d\mu)} \leqq C\|f\|^p_{L^p(d\mu)}<\infty.$$

To prove the inclusion, fix two cubes E, Q with $E\subset Q$ and $|E|=R_1^n$ and let $\varepsilon>0$ be given. Setting

$$f(x) = \prod_{i=1}^n\left\{\sin\frac{r}{j}(x_i-a_i)\Big/\frac{r}{j}(x_i-a_i)\right\}^j$$

where a is the center of E and j is an integer, we know that $f\in\mathbf{E}_r$. Now, using $M_s(r,p)$,

$$\int\frac{d\mu(x)}{(1+r|x-a|)^{\lambda p}} \leqq \|n_{r\lambda}f\|^p_{L^p(d\mu)} \leqq C\|f\|^p_{L^p(d\mu)}.$$

Hence, with $\lambda p=(1+\varepsilon)ns$ and $j\approx 2\lambda$,

$$\int\frac{d\mu(x)}{(1+r|x-a|)^{(1+\varepsilon)ns}} \leqq C\left\{\int_{x\in E}d\mu(x)+\int_{x\notin E}|f(x)|^p d\mu(x)\right\} \leqq$$

$$\leqq C\mu(E)+\frac{1}{2}\int\frac{d\mu(x)}{(1+r|x-a|)^{(1+\varepsilon)ns}}$$

if we choose E (i.e. R_1) large enough. By Lemma 3.3 both sides are finite. Thus

$$\int\frac{d\mu(x)}{(1+r|x-a|)^{(1+\varepsilon)ns}} \leqq C\mu(E).$$

This completes the proof of the theorem.

References

1. BERGH J. and PEETRE J., On the spaces V_p $(0<p\leqq\infty)$. *Boll. Un. Mat. Ital.* **3** (1970), 632—648.
2. BOAS R. P., *Entire functions.* Academic Press, New York, 1954.
3 CAFFARELLI L. A. and CALDERÓN C. P., Weak type estimates for the Hardy—Littlewood maximal function. *Studia Math.* **49** (1973—74), 217—223.
4. COIFMAN R. and FEFFERMAN C., Weighted norm inequalities for maximal functions and singular integrals. *Studia Math.* **51** (1974), 241—250.
5. FEFFERMAN C. and MUCKENHOUPT B., Two nonequivalent conditions for weight functions. *Proc. Amer. Math. Soc.* **45** (1974), 99—104.
6. FEFFERMAN C. and STEIN E. M., H^p spaces of several variables. *Acta Math.* **129** (1972), 137—193.
7. HÖRMANDER L., *Linear partial differential operators.* Springer-Verlag, Berlin—Heidelberg—New York, 1969.
8. JAWERTH B., Om hela funktioner av exponentialtyp. *Technical report*, Lund, 1974.
9. MUCKENHOUPT B., Weighted norm inequalities for classical operators. *To appear.*
10. PEETRE J., *New thoughts on Besov spaces.* Duke University Press, Durham. *To appear.*
11. PEETRE J., On spaces of Triebel—Lizorkin type. *Ark. Mat.* **13** (1975), 123—130.
12. PLANCHEREL M. and POLYA G., Fonctions entières et intégrales de Fourier multiples. *Comment. Math. Helv.* **9** (1937), 224—248.

Received November 15, 1975
in revised form January 15, 1977

Björn Jawerth
Department of Mathematics
University of Lund
Box 725
220 07 Lund
Sweden