Weak estimates on maximal functions with rectangles in certain directions

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§ 1. Introduction

Let $\mathscr{I}$ be the family of intervals in $\mathbf{R}^n$. We use the term interval in the n-dimensional sense for sets of the form $\{(x_1, \ldots, x_n);\ x_i \in I_i,\ i=1, \ldots, n\}$ where I_i, $i=1, \ldots, n$ are one-dimensional intervals.

Consider the maximal function $M_{\mathscr{I}} f$ of the function f defined by

$$M_{\mathscr{I}} f(x) = \sup_{x \in I \in \mathscr{I}} \frac{1}{\mu(I)} \int_I |f(y)|\, dy.$$

Here μ denotes the Lebesgue measure in $\mathbf{R}^n$. It was shown by Jessen, Marcinkiewicz & Zygmund [4] that

$$\mu\{M_{\mathscr{I}} f(x) > \alpha\} < C \int_{\mathbf{R}^n} |f(y)/\alpha| (1+\log^+ |f(y)/\alpha|)^{n-1} dy, \quad \alpha > 0. \tag{1}$$

If we in the definition above replace $\mathscr{I}$ by the larger class R of 'rectangles' obtained by all orthogonal transformations of intervals in $\mathscr{I}$, we get another maximal function $M_R f$. It is possible to find sets E_N for every large $N > 0$ such that the following inequality holds for the characteristic function χ_{E_N}

$$\mu\left\{M_R \chi_{E_N}(x) > \frac{1}{8}\right\} > N \mu(E_N). \tag{2}$$

A construction leading to such sets is the Perron tree. Perron [5] simplified an orginal construction of Besicovitch [1] and the construction has been further simplified by Rademacher [6] and Schoenberg [7]. We will also refer to M. de Guzmán's Lecture Notes [3] where this construction is described and also many other solved and open problems about different kinds of maximal functions are considered.

In this paper we will consider maximal functions on $\mathbf{R}^2$ where the supremum is taken over all rectangles with certain given directions (containing the given point).

The set of permitted directions is countable and the direction will converge in an exponential way to a limit direction.

The main result of this paper is a weak type estimate for this kind of maximal functions. These weak type estimates are slightly weaker (with some logaritms) than weak (2,2) and will thus imply that the maximal operators are bounded in L^p for p larger than 2.

In Section 2 we give the condition for the set of directions and define the corresponding maximal function. The main result is stated in Section 3 in Theorem 1 and Theorem 2. Theorem 1 is restricted to characteristic functions, and Theorem 2 follows from Theorem 1. For the proof of Theorem 1 we use two geometric estimates which are proved in Section 4. The proof of Theorem 1 is given in Section 5. In Section 6 we mention three applications. In Section 7 we state how the main result can be generalized to larger sets of directions. In Section 8 we discuss the sharpness of the estimates and mention some open problems.

Finally, I would like to express my deep gratitude to Professor Lennart Carleson for the helpful discussions which led me to consider this problem, for his advice and interest.

§ 2. Preliminaries

Let $R(\varphi)$ be the family of rectangles S in $\mathbf{R}^2$ such that the angle between the longest side of S and the x_1-axis is φ. We call φ the *direction* of S.

Given a set Φ of directions we define the maximal function $M_\Phi f$ by

$$M_\Phi f(x) = \sup_S \frac{1}{\mu(S)} \int_S |f(y)|\, dy,$$

where the supremum is taken over all rectangles S in the families $R(\varphi)$, $\varphi \in \Phi$, containing the point x. (μ is the Lebesgue measure in $\mathbf{R}^2$.)

Condition on the set Φ. We consider only countable $\Phi = \{\varphi_i\}_{i=1}^\infty$ where φ_i converge to some direction φ_∞ as $i \to \infty$. Further we assume that $\{\varphi_i\}_{i=1}^\infty$ satisfies the following condition:

$$(3) \qquad |\varphi_j - \varphi_i| > C|\varphi_i - \varphi_\infty|, \quad i \neq j$$

for some $c>0$. (In this paper C and c are used for constans that may differ from place to place.) We say that such sets Φ are *exponential.*

Remark. If φ_i converge monotonically to φ_∞ and also $|\varphi_{i-1} - \varphi_i|$ decrease as $i \to \infty$, then either (3) will hold, or it is for every large $N>0$ possible to find sets E_N such that (2) holds (with $M_{\mathscr{I}}$ replaced by M_Φ).

We will in the following use the symbol χ for characteristic functions, with the corresponding set as an index.

§ 3. Main result

Let Φ be an exponential set of directions. Then we have the following weak type estimates for the corresponding maximal functions.

Theorem 1. *Let χ_E be the characteristic function of the measurable set E in $\mathbf{R}^2$. Then*

$$\mu\{M_\Phi \chi_E(x) > \alpha\} \leqq C\alpha^{-2}(1+\log(1/\alpha))^2\mu(E)$$

for all $0<\alpha\leqq 1$. The constant C depends only on Φ.

Theorem 2. *Let $\varepsilon>0$ and let f be a function on $\mathbf{R}^2$. Then*

$$\mu\{M_\Phi f(x) > \alpha\} \leqq C\int_{\mathbf{R}^2} |f(y)/\alpha|^2(1+\log^+|f(y)/\alpha|)^{4+\varepsilon}dy$$

for all $\alpha>0$, if the integral is finite. The constant C depends only on ε and Φ.

Theorem 2 follows from Theorem 1: Set $a=\sum_{j=0}^{\infty}(1+j)^{-1-(\varepsilon/3)}$ and let $E_0=\{|f(x)|\leqq\alpha/a\}$ and $E_j=\{2^{j-1}\alpha/a<|f(x)|\leqq 2^j\alpha/a\}$, $0<j<\infty$. Then the set $\{M_\Phi f(x)>\alpha\}$ is contained in the union of the sets $\{M_\Phi\chi_{E_j}(x)>2^{-j}j^{-1-(\varepsilon/3)}\}$. From Theorem 1 follows that the measure of this union is less than $C\sum_{j=1}^{\infty}2^{2j}(1+j)^{+4+\varepsilon}\mu(E_j)$ which is bounded by the right side of the inequality in Theorem 2.

§ 4. Two auxiliary geometrical estimates

We shall in this section state two geometrical estimates which are needed in the proof of Theorem 1. The first one, Lemma 1 follows from a rather simple geometric observation. The second one, Lemma 2 is shown by means of Lemma 2 and (1). In the final step of the proof we get Theorem 1 from Lemma 2 and (1).

Let us first make some more assumptions on Φ. By splitting the set Φ into finitely many subsets we can without loss of generality assume that $\varphi_\infty=0$, $0<\varphi_i<\pi/4$ and $2^i\beta \operatorname{tg}\varphi_i\in[19/20, 21/20]$ for some $1\leqq\beta<2$. We also assume that $\beta=1$ as β has no important role in the proof.

The following lemma concerns the intersections of rectangles with two different directions.

Lemma 1. *Let T be an interval and let φ_0 be the direction of one of its diagonals. Further let φ_i and φ_j be two different directions in Φ (we may also have $j=\infty$), one of which is close to φ_0; more precisely, $|\varphi_i-\varphi_0|<\varphi_i/9$ or $|\varphi_j-\varphi_0|<\varphi_j/9$. Further, let d_i and d_j be the distance from the lower left hand corner of T along the φ_i (resp. φ_j) direction to one of the opposite sides.*

Let $S\in R(\varphi_i)$ of length $>d_i/64$ and F_j be a union of some $S'\in R(\varphi_j)$ of lengths $>d_j/64$. Then

$$\mu(F_j\cap S\cap T)\leqq C\mu(S)\mu(F_j\cap 2T)/\mu(T)$$

($2T$ is the interval with same center as T but with doubled sides).

The following lemma concerns the intersections of rectangles with infinitely many directions.

Lemma 2. *Let T be an interval with one diagonal in the direction $\varphi_i\in\Phi$, and let d_j be the shortest distance between parallel sides of T in the φ_j direction for $\varphi_j\in\Phi$.*

Let $S\in R(\varphi_i)$ with length $>d_i/8$ and F_j, $j\neq i$, be the unions of some $S'\in R(\varphi_j)$ of lengths $>d_j/8$ and F be the union of the sets F_j, $j\neq i$. Then

$$\mu(F\cap S\cap T)\leqq C\mu(S)\mu(F\cap 2T)/\mu(T).$$

Proof of Lemma 1. We make a linear map ϱ from $\mathbf{R}^2$ to $\mathbf{R}^2$ which maps φ_i and φ_j into orthogonal directions such that d_i and d_j correspond to the length 1. Now let us first show that $\mu(\varrho(T))<C$. We can estimate $\mu(T)$ by $Cd_id_j\sin\max(\varphi_i,\varphi_j)$ and since ϱ changes the area by $[d_id_j\sin|\varphi_i-\varphi_j|]^{-1}$ we get $\mu(\varrho(T))<C\sin\max(\varphi_i,\varphi_j)/\sin|\varphi_i-\varphi_j|$. From (3) we see that this quotient is bounded, and here is the only time in the proof of Theorem 1 where we really use that Φ is exponential. We observe that $\varrho(S)$ and $\varrho(F\cap 2T)$ consist of line segments of lengths $>1/64$ which are orthogonal. From this we conclude that $\mu(\varrho(F\cap T\cap S))\leqq C\mu(\varrho(F\cap 2T))\mu(\varrho(S))$. Hence

$$\frac{\mu(F\cap T\cap S)}{\mu(S)}=\frac{\mu(\varrho(F\cap T\cap S))}{\mu(\varrho(S))}\leqq C\frac{\mu(\varrho(F\cap 2T))\mu(\varrho(S))}{\mu(\varrho(S))}$$
$$\leqq C\frac{\mu(\varrho(F\cap 2T))}{\mu(\varrho(T))}\leqq C\frac{\mu(F\cap 2T)}{\mu(T)}$$

which proves Lemma 1.

Proof of Lemma 2. We assume that $j<i$ (the case $j>i$ could be proved in the same way).

Let I_T^1 and I_T^2 be the one dimensional intervals defined by $T=I_T^1\times I_T^2$ and let $\mathscr{I}_m$ consist of intervals of the form $I_m=I_m^1\times 2I_T^2$ where I_m^1 belongs to the dyadic decomposition of $2I_T^1$ in 2^m equal intervals.

For a rectangle S' in the union F_j with $S'\cap T\neq\emptyset$ one of the following cases will occur

Case 1. $S'\cap 2T$ is contained in an interval $I^{S'}$ with $\mu(I^{S'})\leqq 100\mu(S'\cap 2T)$,

Case 2. $S'\cap T\subset V^{S'}\subset S'\cap 2T$ for some $V^{S'}\in R(\varphi_j)$ with length $>d_j/8$ such that $V^{S'}\subset 4I$ for every $I\in\mathscr{I}_{i-j}$ intersecting $V^{S'}$. We use the notation $I_{i-j}^{S'}$ for such a dyadic interval.

Case 1 will occur when the length of S' in the φ_∞-direction is large, say larger than 2^{j-i-1} times the length of I_T^1. Otherwise Case 2 will occur.

We will now split the rectangles in the union F into the following families R_m, $m \geqq 0$:

R_0 consists of of those rectangles S' for which Case 1 occurs,

R_m, $0<m<i$, consists of all rectangles in F_{i-m} with non-empty intersection with T for which Case 2 occurs,

R_m, $m \geqq i$ is defined to be empty for convenience.

Now we shall define the subfamilies $R_m^* \subset R_m$, $m>0$, by induction over m. Set

$$R_1^* = R_1,$$

and for $m>1$ set $R_m^* = \{S' \in R_m$ such that $V^{S'}$ satisfies (4) below$\}$;

$$\mu\Big\{V^{S'} \cap \Big(\bigcup_{\nu<m} \bigcup_{S'' \in R_\nu^*} V^{S''}\Big)\Big\} < \frac{1}{2}\mu\{V^{S'}\}. \tag{4}$$

We now consider the following unions of rectangles

$$G_0 = \bigcup_{S' \in R_0} I^{S'}, \quad G_m = \bigcup_{S' \in R_m^*} V^{S'}, \quad 0 < m < \infty,$$

$$G_\infty = \bigcup_{0<m<\infty} \bigcup_{S' \in (R_m \setminus R_m^*)} 4I_m^{S'}.$$

We observe that the set $F \cap T$ is contained in the union of these unions G_m, $0 \leqq m \leqq \infty$, and that each of these G_m is a union of rectangles with the same direction. Thus $\mu(F \cap S \cap T) \leqq \sum_{0 \leqq m \leqq \infty} \mu(S \cap T \cap G_m)$ and applying Lemma 1 we get the desired estimate in Lemma 2 if we can show that

$$\sum_{0 \leqq m \leqq \infty} \mu(G_m) \leqq C\mu(F \cap 2T). \tag{5}$$

Let us show (5). First we shall estimate $\mu(G_0)$. Since $\mu(I^{S'}) \leqq 100\mu(S' \cap 2T)$ for $S' \in R_0$ we observe that the union G_0 of these intervals $I^{S'}$ is contained in the set where the maximal function $M_{\mathscr{I}} \chi_{F \cap 2T}$ is not less than 1/100. Hence by (1) we get $\mu(G_0) \leqq C\mu(F \cap 2T)$.

In order to estimate the sum $\sum_{0<m<\infty} \mu(G_m)$ we define the disjoint subsets H_m of $F \cap 2T$, $0<m<\infty$, by

$$H_1 = G_1,$$
$$H_m = G_m \setminus \Big(\bigcup_{\nu<m} G_\nu\Big), \quad 1 < m < \infty.$$

By (4) we get that $\mu(V^{S'} \cap H_m) > (1/2)\mu(V^{S'})$ for $S' \in R_m^*$, $0<m<i$. Hence we conclude that the union G_m of these rectangles $V^{S'}$ is contained in the set where the maximal function with rectangles in the φ_{i-m} direction $M_{\varphi_{i-m}} \chi_{H_m}$ is larger than 1/2. Since this is a maximal function with rectangles in only one direction, we can use (1) again to get $\mu(G_m) \leqq C\mu(H_m)$, $0<m<i$. Since G_m, $i<m<\infty$, is empty and H_m

are disjoint subsets of $F\cap 2T$ we get by summation the estimate $\sum_{0<m<\infty}\mu(G_m)\leqq \leqq C\mu(F\cap 2T)$.

It remains now to show that $\mu(G_\infty)\leqq C\mu(F\cap 2T)$. We consider a rectangle $V^{S'}$ with $S'\in(R_m\setminus R_m^*)$, $0<m<\infty$. It does not satisfy (4), i.e. $\mu(\bigcup_{v<m}(G_v\cap V^{S'}))\geqq \geqq(1/2)\mu(V^{S'})$. Thus we have

$$\sum_{v<m}\frac{\mu(G_v\cap V^{S'})}{\mu(V^{S'})}\geqq\frac{1}{2}\quad\text{for}\quad S'\in(R_m\setminus R_m^*),\quad 0<m<\infty. \tag{6}$$

Let $I_v^{S'}\in\mathscr{I}_v$, $v<m$, be the dyadic intervals containing $I_m^{S'}$. Then $V^{S'}$ is contained in $4I_v^{S'}$ for every $v\leqq m$. Applying Lemma 1 to each term in the sum in (6) we get

$$\sum_{v<m}\frac{\mu(G_v\cap 8I_v^{S'})}{\mu(4I_v^{S'})}\geqq C\sum_{v<m}\frac{\mu(G_v\cap V^{S'})}{\mu(V^{S'})}\geqq C<0, \tag{7}$$
$$S'\in(R_m\setminus R_m^*),\quad 0<m<\infty.$$

Now we define the function ψ by

$$\psi(x)=\sum_{0<v<\infty}\sum_{I_v\in J_v}\frac{\mu(G_v\cap 8I_v)}{\mu(4I_v)}\chi_{4I_v}(x).$$

Here the sum is taken over all dyadic intervals I_v in $\mathscr{I}_v$ for all v. From (7) we see that ψ is larger than a positive constant on $I_m^{S'}$ for $S'\in(R_m\setminus R_m^*)$, $0<m<\infty$. Thus the union G_∞ is contained in the set $\{\psi(x)\geqq c\}$ for a constant $c>0$ and we get

$$\mu(G_\infty)\leqq\mu\{\psi(x)\geqq c\}\leqq C\int_{\mathbf{R}^2}\psi(x)\,dx\leqq C\sum_{0<v<\infty}\sum_{I_v\in\mathscr{I}_v}\mu(G_v\cap 8I_v)$$
$$\leqq C\sum_{0<v<\infty}\mu(G_v)\leqq C\mu(F\cap 2T).$$

By summation we get (5). The proof of Lemma 2 is complete.

§ 5. Proof of Theorem 1

As before we make the restriction $2^j\operatorname{tg}\varphi_j\in[19/20, 21/20]$.

Let $R_{j,k}$ $j=1,2,\dots$, $k=0,\pm1,\pm2,\dots$, be the families of rectangles $S\in R(\varphi_j)$ with $2^{-k-1}<$ length of $S\leqq 2^{-k}$, such that

$$\mu(S\cap E)>\alpha\mu(S). \tag{8}$$

Then

$$\{M_\Phi\chi_E(x)>\alpha\}=\bigcup_{j,k}\bigcup_{S\in R_{j,k}}S.$$

By a change of scale it is enough to estimate the measure of

$$\bigcup_{0<j,k\leqq N}\bigcup_{S\in R_{j,k}}S \tag{9}$$

for large $N>0$. From now on N is fixed and $j,k=1,2,\dots,N$.

Let $m=1, 2, N^2$ be an enumeration of the index pair (j, k) given by $m=N(k-1)+j$, and let $\mathcal{I}_m$ be a decomposition of $\mathbf{R}^2$ in intervals for each of which one diagonal has length 2^{-k} and the direction is φ_j.

For $S\in R_{j,k}$ one of the following two cases occurs

Case 1. S is contained in an interval I^S with $\mu(I^S)\leqq 10\mu(S)$.

Case 2. S is contained in $4I$ for every interval $I\in\mathcal{I}_m$ intersecting S. We use the notation I^S_m for such an interval.

Now we split up the rectangles in the following families R_m $m=0, 1, \dots, N^2$, R_0 consists of those rectangles in $R_{j,k}$, all j, k, for which Case 1 occurs,

R_m, $0<m\leqq N^2$, consists of those rectangles in $R_{j,k}$, $N(k-1)+j=m$, for which Case 2 occurs.

As in the proof of Lemma 2 we define some subfamilies $R^*_m\subset R_m$, $m>0$, by induction over m.

Set

$$R^*_1 = R_1,$$

for $m>1$ set $R^*_m=\{S\in R_m$ such that S satisfies (10) below$\}$;

$$\mu\{S\cap F^{(m)}\} < \frac{\alpha}{2}\mu(S) \quad \text{where we have put} \quad F^{(m)} = \bigcup_{\substack{0<\nu<m \\ \nu\neq m \bmod N}} \bigcup_{S\in R^*_\nu} S. \tag{10}$$

We now consider the following unions of rectangles

$$G_0 = \bigcup_{S\in R_0} I^S, \quad G_j = \bigcup_{\substack{0<m\leqq N^2 \\ m=j \bmod N}} \bigcup_{S\in R^*_m} S, \quad 0<j\leqq N,$$

$$G_\infty = \bigcup_{0<m\leqq N^2} \bigcup_{S\in(R_m\setminus R^*_m)} 4I^S_m.$$

We observe that the union in (9) is contained in the union of the sets G_m, $m=0, 1, \dots, N^2, \infty$. Thus we get the desired estimate of Theorem 1 if we can show that

$$\mu(G_0)+\textstyle\sum_{0<m\leqq N}\mu(G_j)+\mu(G_\infty) \leqq C\alpha^{-2}(1+\log(1/\alpha))^2\mu(E). \tag{11}$$

Let us show (11). First we shall estimate $\mu(G_0)$. Since $\mu(I^S)\leqq\mu(S)$ for $S\in R_0$ we see by (8) that $\mu(I^S\cap E)>(\alpha/10)\mu(I^S)$, $S\in R_0$. Thus the union G_0 of these intervals I^S is contained in the set where the maximal function $M_{\mathcal{I}}\chi_E$ is not less than $\alpha/10$. Hence we get by (1) $\mu(G_0)\leqq C\alpha^{-1}(1+\log(1/\alpha))\mu(E)$.

In order to estimate the sum $\sum_{0<j\leqq N}\mu(G_j)$ we define the sets H_m, $0<m\leqq N^2$ by

$$H_1 = \bigcup_{S\in R^*_1} S$$

$$H_m = \Big(\bigcup_{S\in R^*_m} S\Big)\setminus F^{(m)}, \quad 1<m\leqq N^2.$$

Let E_j, $j=1, \ldots, n$, be subsets of E defined by

$$E_j = E \cap \Big(\bigcup_{\substack{0<m\leqq N^2 \\ m=j \bmod N}} H_m \Big).$$

It is easy to check that the sets E_j are disjoint: If x is a point in E_j then there is a least m such that x is in $\bigcup_{S\in R_m^*} S$, and if x is in $H_{m'}$ then $m'=m \pmod N$. As x is in E_j we must have $m'=j \pmod N$ which implies that $E_{j'}$ does not contain x for $j'\neq j$.

Now we consider a rectangle $S\in R_m^*$ and the set E_j, $m=j \bmod N$. The following set inclusions yield

$$S \cap E_j \supset S \cap H_m \cap E \supset (S \cap E)\setminus(S \cap F^{(m)}).$$

Hence we get by (8) and (10)

$$\mu(S \cap E_j) > \alpha/2\mu(S) \quad \text{for} \quad S\in R_m^*, m = j \bmod N, \quad 0 < m \leqq N^2, \quad 0<j\leqq N. \tag{12}$$

Since a union G_j, $0<j\leqq N$ consists of rectangles only in the direction φ_j we conclude by (12) that G_j is contained in the set where the maximal function with rectangles in this direction $M_{\varphi_j}\chi_{E_j}$ is larger than $\alpha/2$. Since this is a maximal function with rectangles in only one direction we can use (1) to get

$$\mu(G_j) \leqq C\alpha^{-1}\big(1+\log(1/\alpha)\big)\mu(E_j).$$

By summation, using that E_j are disjoint subsets of E, we get

$$\bigcup_{0<j<N} \mu(G_j) \leqq C\alpha^{-1}\big(1+\log(1/\alpha)\big)\mu(E). \tag{13}$$

Now it remains to estimate $\mu(G_\infty)$. We consider a rectangle $S\in(R_m\setminus R_m^*)$, $0<m\leqq N^2$. It does not satisfy (10), i.e.

$$\frac{\mu(F^{(m)} \cap S)}{\mu(S)} \geqq \frac{\alpha}{2}. \tag{14}$$

Now we shall use Lemma 2 with T replaced by $4I_m^S$ and with the rectangle S and the union $F^{(m)}$ of rectangles S'. We observe that the directions of the rectangles S' in $F^{(m)}$ are different from the direction of S and that the lengths of the rectangles S' are large enough compared to the interval $4I_m^S$ in order to fulfil the conditions in Lemma 2. We get the following inequalities

$$\frac{\mu\Big\{8I_m^S \cap \Big(\bigcup_{0<\nu<\infty} G_j\Big)\Big\}}{\mu\{8I_m^S\}} \geqq \frac{1}{4}\,\frac{\mu\{8I_m^S \cap F^{(m)}\}}{\mu\{4I_m^S\}} \geqq \frac{1}{C}\,\frac{\mu(F^{(m)} \cap S)}{\mu(S)} \geqq \frac{\alpha}{C} \quad \text{for} \quad S\in(R_m\setminus R_m^*), \quad 0<m<\infty. \tag{15}$$

In the first inequality of (15) we have only used that $F^{(m)}$ is contained in the union of the sets G_j, $0<j<\infty$, for the second inequality we use Lemma 2 and also that S is contained in I_m^S, the third inequality is (14).

From (14) we conclude that G_∞ is contained in the set where the maximal function $M_{\mathcal{J}}\chi_{\bigcup_{0<j<\infty}G_j}$ is not less than α/C, if the constant C is large enough. By (1) we now get

$$\mu(G_\infty) \leqq C\alpha^{-1}(1+\log(1/\alpha)) \sum\nolimits_{0<j<\infty} \mu(G_j),$$

and by (13) we now get

$$\mu(G_\infty) \leqq C\alpha^{-2}(1+\log(1/\alpha))^2\mu(E). \tag{16}$$

By summation of (13), (16) and our estimate of $\mu(G_0)$ we get (11).

This completes the proof of Theorem 1.

§ 6. Application

We give one application concerning maximal functions defined by means of polygons. We also mention two applications given by A. Cordoba and R. Fefferman in [2], the second one also together with C. Fefferman. The main result of this paper is used to get two estimates for multipliers.

I. Let P be a polygon in $\mathbf{R}^2$ with infinitely many sides L_j, $j=0, \pm 1, \pm 2, \ldots$, given by their endpoints $(2^{-j}, 2^{-2j})$ and $(2^{-j-1}, 2^{-2(j+1)})$. Let $P_t=\{x\in P;\ |x|<t\}$, $t>0$, and let $d\sigma$ be the length measure on P.

We define the maximal function $M_P f$ of the function f by

$$M_P f(x) = \sup_{t>0} \frac{1}{\sigma(P_t)} \int_{P_t} |f(x+y)|\, d\sigma(y).$$

Then we get the following estimate

Theorem 3. $\mu\{M_P\chi_E(x)>\alpha\} \leqq C\alpha^{-2}(1+\log(1/\alpha))\mu(E)$ *for all* $0<\alpha\leqq 1$ *and all measurable sets* E *in* $\mathbf{R}^2$.

Proof: Let φ_j be the direction of L_j. Then $\Phi=\{\varphi_j\}_1^\infty$ is an exponential set of directions. Define

$$M_P^* f(x) = \sup_j \frac{1}{\sigma(L_j)} \int_{L_j} |f(x+y)|\, d\sigma(y).$$

We observe that $M_P f \leqq C M_P^* f$. The middle points a_j of the sides L_j are different from the origin. If a_j had been at the origin for all j, then $M_P^* f$ would be domainted by $M_\Phi f$ a.e. and Theorem 1 could be used directly. Now, however we have to take care of the translations of sets by a_j in the proof of Theorem 1. It is to consider inequality (1) and to restrict this inequality to those rectangles $S\in R(\varphi_j)$ which are very thin, in fact, approaching line segments of length $\sigma(L_j)$, $j=0, 1, \dots$. Then the first term of (11) vanishes and the sum is unchanged since the Lebesgue measure is translation invariant. In order to see how the translations effect the term $\mu(G_\infty)$ we observe that the origin is contained in the interval $4I^{L_j}$, where I^{L_j} is the interval given by the diagonal L_j. Hence it is enough to replace $4I_m^S$ by $8I_m^S$ in the definition of G_∞. This changes the estimate of $\mu(G_\infty)$ only by a constant.

Since we are considering line segments instead of rectangles we can use the weak type (1, 1) estimate of the one-dimensional Hardy—Littlewood maximal operator instead of (1) to get a better estimate in (13). Because of that, the estimate in Theorem 3 will be a factor $1/(1+\log(1/\alpha))$ better then the estimate in Theorem 1.

II. Let D be a set in the (x, y)-plane, whose boundary ∂D is a polygon with infinitely many sides $Z_0, Z_1, \dots$. Let Z_0 be on the line $x=1$ and Z_j, $j=1, 2, \dots$ has the direction 2^{-j} and the endpoints on the lines $x=2^{1-j}$ and $x=2^{-j}$.

Let T_D be the operator on L^p defined on the Fourier transform side by

$$(T(f))^\frown(\xi) = \chi_D(\xi)\hat{f}(\xi).$$

Then T_D is a bounded operator on L^p, $4/3<p<4$. (We refer to [2] for the proof.)

III. In $\mathbf{R}^2$ denote by Ω_j the sector $\{z\in\mathbf{R}^2;\ 2^{-j-1}\leqq\arg(z)<2^{-j}\}$, $j=1, 2, \dots$. Let the operator T_j be defined by

$$(T_j(f))^\frown(\xi) = \chi_{\Omega_j}(\xi)\hat{f}(\xi).$$

Then

$$\|(\textstyle\sum_{j=1}^\infty |T_j(f)|^2)^{1/2}\|_p \sim \|f\|_p$$

for $4/3<p<4$. (For the proof we refer to [2]).

§ 7. Generalization to larger sets of directions

We shall generalize the concept of exponential sets which we have used above. We say that a set of directions Φ is *exponential of the 0-generation* if Φ consists of a single direction φ. We say that a set of directions Φ is *exponential of the m-generation* if $\Phi=\{\Phi_{i_1,\dots,i_m}\}_{i_1,\dots,i_m=1}^{\infty}$ satisfy

(i) $\lim_{i_m\to\infty} \varphi_{i_1,\dots,i_{m-1},i_m}=\varphi_{i_1,\dots,i_{m-1}}$ where $\{\varphi_{i_1,\dots,i_{m-1}}\}_{i_1,\dots,i_{m-1}=1}^\infty$ is exponential in the $(m-1)$-generation.

(ii) There is a constant $c>0$ such that for $(i_1, \dots, i_m)\neq(j_1, \dots, j_m)$ the following inequality holds

$$|\varphi_{j_1,\dots,j_m}-\varphi_{i_1,\dots,i_m}| > c\,|\varphi_{i_1,\dots,i_m}-\varphi_{i_1,\dots,i_k}|$$

where $0\leqq k<m$ such that $(i_1, \dots, i_k)=(j_1, \dots, j_k)$ and $i_{k+1}\neq j_{k+1}$.

We see that to say that a set is exponential of the 1-generation is the same as to say that it is exponential.

A generalization of Theorem 1 and Theorem 2 is the following

Theorem 1′. *Let* $m\geqq 0$, *and let* Φ *be exponential of the m-generation. Let* χ_E *be the characteristic function of the measurable set E in* $\mathbf{R}^n$. *Then*

$$\mu\{M_\Phi\chi_E(x)>\alpha\}\leqq C\alpha^{-1-m}(1+\log(1/\alpha))^{1+m}\mu(E)$$

for all $0<\alpha<1$. *The constant C depends only on* Φ.

Theorem 2′. *Let* $m\geqq 0$, *and let* Φ *be exponential of the m-generation. Let* $\varepsilon>0$, *and let f be a function on* $\mathbf{R}^2$. *Then*

$$\mu\{M_\Phi f(x)>\alpha\}\leqq C\int_{\mathbf{R}^2}|f(y)/\alpha|^{-m-1}(1+\log|f(y)/\alpha|)^{2m+2+\varepsilon}dy$$

for all $\alpha>0$, *if the integral is finite. The constant C depends only on* ε *and* Φ.

Theorem 2′ follows from Theorem 1′ in the same way as before. We shall not give the proof of Theorem 1′. We will only say that the method used in the proof of Theorem 1 can be repeated in such a way that Theorem 1′ can be shown for Φ exponential of the m-generation, if it is already shown for Φ exponential of the $(m-1)$-generation.

§ 8. Some remarks

I. There is no indication that the estimate in Theorem 1 is sharp. If we let the set E be a ball and let $\Phi=\{2^{-i}\}_1^\infty$ than we see that it is impossible to get an estimate better than

$$\mu\{M_\Phi\chi_E(x)>\alpha\}\leqq C\alpha^{-1}(1+\log(1/\alpha))^2\mu(E). \tag{17}$$

In fact, one does not know any example of a set E that violates (17).

More generally the estimate in Theorem 1′ is very likely not sharp. If we let the set E be a ball again and let $\Phi=\{\varphi_{i_1,\dots,i_m}\}$ with

$$\varphi_{i_1,\dots,i_m}=\sum_{1\leq k\leq m}\exp\left(-k-\sum_{1\leq l\leq k}i_l\right)$$

then we see that it is impossible to get an estimate better

$$\mu\{M_\Phi\chi_E(x)>\alpha\}\leqq C\alpha^{-1}(1+\log(1/\alpha))^{1+m}\mu(E). \tag{18}$$

The author has not been able to disapprove (18) by any counterexample.

II. It is an open problem to find weak type estimates when we have more general sets of directions than those considered in Theorem 1′. An important condition for the sets of directions is the following condition proposed by P. Sjögren (c.f. the condition in Theorem 1 of [8]):

There is an $\varepsilon > 0$ with the following property: Any segment I_1 of the unit-circle contains a subsegment I_2 which is disjoint from Φ such that the ratio between the lengths of I_2 and I_1 equals ε.

The condition means, roughly speaking, that Φ is contained in a Cantor set with constant ratio. If Φ does not satisfy this condition then it is possible to make the construction of sets E_N giving the inequality (2), i.e.

$$\mu\{M_\Phi \chi_{E_N}(x) > 1/8\} > N\mu(E_N)$$

for every large $N > 0$.

It is an open problem to find any weak type estimate for the maximal functions when the set of directions is a Cantor set with constant ratio.

Added in proof Recently A. Cordoba and R. Fefferman has proved a weak L^2-estimate for the maximal function in Theorem 2. Using their methods we can also get a weak L^2-estimate in Theorem 3. Concerning Theorem 1′ and 2′ it is possible to prove a weak L^2-estimate for all m using some additional arguments.

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