On the instability of capacity

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§ 1. Introduction

Let E be a Borel set in the space $\mathbf{R}^d$. It is well-known that the Lebesgue measure m is unstable in the sense that

$$\lim_{\delta\to 0} m(B(x,\delta))^{-1} m(E\cap B(x,\delta)) = 1 \quad\text{or}\quad \lim_{\delta\to 0} m(B(x,\delta))^{-1} m(E\cap B(x,\delta)) = 0$$

almost everywhere on $\mathbf{R}^d$, where $B(x,\delta)$ is the open ball of radius δ with center at x. Vitushkin discovered that the continuous analytic capacity α has a similar property, namely $\lim_{\delta\to 0}\delta^{-2}\alpha(E\cap B(x,\delta))=0$ or $\lim_{\delta\to 0}\delta^{-1}\alpha(E\cap B(x,\delta))=1$ with the exception of a set of zero area, where E is an arbitrary subset of the complex plane (see [8]). In [6] Lysenko and Pisarevskiĭ investigated the classical Newtonian capacity, here denoted by $C_{1,2}$, in this direction. They proved that $\lim_{\delta\to 0}\delta^{-3}C_{1,2}(E\cap B(x,\delta))=0$ or $\lim_{\delta\to 0}C_{1,2}(B(x,\delta))^{-1}C_{1,2}(E\cap B(x,\delta))=1$ almost everywhere on $\mathbf{R}^3$, if E is a Borel set. See also in this connection Gonchar [3] and [4]. L. I. Hedberg discovered in [5] that many capacities C are unstable in a certain sense. He proved that for all Borel sets E the following two relations are equivalent:

(a) $C(E\cap\Omega)=C(\Omega)$ for all open sets Ω,

(b) $\overline{\lim}_{\delta\to 0}\delta^{-d}C(E\cap B(x,\delta))>0$ almost everywhere on $\mathbf{R}^d$.

The purpose of this paper is to generalize the theorem of Lysenko and Pisarevskiĭ to $\mathbf{R}^d$ and to more general capacities $C_{\alpha,q}$ (see Section 2 for a definition of $C_{\alpha,q}$). Our result can be found in Section 4, Theorem 4.1. In Section 4 we also prove that there is a similar gap, if we replace "almost everywhere" by "$C_{\alpha,q}$—a.e.". See Theorem 4.2 and Theorem 4.3. In Section 3, Theorem 3.2 we show that

$$C_{\alpha,q}(B(x,\delta))^{-1}C_{\alpha,q}(E\cap B(x,\delta))\to 1 \quad\text{when}\quad \delta\to 0,$$

if E is a Borel set and if x is a density point for E with respect to the Lebesgue measure.

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§ 2. Preliminaries

The underlying space in this paper is the Euclidean space $\mathbf{R}^d$. Let p and q be real numbers such that $1<p<\infty$, $1<q<\infty$ and $p^{-1}+q^{-1}=1$. Let $\mathcal{M}$ be the set of all positive Borel measures μ such that $\mu(\mathbf{R}^d)<\infty$ and let

$$L^q(\mathbf{R}^d) = \left\{f;\ \|f\|_q^q = \int_{\mathbf{R}^d} |f(x)|^q dm(x) < \infty\right\},$$

where m denotes the d-dimensional Lebesgue measure. The set of all non-negative functions $f \in L^q(\mathbf{R}^d)$ is denoted by L_+^q.

For $f \in L^q(\mathbf{R}^d)$ and $\alpha>0$ we define a potential

$$U_\alpha^f(x) = \int |x-y|^{\alpha-d} f(y)\, dm(y),$$

and for $\mu \in \mathcal{M}$ we similarly define

$$U_\alpha^\mu(x) = \int |x-y|^{\alpha-d} d\mu(y).$$

Definition 2.1. *Let E be an arbitrary set and let $\alpha>0$. Then $C_{\alpha,q}(E)=\inf\|f\|_q^q$, where the infimum is taken over all $f \in L_+^q$ such that $U_\alpha^f(x) \geqq 1$ for all $x \in E$.*

The classical Riesz capacities are obtained by setting $q=2$.

Let $B(x, \delta)$ denote the open ball of radius δ with center at x. Various constants are denoted by A. The complementary set of a set E is denoted by $\complement E$.

It follows from Definition 2.1 that

$$C_{\alpha,q}\big(B(x,\delta)\big) = A\delta^{d-\alpha q}, \tag{2.1}$$

where A is independent of δ and x. It is easy to see that $A>0$ if and only if $\alpha q<d$.

We always assume in the rest of this paper that the capacities are not identically equal to zero.

The following theorem will be used several times. For a proof see Meyers [7, p. 273].

Theorem 2.2. *Let E be a Borel set. Then*

$$C_{\alpha,q}(E)^{1/q} = \sup \nu(\mathbf{R}^d),$$

where the supremum is taken over all $\nu \in \mathcal{M}$, such that ν is concentrated on E and $\|U_\alpha^\nu\|_p \leqq 1$.

A property, which holds for all points on $E \setminus E_1$ with $C_{\alpha,q}(E_1)=0$ $\big(m(E_1)=0\big)$, is said to hold $C_{\alpha,q}$-a.e. on E (a.e. on E).

§ 3. Density points

Definition 3.1. *Let E be a Borel set. Then x is a density point for E if*

$$\lim_{\delta\to 0}\frac{m(E\cap B(x,\delta))}{m(B(x,\delta))}=1.$$

The purpose of this section is to prove the following theorem.

Theorem 3.2. *Let E be a Borel set and let x be a density point for E. Then*

$$\lim_{\delta\to 0}\frac{C_{\alpha,q}(E\cap B(x,\delta))}{C_{\alpha,q}(B(x,\delta))}=1.$$

The following corollary follows directly from Theorem 3.2.

Corollary 3.3. *Let E be a Borel set. Then*

$$\lim_{\delta\to 0}\frac{C_{\alpha,q}(E\cap B(x,\delta))}{C_{\alpha,q}(B(x,\delta))}=1 \text{ a.e. } on\ E.$$

Remark 3.4. Let $K(r)$, $r>0$ be a non-negative, decreasing, continuous function such that $K(r)\to\infty$ when $r\to 0$ and $K(r)\to 0$ when $r\to\infty$. For $x\in\mathbf{R}^d$, $x\neq 0$, we define $K(x)=K(|x|)$, and we assume

$$\int_{|x|<1}K(x)\,dm(x)<\infty. \tag{3.1}$$

We call such a function a kernel.

Let K be a kernel such that for all $\varepsilon>0$ there are $\delta>0$ and γ, $1>\gamma>0$, such that $K((1-\gamma)x)\leqq(1+\varepsilon)K(x)$ for all x, $|x|\leqq\delta$. Then Theorem 3.2 and Corollary 3.3 remain true if we replace $|x|^{\alpha-d}$ by $K(x)$. For a proof see [2].

Remark 3.5. Let $H_{d-\beta}$ denote the classical Hausdorff measure with respect to the function $t^{d-\beta}$. For every β, $0<\beta<d$, there exists a compact set E with $H_{d-\beta}(E)=\infty$, such that

$$\varliminf_{\delta\to 0}\frac{C_{\alpha,2}(E\cap B(x,\delta))}{C_{\alpha,2}(B(x,\delta))}=0 \quad \text{for all} \quad x.$$

The proof can be found in [2].

Proof of Theorem 3.2 Suppose that 0 is a density point for E. Let $\varepsilon>0$ and let $0<\gamma<1$. Theorem 2.2 gives that we can choose $\mu_\delta\in\mathcal{M}$, $\delta>0$, such that μ_δ is concentrated on $B(0,(1-\gamma)\delta)$, $\mu_\delta(\mathbf{R}^d)=1$ and

$$\|U_\alpha^{\mu_\delta}\|_p\leqq\left\{\frac{1+\varepsilon}{C_{\alpha,q}(B(0,(1-\gamma)\delta))}\right\}^{1/q}. \tag{3.2}$$

Choose non-negative, continuous functions φ_δ such that $\varphi_\delta(x)=0$ for all x, $|x|\geqq\frac{1}{2}\gamma\delta$,

(3.3) $$\varphi_\delta(x)\leqq A\gamma^{-d}\delta^{-d} \quad \text{for all} \quad x,$$

where A is independent of δ, and

(3.4) $$\int \varphi_\delta(x)\,dm(x)=1.$$

Put $d\gamma_\delta(x)=\left(\int\varphi_\delta(x-y)\,d\mu_\delta(y)\right)dm(x)$. Then $\nu_\delta\in\mathcal{M}$, $\nu_\delta(\mathbf{R}^d)=1$ and ν_δ is concentrated on $B(0,\delta)$. Now (3.3) gives

$$\nu_\delta(\complement E\cap B(0,\delta))=\int_{\complement E\cap B(0,\delta)}\left(\int\varphi_\delta(x-y)\,d\mu_\delta(y)\right)dm(x)$$
$$\leqq Am(\complement E\cap B(0,\delta))\gamma^{-d}\delta^{-d}.$$

If we use that 0 is a density point for E, we obtain

$$\nu_\delta(\complement E\cap B(0,\delta))\to 0 \quad \text{when} \quad \delta\to 0.$$

Since ν_δ is concentrated on $B(0,\delta)$ and $\nu_\delta(\mathbf{R}^d)=1$, we get

(3.5) $$\nu_\delta(E\cap B(0,\delta))\to 1 \quad \text{when} \quad \delta\to 0.$$

From the definition of ν_δ we see that

$$\|U_\alpha^{\nu_\delta}\|_p=\left\|\int|x-y|^{\alpha-d}d\nu_\delta(y)\right\|_p=\left\|\iint|x-y|^{\alpha-d}\varphi_\delta(y-z)\,d\mu_\delta(z)\,dm(y)\right\|_p.$$

If we put $y-z=t$, we find that

$$\|U_\alpha^{\nu_\delta}\|_p=\left\|\iint|x-z-t|^{\alpha-d}\varphi_\delta(t)\,d\mu_\delta(z)\,dm(t)\right\|_p=\left\|\int U_\alpha^{\mu_\delta}(x-t)\,\varphi_\delta(t)\,dm(t)\right\|_p.$$

Minkowski's inequality and (3.4) now give

(3.6) $$\|U_\alpha^{\nu_\delta}\|_p\leqq\|U_\alpha^{\mu_\delta}\|_p.$$

From (3.2), (3.5), (3.6) and Theorem 2.2 we get that $C_{\alpha,q}(E\cap B(0,\delta))>0$. Choose $f_\delta\in L_+^q$ such that

(3.7) $$U_\alpha^{f_\delta}(x)\geqq 1 \quad \text{for all} \quad x\in E\cap B(0,\delta),$$

and

(3.8) $$\|f_\delta\|_q\leqq\left\{(1+\varepsilon)\,C_{\alpha,q}(E\cap B(0,\delta))\right\}^{1/q}.$$

Now (3.7) gives

(3.9) $$\int U_\alpha^{f_\delta}(x)\,d\nu_\delta(x)\geqq\nu_\delta(E\cap B(0,\delta)).$$

The Hölder inequality, (3.6), (3.2) and (3.8) give

$$\int U_\alpha^{f_\delta}(x)\,d\nu_\delta(x)=\int U_\alpha^{\nu_\delta}(x)f_\delta(x)\,dm(x)\leqq\|U_\alpha^{\nu_\delta}\|_p\|f_\delta\|_q\leqq\|U_\alpha^{\mu_\delta}\|_p\|f_\delta\|_q$$
$$\leqq\left\{\frac{(1+\varepsilon)^2C_{\alpha,q}(E\cap B(0,\delta))}{C_{\alpha,q}(B(0,(1-\gamma)\delta))}\right\}^{1/q}.$$

Thus by (3.9)

$$\left\{(1+\varepsilon)^2 \frac{C_{\alpha,q}(E\cap B(0,\delta))}{C_{\alpha,q}(B(0,(1-\gamma)\delta))}\right\}^{1/q} \geqq \nu_\delta(E\cap B(0,\delta)).$$

Using (2.1) we find that

$$\left\{(1+\varepsilon)^2(1-\gamma)^{\alpha q-d} \frac{C_{\alpha,q}(E\cap B(0,\delta))}{C_{\alpha,q}(B(0,\delta))}\right\}^{1/q} \geqq \nu_\delta(E\cap B(0,\delta)).$$

The theorem now follows from (3.5) and from the fact that ε, $\varepsilon>0$, and γ, $0<\gamma<1$, may be chosen arbitrarily small.

§ 4. The instability of capacity

In this section we prove the following three theorems.

Theorem 4.1. *Let E be a Borel set. Then* a.e. *on $\mathbf{R}^d$ one of the following relations holds:*

$$\lim_{\delta\to 0} \frac{C_{\alpha,q}(E\cap B(x,\delta))}{C_{\alpha,q}(B(x,\delta))} = 1$$

or

$$\lim_{\delta\to 0} \frac{C_{\alpha,q}(E\cap B(x,\delta))}{\delta^d} = 0.$$

Theorem 4.2. *Let $0<\beta\leqq\alpha$ and let E be a Borel set. Suppose that $h(\delta)$ is an increasing function such that*

$$\int_0^1 h(\delta)^{p-1}\delta^{-1}d\delta < \infty.$$

*Then $C_{\beta,q}$-*a.e. *on $\mathbf{R}^d$ one of the following relations holds:*

$$\overline{\lim_{\delta\to 0}} \frac{C_{\alpha,q}(E\cap B(x,\delta))}{h(\delta)C_{\alpha,q}(B(x,\delta))} = \infty$$

or

$$\lim_{\delta\to 0} \frac{C_{\alpha,q}(E\cap B(x,\delta))}{\delta^{d-\beta q}} = 0.$$

Theorem 4.3. *Let $0<\beta\leqq\alpha$ and let E be a Borel set. Suppose that $q>2-\beta/d$. Then $C_{\beta,q}$-*a.e. *on $\mathbf{R}^d$ one of the following relations holds:*

$$\int_0^1 \left\{\frac{C_{\alpha,q}(E\cap B(x,\delta))}{C_{\alpha,q}(B(x,\delta))}\right\}^{p-1} \frac{d\delta}{\delta} = \infty$$

or

$$\lim_{\delta\to 0} \frac{C_{\alpha,q}(E\cap B(x,\delta))}{\delta^{d-\beta q}} = 0.$$

Remark 4.4. Let K be a kernel with the same properties as in Remark 3.4. Then Theorem 4.1 remains true if we replace $|x|^{\alpha-d}$ by $K(x)$. See [2].

Remark 4.5. Let K be a kernel. Suppose that there is a constant A such that $K(x) \leqq AK(2x)$ for all x, $|x| \leqq 1$. If we furthermore assume that $K(r)r^{d-\beta}$ is an increasing function for all r, $0<r\leqq r_0$, we may replace $|x|^{\alpha-d}$ by $K(x)$ in Theorem 4.2 and Theorem 4.3. See [2].

In order to prove the theorems, we need some lemmas. The first lemma, which can be found in Bagby and Ziemer [1], is essential for the following.

Lemma 4.6. *Suppose that* $f \in L^q(\mathbf{R}^d)$. *Then*

(i) $$\lim_{\delta\to 0} \frac{1}{\delta^d} \int_{B(x,\delta)} |f(y)-f(x)|^q \, dm(y) = 0 \quad \text{a.e.} \quad on \quad \mathbf{R}^d$$

and

(ii) $$\lim_{\delta\to 0} \frac{1}{\delta^{d-\alpha q}} \int_{B(x,\delta)} |f(y)|^q \, dm(y) = 0 \quad C_{\alpha,q}\text{-a.e.} \quad on \quad \mathbf{R}^d.$$

Before proving the next lemma we need some notation. Let E be an arbitrary set. Then we define

$$E(C_{\alpha,q};\, \delta^\beta) = \left\{x;\, \overline{\lim_{\delta\to 0}}\, \frac{C_{\alpha,q}(E \cap B(x,\delta))}{\delta^\beta} > 0\right\}.$$

Lemma 4.7. *Let* $\beta>0$ *and let* E *be an arbitrary set. Then* $E(C_{\alpha,q};\, \delta^\beta)$ *is a Borel set.*

Proof. Put

$$E_n = \left\{x;\, \overline{\lim_{\delta\to 0}}\, \frac{C_{\alpha,q}(E \cap B(x,\delta))}{\delta^\beta} > \frac{1}{n}\right\} \quad \text{for} \quad n = 1, 2, 3, \ldots.$$

Let $x \in E_n$. Choose $\delta_i(x)$, $i=1, 2, 3, \ldots$, such that $\delta_i(x) \leqq 2^{-i}$ and

(4.1) $$\frac{C_{\alpha,q}(E \cap B(x, \delta_i(x)))}{\delta_i(x)^\beta} > \frac{1}{n} \quad \text{for} \quad n = 1, 2, 3, \ldots.$$

Put

$$A_n^{(i)} = \bigcup_{x\in E_n} B(x, \delta_i(x)), \quad B_n = \bigcap_{i=1}^{\infty} A_n^{(i)} \quad \text{and} \quad B = \bigcup_{n=1}^{\infty} B_n.$$

Then B is a Borel set. It is enough to show that $E(C_{\alpha,q};\, \delta^\beta) = B$. We have the following chain of implications:

$z \in E(C_{\alpha,q};\, \delta^\beta) \Rightarrow z \in E_n$ for some $n \Rightarrow z \in A_n^{(i)}$ for some n and all $i \Rightarrow z \in B_n$ for some $n \Rightarrow z \in B$.

Thus

(4.2) $$E(C_{\alpha,q};\, \delta^\beta) \subset B.$$

On the other hand $z\in B$ gives that $z\in A_m^{(i)}$ for some m and all i. For every i there exists $x_i\in E_m$ such that $z\in B(x_i,\,\delta_i(x_i))$. If we now use that $C_{\alpha,q}$ is an increasing set function and (4.1), we get

$$\frac{C_{\alpha,q}(E\cap B(z,2\delta_i(x_i)))}{\{2\delta_i(x_i)\}^\beta}\geqq\frac{1}{2^\beta}\frac{C_{\alpha,q}(E\cap B(x_i,\delta_i(x_i)))}{\{\delta_i(x_i)\}^\beta}>\frac{1}{2^\beta m}.$$

If we use that $\delta_i(x_i)\leqq 2^{-i}$, we obtain

$$\overline{\lim_{\delta\to 0}}\frac{C_{\alpha,q}(E\cap B(z,\delta))}{\delta^\beta}>0.$$

Thus

$$B\subset E(C_{\alpha,q};\,\delta^\beta),$$

and (4.2) gives $B=E(C_{\alpha,q};\,\delta^\beta)$.

Lemma 4.8. *Let E be a Borel set and let $f\in L_+^q$. Suppose that $U_\alpha^f(x)\geqq 1$ for all $x\in E$. Then*

(i) $U_\alpha^f(x)\geqq 1$ a.e. *on* $E\cup E(C_{\alpha,q};\,\delta^d)$,

and

(ii) $U_\alpha^f(x)\geqq 1$ $C_{\beta,q}$-a.e. *on* $E\cup E(C_{\alpha,q};\,\delta^{d-\beta q})$.

Proof. The proof follows an idea used by L. I. Hedberg in [5]. We prove (ii). The proof of (i) is similar. The proof of (i) can also be found in the proof of Theorem 9 in [5].

Let $x_0\in E(C_{\alpha,q};\,\delta^{d-\beta q})$. It is no restriction to assume that $U_\alpha^f(x_0)<\infty$. Applying Lemma 4.6 we may also assume that

$$\lim_{\delta\to 0}\frac{1}{\delta^{d-\beta q}}\int_{B(x_0,\delta)}|f(x)|^q\,dm(x)=0. \tag{4.3}$$

Theorem 2.2. gives that we can choose $\nu_\delta\in\mathcal{M}$ such that ν_δ is concentrated on $E\cap B(x_0,\delta)$, $\nu_\delta(\mathbf{R}^d)=1$ and

$$\|U_\alpha^{\nu_\delta}\|_p\leqq\left\{\frac{2}{C_{\alpha,q}(E\cap B(x_0,\delta))}\right\}^{1/q}. \tag{4.4}$$

If we use that $U_\alpha^f(x)\geqq 1$ on E, we find

$$1\leqq\int U_\alpha^f(x)\,d\nu_\delta(x)=\int U_\alpha^{\nu_\delta}(x)f(x)\,dm(x).$$

Thus it is enough to prove that

$$\lim_{\delta\to 0}\left|\int U_\alpha^{\nu_\delta}(x)f(x)\,dm(x)-U_\alpha^f(x_0)\right|=0. \tag{4.5}$$

Let $\varepsilon>0$. Since $U_\alpha^f(x_0)<\infty$, it is possible to choose $\varrho>0$ such that

$$\int_{B(x_0,\varrho)}|x-x_0|^{\alpha-d}f(x)\,dm(x)<\varepsilon. \tag{4.6}$$

Let δ, $0<\delta<\varrho/2$ be arbitrary. Then

$$|\int U_\alpha^{\nu_\delta}(x)f(x)\,dm(x)-U_\alpha^f(x_0)| \leqq \int |U_\alpha^{\nu_\delta}(x)-|x-x_0|^{\alpha-d}|f(x)\,dm(x)$$

$$\leqq \int_{B(x_0,\varrho)} |x-x_0|^{\alpha-d}f(x)\,dm(x)+\int_{B(x_0,2\delta)} U_\alpha^{\nu_\delta}(x)f(x)dm(x)+$$

$$+\int_{2\delta\leqq|x-x_0|\leqq\varrho} U_\alpha^{\nu_\delta}(x)f(x)\,dm(x)+$$

$$+\int_{\varrho\leqq|x-x_0|} \left|U_\alpha^{\nu_\delta}(x)-|x-x_0|^{\alpha-d}\right| f(x)\,dm(x) = I_1+I_2+I_3+I_4.$$

From (4.6) we get

$$I_1 < \varepsilon. \tag{4.7}$$

The Hölder inequality and (4.4) give

$$I_2 \leqq \|U_\alpha^{\nu_\delta}\|_p \left\{\int_{B(x_0,2\delta)} |f(x)|^q dm(x)\right\}^{1/q}$$

$$\leqq \left\{\frac{2\delta^{d-\beta q}}{C_{\alpha,q}(E\cap B(x_0,\delta))}\cdot\frac{1}{\delta^{d-\beta q}}\int_{B(x_0,2\delta)} |f(x)|^q dm(x)\right\}^{1/q}.$$

Now (4.3) and the fact that $x_0\in E(C_{\alpha,q};\ \delta^{d-\beta q})$ give

$$\varliminf_{\delta\to 0} I_2 = 0. \tag{4.8}$$

If we use that ν_δ is concentrated on $B(x_0,\delta)$, $\nu_\delta(\mathbf{R}^d)=1$ and (4.6), we obtain

$$I_3 = \int d\nu_\delta(y)\int_{2\delta\leqq|x-x_0|\leqq\varrho} |x-y|^{\alpha-d}f(x)\,dm(x)$$

$$\leqq \int d\nu_\delta(y)\int_{B(x_0,\varrho)} |(x-x_0)/2|^{\alpha-d}f(x)\,dm(x)$$

$$= 2^{d-\alpha}\int_{B(x_0,\varrho)} |x-x_0|^{\alpha-d}f(x)\,dm(x) < 2^{d-\alpha}\varepsilon.$$

Thus

$$I_3 < 2^{d-\alpha}\varepsilon. \tag{4.9}$$

If $|x-x_0|\geqq\varrho$, it is easy to prove that $U_\alpha^{\nu_\delta}(x)$ tends uniformly to $|x-x_0|^{\alpha-d}$ when δ tends to zero. Thus

$$\lim_{\delta\to 0} I_4 = 0. \tag{4.10}$$

Now (4.7), (4.8), (4.9) and (4.10) give (4.5). This finishes the proof.

Lemma 4.9. *Let E be a Borel set.*

(i) *There exists for all $x\in\mathbf{R}^d$ and for all $\delta>0$ a Borel set $O_{x,\delta}$ such that $m(O_{x,\delta})=0$ and*

$$C_{\alpha,q}(E\cap B(x,\delta)) = C_{\alpha,q}\big((E\cup(E(C_{\alpha,q};\ \delta^d)\setminus O_{x,\delta}))\cap B(x,\delta)\big).$$

(ii) *There exists for all $x\in\mathbf{R}^d$ and for all $\delta>0$ a set $O_{x,\delta}$ such that $C_{\beta,q}(O_{x,\delta})=0$ and*

$$C_{\alpha,q}(E\cap B(x,\delta)) = C_{\alpha,q}\big((E\cup(E(C_{\alpha,q};\ \delta^{d-\beta q})\setminus O_{x,\delta}))\cap B(x,\delta)\big).$$

Proof. We prove (i). The proof of (ii) is the same. Put $E(C_{\alpha,q};\, \delta^d)=E_0$. Let $\delta>0$ and $x\in\mathbf{R}^d$ be fixed. Choose $f_n\in L^q_+$ such that

(4.11) $$U^{f_n}_\alpha(z) \geqq 1 \quad \text{for all} \quad z\in E\cap B(x,\delta),$$

and

(4.12) $$\|f_n\|^q_q \leqq C_{\alpha,q}(E\cap B(x,\delta))+1/n \quad \text{for} \quad n=1,2,3,\ldots.$$

Let $y\in E_0\cap B(x,\delta)$. Then $B(y,\varepsilon)\subset B(x,\delta)$ for small ε. Now using $y\in E_0$ we find

$$\varlimsup_{\varepsilon\to 0}\frac{C_{\alpha,q}\big((E\cap B(x,\delta))\cap B(y,\varepsilon)\big)}{\varepsilon^d} = \varlimsup_{\varepsilon\to 0}\frac{C_{\alpha,q}(E\cap B(y,\varepsilon))}{\varepsilon^d} > 0.$$

Thus

(4.13) $$E_0\cap B(x,\delta)\subset \big(E\cap B(x,\delta)\big)(C_{\alpha,q};\,\delta^d) = \big(E\cap B(x,\delta)\big)_0.$$

Lemma 4.8 (i) and (4.11) give that there are Borel sets O_n such that $m(O_n)=0$ and

$$U^{f_n}_\alpha(z)\geqq 1 \quad \text{on} \quad \big(E\cap B(x,\delta)\big)\cup\big((E\cap B(x,\delta))_0\setminus O_n\big) \quad \text{for} \quad n=1,2,3,\ldots.$$

Now (4.13) gives

$$\big(E\cap B(x,\delta)\big)\cup\big((E\cap B(x,\delta))_0\setminus O_n\big)\supset \big(E\cup(E_0\setminus O_n)\big)\cap B(x,\delta).$$

Put $O=\bigcup_{n=1}^\infty O_n$. Then $m(O)=0$ and

$$U^{f_n}_\alpha(z)\geqq 1 \quad \text{on} \quad \big(E\cup(E_0\setminus O)\big)\cap B(x,\delta) \quad \text{for} \quad n=1,2,3,\ldots.$$

Using the definition of $C_{\alpha,q}$ and (4.12) we get

$$C_{\alpha,q}\big((E\cup(E_0\setminus O))\cap B(x,\delta)\big)\leqq C_{\alpha,q}\big(E\cap B(x,\delta)\big).$$

But $(E\cup(E_0\setminus O))\cap B(x,\delta)\supset E\cap B(x,\delta)$. Thus

$$C_{\alpha,q}\big((E\cup(E_0\setminus O))\cap B(x,\delta)\big) = C_{\alpha,q}\big(E\cap B(x,\delta)\big).$$

Proof of Theorem 4.1. Lemma 4.7 gives that $E(C_{\alpha,q};\,\delta^d)=E_0$ is a Borel set. Let x be a density point for $E\cup E_0$. It is enough to prove that

$$\lim_{\delta\to 0}\frac{C_{\alpha,q}(E\cap B(x,\delta))}{C_{\alpha,q}(B(x,\delta))}\geqq 1.$$

Let δ_i, $i=1,2,3,\ldots$, be a sequence of positive numbers such that $\delta_i\to 0$ when $i\to\infty$. Lemma 4.9 (i) gives that there are Borel sets O_i, $m(O_i)=0$, such that

(4.14) $$C_{\alpha,q}\big(E\cap B(x,\delta_i)\big) = C_{\alpha,q}\big((E\cup(E_0\setminus O_i))\cap B(x,\delta_i)\big).$$

Put $O=\bigcup_{i=1}^\infty O_i$. Then $m(O)=0$. Since x is a density point for $E\cup E_0$, x is a density point for $E\cup(E_0\setminus O)$. Now (4.14) and Theorem 3.2 give

$$\varliminf_{i\to\infty}\frac{C_{\alpha,q}(E\cap B(x,\delta_i))}{C_{\alpha,q}(B(x,\delta_i))}\geqq \varliminf_{i\to\infty}\frac{C_{\alpha,q}\big((E\cup(E_0\setminus O))\cap B(x,\delta_i)\big)}{C_{\alpha,q}(B(x,\delta_i))}\geqq 1.$$

Since the sequence δ_i was chosen arbitrarily, the theorem follows.

Lemma 4.10. *Let E be an arbitrary set. Suppose that $0<\beta\leqq\alpha$. Then*

$$\frac{C_{\beta,q}(E\cap B(x,\delta))}{\delta^{d-\beta q}}\leqq A\frac{C_{\alpha,q}(E\cap B(x,\delta))}{C_{\alpha,q}(B(x,\delta))},$$

where A is independent of x and δ.

Proof: Let x be fixed. It is easy to see that there is a constant N independent of x and δ such that

$$\left\{\int_{|y|\geqq N\delta}|y|^{(\alpha-d)p}dm(y)\right\}^{1/p}\left\{2C_{\alpha,q}(B(x,\delta))\right\}^{1/q}<\frac{1}{2}. \tag{4.15}$$

Let ε, $0<\varepsilon<1$, be arbitrary. Now choose $f_\delta\in L^q_+$ such that

$$U_\alpha^{f_\delta}(z)\geqq 1 \quad \text{for all} \quad z\in E\cap B(x,\delta) \tag{4.16}$$

and

$$\|f_\delta\|_q^q\leqq C_{\alpha,q}(E\cap B(x,\delta))+\varepsilon C_{\alpha,q}(B(x,\delta)). \tag{4.17}$$

If we use the Hölder inequality, (4.17) and (4.15), we get

$$\int_{|y|\geqq N\delta}|y|^{\alpha-d}f_\delta(z-y)\,dm(y)\leqq\left\{\int_{|y|\geqq N\delta}|y|^{(\alpha-d)p}dm(y)\right\}^{1/p}\|f_\delta\|_q$$

$$\leqq\left\{\int_{|y|\geqq N\delta}|y|^{(\alpha-d)p}dm(y)\right\}^{1/p}\left\{2C_{\alpha,q}(B(x,\delta))\right\}^{1/q}<\frac{1}{2}.$$

Now (4.16) gives

$$\int_{|y|\leqq N\delta}|y|^{\alpha-d}2f_\delta(z-y)\,dm(y)\geqq 1 \quad \text{if} \quad z\in E\cap B(x,\delta).$$

Thus

$$\int_{|y|\leqq N\delta}|y|^{\alpha-d}|y|^{d-\beta}|y|^{\beta-d}2f_\delta(z-y)\,dm(y)\geqq 1 \quad \text{if} \quad z\in E\cap B(x,\delta).$$

If we use that $r^{\alpha-d}r^{d-\beta}$ is an increasing function for $r>0$, we find

$$\int|y|^{\beta-d}(N\delta)^{\alpha-\beta}2f_\delta(z-y)\,dm(y)\geqq 1 \quad \text{for all} \quad z\in E\cap B(x,\delta).$$

The definition of $C_{\beta,q}$ and (4.17) give

$$C_{\beta,q}(E\cap B(x,\delta))\leqq 2^q N^{q(\alpha-\beta)}\delta^{\alpha q-\beta q}\|f_\delta\|_q^q$$

$$\leqq 2^q N^{q(\alpha-\beta)}\delta^{\alpha q-d}\delta^{d-\beta q}\big(C_{\alpha,q}(E\cap B(x,\delta))+\varepsilon C_{\alpha,q}(B(x,\delta))\big).$$

Since ε may be chosen arbitrarily small, we find

$$\frac{C_{\beta,q}(E\cap B(x,\delta))}{\delta^{d-\beta q}}\leqq 2^q N^{q(\alpha-\beta)}\frac{C_{\alpha,q}(E\cap B(x,\delta))}{\delta^{d-\alpha q}},$$

which gives the lemma.

Lemma 4.11. *Let E be a Borel set. Suppose that g is an increasing function such that*

$$\int_0^1 \left\{\frac{g(\delta)}{\delta^{d-\beta q}}\right\}^{p-1} \frac{d\delta}{\delta} < \infty.$$

Then $C_{\beta,q}$-a.e. on E

$$\overline{\lim_{\delta\to 0}} \frac{C_{\beta,q}(E \cap B(x,\delta))}{g(\delta)} = \infty.$$

Proof. See L. I. Hedberg [5], Theorem 8.

Lemma 4.12. *Let E be a Borel set. Suppose that $q>2-\beta/d$. Then $C_{\beta,q}$-a.e. on E*

$$\int_0^1 \left\{\frac{C_{\beta,q}(E \cap B(x,\delta))}{\delta^{d-\beta q}}\right\}^{p-1} \frac{d\delta}{\delta} = \infty.$$

Proof. See L. I. Hedberg [5], Theorem 4 and Theorem 6.

Proof of Theorem 4.2. Put $E(C_{\alpha,q};\, \delta^{d-\beta q})=E_0$. It is enough to prove that $C_{\beta,q}$-a.e. on $E\cup E_0$.

$$\overline{\lim_{\delta\to 0}} \frac{C_{\alpha,q}(E \cap B(x,\delta))}{h(\delta)\, C_{\alpha,q}(B(x,\delta))} = \infty. \tag{4.18}$$

Let $x\in E\cup E_0$ be fixed. The function $g(\delta)=h(\delta)\delta^{d-\beta q}$ fulfils the assumptions in Lemma 4.11. Lemma 4.7. gives that $E\cup E_0$ is a Borel set. Lemma 4.11 now shows that we may assume that

$$\overline{\lim_{\delta\to 0}} \frac{C_{\beta,q}((E \cup E_0) \cap B(x,\delta))}{h(\delta)\delta^{d-\beta q}} = \infty. \tag{4.19}$$

Lemma 4.9. (ii) gives the existence of a set O_δ such that $C_{\beta,q}(O_\delta)=0$ and

$$C_{\alpha,q}(E \cap B(x,\delta)) = C_{\alpha,q}((E \cup (E_0\setminus O_\delta)) \cap B(x,\delta)). \tag{4.20}$$

Applying that $C_{\beta,q}(O_\delta)=0$, Lemma 4.10 and (4.20) we get

$$\frac{C_{\beta,q}((E \cup E_0) \cap B(x,\delta))}{\delta^{d-\beta q}} = \frac{C_{\beta,q}((E \cup (E_0\setminus O_\delta)) \cap B(x,\delta))}{\delta^{d-\beta q}}$$

$$\leqq \frac{AC_{\alpha,q}((E \cup (E_0\setminus O_\delta)) \cap B(x,\delta))}{C_{\alpha,q}(B(x,\delta))} = \frac{AC_{\alpha,q}(E \cap B(x,\delta))}{C_{\alpha,q}(B(x,\delta))}.$$

Thus

$$\frac{C_{\beta,q}((E \cup E_0) \cap B(x,\delta))}{\delta^{d-\beta q}} \leqq \frac{AC_{\alpha,q}(E \cap B(x,\delta))}{C_{\alpha,q}(B(x,\delta))}.$$

Now (4.19) immediately gives (4.18).

Proof of Theorem 4.3. Put $E(C_{\alpha,q};\, \delta^{d-\beta q})=E_0$. It is enough to prove that $C_{\beta,q}$-a.e. on $E\cup E_0$

$$\int_0^1 \left\{\frac{C_{\alpha,q}(E\cap B(x,\delta))}{C_{\alpha,q}(B(x,\delta))}\right\}^{p-1}\frac{d\delta}{\delta}=\infty. \tag{4.21}$$

Let $x\in E\cup E_0$ be fixed. Lemma 4.7 gives that $E\cup E_0$ is a Borel set. Applying Lemma 4.12 we may assume that

$$\int_0^1 \left\{\frac{C_{\alpha,q}((E\cup E_0)\cap B(x,\delta))}{\delta^{d-\beta q}}\right\}^{p-1}\frac{d\delta}{\delta}=\infty. \tag{4.22}$$

In the same way as in the proof of Theorem 4.2 we have for all δ

$$\frac{C_{\beta,q}((E\cup E_0)\cap B(x,\delta))}{\delta^{d-\beta q}}\leqq\frac{AC_{\alpha,q}(E\cap B(x,\delta))}{C_{\alpha,q}(B(x,\delta))}.$$

Now (4.22) gives (4.21).

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