Isometric embedding of a smooth compact manifold with a metric of low regularity

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1. Introduction

Let X be a compact C^∞ manifold of dimension $n>1$ with a C^k Riemannian metric G. By an isometric embedding of X in $\mathbf{R}^N$ we mean an injective function $U \in C^1(X, \mathbf{R}^N)$ which induces the given metric, that is

$$(dU, dU) = G. \tag{1.1}$$

Nash [7] proved that if $G \in C^0$ there is an isometric embedding $U \in C^1(X, \mathbf{R}^N)$ provided that $N \geqq n+2$ and that there is a differentiable embedding of X in $\mathbf{R}^N$, in particular if $N \geqq 2n$. Nash also indicated that the condition $N \geqq n+2$ could be weakened to $N \geqq n+1$, which was proved by Kuiper [6]. It should be observed that (1.1) in local coordinates means $n(n+1)/2$ equations for N variables. For $G \in C^k$, $k \geqq 3$, Nash [8] also showed that there is an embedding $U \in C^k(X, \mathbf{R}^N)$ if $N \geqq n(3n+11)/2$. The condition on N has been improved for smooth metrics to $N \geqq n(n+1)/2+3n+5$ by Gromov and Rokhlin [3], who also gave lower estimates for the embedding dimension of the same order of magnitude for $k \geqq 2$. This result of Nash was extended by Jacobowitz [5] to Hölder classes H^a with $a>2$, and he also showed that there are metrics $G \in H^\beta$, $\beta>2$, such that (1.1) has no solution $U \in H^\alpha(X, \mathbf{R}^N)$, $\alpha>\beta$, for any N.

The result of Nash—Kuiper shows in particular that there is always a local embedding of X in $\mathbf{R}^{n+1}$. Borisov [1] has announced that if G is analytic there is a local isometric embedding $U \in H^\alpha(X, \mathbf{R}^{n+1})$ with any $\alpha<1+1/(n^2+n+1)$. Thus α is close to 1 if n is large. The low regularity seems to be caused by the demand for a low codimension, for by permitting large values of N we shall prove

Theorem 1.1. *If* $G \in H^\beta$, $0<\beta \leqq 2$, *then the equation* (1.1) *has a solution* $U \in H^\alpha(X, \mathbf{R}^N)$ *if* $\alpha<1+\beta/2$ *and* N *is sufficiently large. On the other hand, if* $0 \leqq \beta<2$ *the set of all* $G \in H^\beta$ *for which* (1.1) *has a solution* $U \in H^\alpha(X, \mathbf{R}^N)$ *with* $\alpha>1+\beta/2$ *is of the first category.*

The proof of the first half of Theorem 1.1 uses ideas from Nash [7, 8]. We give a general outline here.

To solve equation (1.1) for given G we want to find an appropriate iteration scheme producing metrics G_k, $k=0, 1, \dots$, $G_k \to G$, and functions $U_k \in C^\infty(X, \mathbf{R}^N)$, $U = \lim U_k \in H^\alpha(X, \mathbf{R}^N)$, $k=0, 1, \dots$, such that

$$(dU_k, dU_k) = G_k + e_k. \tag{1.2}$$

Here the error term e_k is to be so small that it almost can be corrected in the next step.

To construct U_{k+1} from U_k we perturb U_k in normal directions. However, this introduces a difficulty; if $U_k \in C^\nu$ then the normal will only belong to $C^{\nu-1}$ and so will U_{k+1}. Nash [8] overcame this problem by requiring that the perturbation should be normal not to U_k but to $S_{\theta_k} U_k$, where S_θ is a smoothing operator. We therefore define

$$v_{k+1} = U_{k+1} - U_k = \sum_s c_{k,s} \zeta_{k,s}, \tag{1.3}$$

where $\{\zeta_{k,s}\}$ is an orthonormal system of normals to the range of $S_{\theta_k} U_k$ and $C_{k,s}$ are real valued functions on X. In terms of the coefficients $c_{k,s}$ the equation $(dU_{k+1}, dU_{k+1}) = G_k$ can be written in the form

$$\sum_s \left((dc_{k,s})^2 + 2c_{k,s}(dS_{\theta_k} U_k, d\zeta_{k,s})\right) = G_{k+1} - (G_k + e_k) - E_k \tag{1.4}$$

where we shall always neglect the error term

$$E_k = 2\left((dU_k, dv_{k+1}) - \sum^s c_{k,s}(dS_{\theta_k} U_k, d\zeta_{k,s})\right) + (dv_{k+1}, dv_{k+1}) - \sum_s (dc_{k,s})^2.$$

If $G \in H^\beta$, $\beta > 2$ we can simplify (cf. [5, 8]) (1.4), by also omitting the quadratic term, to

$$2\sum_s c_{k,s}(dS_{\theta_k} U_k, d\zeta_{k,s}) = m_k,$$

that is

$$\sum_s c_{k,s}(d^2 S_{\theta_k} U_k, \zeta_{k,s}) = -m_k/2.$$

Here m_k is close to $G_{k+1} - G_k - e_k$. This linear system of equations for $c_{k,s}$ gives an iteration scheme which leads to a solution $u \in H^\beta$ of (1.1). For details in this case see for instance Hörmander [4].

If $G \in H^\beta$, $\beta < 2$, this does not work because now the quadratic term is dominant. Using an idea in [7] we take U_0 to be a C^∞ embedding of X in $\mathbf{R}^N$ such that $g = G - (dU_0, dU_0)$ is positive definite. Then split g into a geometric series with terms

$$g_k = \theta^{-k\gamma}(1-\theta^{-\gamma})g$$

and define

$$G_k = (dU_0, dU_0) + \sum_0^{k-1} g_j.$$

Then $G_{k+1}-G_k-e_k=g_k-e_k$, and with θ_k and θ properly chosen it turns out that g_k-e_k is the dominant term on the right hand side of (1.4) and that it is positive definite. But then

$$m_k = S_{\theta_k}(g_k-e_k)$$

will also be positive definite and we want to solve

$$\sum_s \left((dc_{k,s})^2+2c_{k,s}(dS_{\theta_k}U_k, d\zeta_{k,s})\right) = m_k. \tag{1.5}$$

This non-linear equation we cannot solve exactly but with the accuracy the iteration scheme requires. This can be done because of the following observation made by Nash in [7]. Write (formally) half of the functions $c_{k,s}$ as $a_{k,t}\cos(\theta_k\varphi_t)/\theta_k$ and the other half as $a_{k,t}\sin(\theta_k\varphi_t)/\theta_k$ where φ_t are linear functions in local coordinates. Then

$$\sum_s (dc_{k,s})^2 = \sum_t \left(a_{k,t}^2(d\varphi_t)^2+(da_{k,t}/\theta_k)^2\right)$$

and it turns out that the dominant term of the left hand side of (1.5) is $\sum_t a_{k,t}^2(d\varphi_t)^2$. But since any positive definite matrix is the sum of n squares of linear forms we can solve the system

$$\sum_t a_{k,t}^2(d\varphi_t)^2 = m_k.$$

With a rather heavy use of the inverse function theorem, we can than solve (1.5) with the required accuracy. In this way we obtain an iteration scheme that for any $\alpha<1+\beta/2$ gives a solution $U\in H^\alpha$ of (1.1).

The second part of Theorem 1.1 follows by the usual derivation of the Gauss equation in differential geometry, where derivatives are replaced by smoothed differences.

We leave it as an open question whether (1.1) has a solution $U\in H^\alpha$ with $\alpha=1+\beta/2$ when $G\in H^\beta$, $0<\beta<2$ and also how large the dimension N in Theorem 1.1 has to be.

Finally I want to express my gratitude to Professor Hörmander for helping me constantly with the following work.

2. Preliminaries

In this section we shall collect some facts that will be needed in the proof of Theorem 1.1. First we shall review briefly some classical facts on Hölder classes (cf. Hörmander [4]). Then, in Lemma 2.3., we shall define a special covering of the manifold X and decompose Riemannian metrics in a way that will suit the iteration scheme in the proof of Theorem 1.1. This iteration scheme also requires the existence of a family of globally defined normal vector fields for embeddings of X. Such fields will be constructed in Lemma 2.4.

We start with discussing Hölder classes. Let B be a fixed convex compact set in $\mathbf{R}^n$ with interior points. For a continuous real valued function defined in B we set

$$|u|_a = \sup_{x, y \in B} |u(x)-u(y)|/|x-y|^a$$

if $0<a\leqq 1$. If instead $k<a\leqq k+1$ where k is a positive integer, we set for $u\in C^k(B, \mathbf{R})$, the space of k times continuously differentiable real valued functions in B,

$$|u|_a = \sum_{|\alpha|=k} |\partial^\alpha u|_{a-k}.$$

Here ∂^α denotes an arbitrary partial derivative of order $|\alpha|$.

Definition 2.1. If $k<a\leqq k+1$ where k is an integer $\geqq 0$, then the Hölder class $H^a(B, \mathbf{R})$ is the set of all $u\in C^k(B, \mathbf{R})$ with $|u|_a<\infty$ and the norm $\|u\|_a=|u|_a+\sup|u|$. We set $H^0(B, \mathbf{R})=C^0(B, \mathbf{R})$ and $\|u\|_0=\sup|u|$.

For functions $u=(u_1, \dots, u_m)$ with values in $B'\subseteqq\mathbf{R}^m$ we write $u\in H^a(B, B')$ if all coordinate functions $u_j\in H^a(B, \mathbf{R})$. We then set

$$\|u\|_a = \sum_{j=1}^m \|u_j\|_a.$$

These Hölder classes have the following six properties. **H1**—**H3** and **H5** were proved in Hörmander [4]. **H6** is a discrete version of Theorem A.11 in [4] and will be proved here. **H4** is an easy consequence of **H2** and **H3** which we shall also prove.

H1. *H^a is a Banach space* which decreases when a increases. For $0\leqq a\leqq b$ and b bounded, $0<t<1$, there is a constant C such that

$$\|u\|_a \leqq C\|u\|_b, \quad \|u\|_{ta+(1-t)b} \leqq C\|u\|_a^t\|u\|_b^{1-t}.$$

H2. *H^a is a ring.* When a is bounded there is a constant C such that

$$\|uv\|_a \leqq C(\|u\|_a\|v\|_0+\|u\|_0\|v\|_a).$$

H3. *H^a is closed under composition.* Let B_i be a compact convex subset of $\mathbf{R}^{n_i}$. If $g\in H^a(B_1, B_2)$ and $f\in H^a(B_2, \mathbf{R}^m)$ then $f\circ g\in H^a(B_1, \mathbf{R}^m)$ and we have the following estimates

$$\|f\circ g\|_a \leqq C_a(\|f\|_a\|g\|_1^a+\|f\|_1\|g\|_a+\|f\|_0), \quad a\geqq 1;$$

$$\|f\circ g\|_a \leqq \min(\|f\|_1\|g\|_a, \|f\|_a\|g\|_1^a)+\|f\|_0, \quad 0\leqq a\leqq 1.$$

Properties **H2** and **H3** allow one to define $H^a(X, \mathbf{R}^m)$ if X is any compact C^∞ manifold. To do so we cover X by coordinate patches Ω_j and take a partition of unity $\sum\chi_j=1$ with $\chi_j\in C_0^\infty(\Omega_j, \mathbf{R})$. A function u on X with values in $\mathbf{R}^m$ is then said to belong to $H^a(X, \mathbf{R}^m)$ if $\chi_j u$ for every j is in H^a as a function of the local coordinates, and $\|u\|_a$ is then defined as $\sum \|\chi_j u\|_a$ with the terms defined by means of local coordinates. The definition of $H^a(X, \mathbf{R}^m)$ does not depend on the choice of covering, local coordinates or partition of unity, and the norm is well defined up to equivalence. Similarily we can define $H^a(X, E)$ for sections of a C^∞ vector bundle E over X; the only new feature is a change of trivializations of the bundle over coordinate patches.

H4. *Estimates of a non-linear differential operator.* Let $F(x, U)$ be a smooth function of $x \in B$ and $U = \{u_\alpha\}_{|\alpha| \leqq m}$ where B is a compact convex subset of $\mathbf{R}^n$ with interior points, $u_\alpha \in \mathbf{R}^N$ and $\alpha = (\alpha_1, \dots, \alpha_n)$ is an n-tuple of non-negative integers with sum $|\alpha|$. Let Φ be the corresponding partial differential operator acting on functions u defined in B and with values in $\mathbf{R}^N$ defined by $\Phi(u) = F(\cdot, \{\partial^\alpha u(\cdot)\}_{|\alpha| \leqq m})$. For $u, v \in H^{m+a}(B, \mathbf{R}^N)$ with $\|u\|_m \leqq C$, where C is a fixed constant, we then have the estimates

(i) $\|\Phi(u)\|_a \leqq C_a(1 + \|u\|_{m+a})$

(ii) $\|\Phi(u+v) - \Phi(u)\|_a \leqq C_a(\|v\|_{m+a} + \|u\|_{m+a}\|v\|_m)$.

Here (i) follows from **H3** with $f = F$ and $g\colon B \ni x \to (x, \{\partial^\alpha u(x)\})$ if we observe that $\|u\|_{m+1} \leqq C\|u\|_{m+a}^{1/a}$ when $a > 1$. (ii) then follows from the mean value theorem, (i) and **H2**. These estimates easily carry over to the case where Φ is a differential operator of order m carrying sections of a vector bundle E over X to sections of another vector bundle F over X. Such an operator is defined to be a functional which over every coordinate patch where E and F are trivial has the above form with respect to the local coordinates and trivializations of the bundles.

H5. *Existence of a smoothing operator.* Let E be a C^∞ vector bundle over a C^∞ compact manifold X. Then there is a smoothing operator S_θ, $\theta > 1$, such that for $u \in H^a(X, E)$

(i) $\|S_\theta u\|_b \leqq C_a \|u\|_a, \quad 0 \leqq b \leqq a;$

(ii) $\|S_\theta u\|_b \leqq C_b \theta^{b-a} \|u\|_a, \quad 0 \leqq a \leqq b;$

(iii) $\|u - S_\theta u\|_b \leqq C_a \theta^{b-a} \|u\|_a, \quad 0 \leqq b \leqq a.$

H6. *A characterisation of H^α when α is not an integer.* Let E be a C^∞ vector bundle over a C^∞ compact manifold X, and assume that the interval $I = [\alpha - \varepsilon, \alpha + \varepsilon]$ does not contain an integer. Let $v_j \in C^\infty(X, E)$ be sections and assume for all $a \in I$ that

$$\|v_j\|_a \leqq K\theta_j^{a-\alpha}, \quad j = 0, 1, \dots$$

where $\theta_j = \theta_0 \theta^j$ with $\theta > 1$. Then it follows that

$$U = \textstyle\sum_0^\infty v_j \in H^\alpha(X, E), \quad \|U\|_\alpha \leqq C_\alpha K/(1 - \theta^{-\varepsilon}) \tag{2.1}$$

where C_α is independent of θ and θ_0.

Proof. We can assume that $v_j \in C^\infty(B, \mathbf{R})$ where $B \subseteqq \mathbf{R}^n$ is compact and convex and that $I \subseteqq]0, 1[$. If k is an integer $\geqq -1$, then we have

$$\text{(2.2)} \qquad \sum_0^k \|v_j\|_{\alpha+\varepsilon} \leqq K \sum_0^k \theta_j^\varepsilon \leqq K\theta_k^\varepsilon/(1-\theta^{-\varepsilon})$$

$$\sum_{k+1}^\infty \|v_j\|_{\alpha-\varepsilon} \leqq K \sum_{k+1}^\infty \theta_j^{-\varepsilon} \leqq K\theta_{k+1}^{-\varepsilon}/(1-\theta^{-\varepsilon}).$$

Set $d=|x-y|$ and assume first that $d<1/\theta_0$. We can then find an integer $k\geqq 0$ such that $1/(d\theta) \leqq \theta_k < 1/d$. If we use this value of k in (2.2) we get

$$|u(x)-u(y)| \leqq 2Kd^\alpha/(1-\theta^{-\varepsilon}).$$

If $d \geqq 1/\theta_0$ we choose $k=-1$ in (2.2) and get

$$|u(x)-u(y)| \leqq d^{\alpha-\varepsilon}K\theta_0^{-\varepsilon}/(1-\theta^{-\varepsilon}) \leqq Kd^\alpha/(1-\theta^{-\varepsilon}).$$

Summing up we have

$$\sup_{x, y \in B} |u(x)-u(y)|/|x-y|^\alpha \leqq 2K/(1-\theta^{-\varepsilon})$$

which proves the statement.

Our next aim is to decompose metrics close to a given one. The corresponding algebraic decomposition is given first in the following lemma.

Lemma 2.2. *Let g be a positive definite quadratic form in $\mathbf{R}^n$. Then one can find linear forms L_t, $t=0, \dots, s_n=n(n+1)/2$ with affinely independent squares and*

$$g = \sum_0^{s_n} L_t^2/(s_n+1).$$

Proof. We can first write $g=\sum_0^{n-1} l_j^2$ with linearly independent linear forms l_j and then choose additional linear forms L_t, $t=n, \dots, s_n$ such that all the squares l_j^2 and L_t^2 are affinely independent. For small $\varepsilon>0$ we have

$$g-\varepsilon \sum_n^{s_n} L_t^2 = \sum_0^{n-1} L_j^2$$

where L_j is close to l_j for $j<n$ if ε is small. But then the squares of the forms L_t for $t=0, \dots, s_n$ will be affinely independent, and if we multiply the forms by $(s_n+1)^{1/2}$ or $(s_n+1)^{1/2}\varepsilon^{-1/2}$ the lemma is proved.

Note that the affine independence means that any quadratic form h can be written in a unique way

$$h = \sum_0^{s_n} \lambda_t(h) L_t^2, \quad \sum_0^{s_n} \lambda_t(h) = 1.$$

Here $\lambda_t(h)$ is an affine linear function of h with $\lambda_t(g)=1/(s_n+1)$.

In the following lemma we denote by $S^2(T^*X)$ the vector bundle over X whose fiber over $x\in X$ is the vector space of symmetric bilinear forms on T_xX. Recall that we have introduced the notation s_n for the fiber dimension.

Lemma 2.3. *Let g be a given positive definite continuous section of $S^2(T^*X)$. We can then find*

(i) *a covering of X by finitely many coordinate patches Ω_j, $j\in J$, such that the index set J is a disjoint union $J_1\cup\ldots\cup J_{n+1}$ and*

$$\Omega_i\cap\Omega_j=\emptyset \quad \text{if} \quad i\neq j \quad \text{and} \quad i,j\in J_k, \tag{2.3}$$

(ii) *functions $\chi_j\in C_0^\infty(\Omega_j,\mathbf{R})$ with $\sum\chi_j^2=1$,*

(iii) *functions $\varphi_t^j\in C_0^\infty(X,\mathbf{R})$, $j\in J$, $t=0,\ldots,s_n$, which are linear in the local coordinates in Ω_j,*

(iv) *a neighborhood W_1 of g in $H^0(X, S^2(T^*X))$ and a neighborhood W_2 of the zero section of $S^2(T^*X)$ in $H^0(X,S^2(T^*X))$, with the following properties: If $m\in W_1$ and $M_t^j\in W_2$ for all j and t we can find real valued continuous functions a_t^j on X such that*

$$m(x)=\textstyle\sum_{j\in J}\sum_{t=0}^{s_n}\big((\chi_j(x)a_t^j(x))^2(d\varphi_t^j)^2+\chi_j(x)a_t^j(x)M_t^j(x)\big),\quad x\in X. \tag{2.4}$$

Here

$$a_t^j=F_t^j(\{M_s^i\},m),\quad F_t^j\in C^\infty; \tag{2.5}$$

$$\|a_t^j\|_0\leqq 1. \tag{2.6}$$

H 3 then implies the estimates

$$\|a_t^j\|_b\leqq C_b\big(1+\|m\|_b+\textstyle\sum_{i,s}\|M_s^i\|_b\big),\quad b\geqq 0. \tag{2.7}$$

Proof. We shall first show that the choices (i)—(iii) can be made so that there is a neighborhood V of $g(X)$ in $S^2(T^*X)$ and $b_t^j\in C^\infty(V,\mathbf{R})$ such that

$$m=\textstyle\sum_{j\in J}\sum_{t=0}^{s_n}(\chi_j(x)b_t^j(m))^2(d\varphi_t^j)^2 \quad \text{if} \quad m\in V\cap S^2(T^*X)_x. \tag{2.8}$$

For any point $x\in X$ we can choose a coordinate neighborhood ω_x with local coordinates $y_1,\ldots,y_n$ vanishing at x, and in ω_x we can write

$$g(x)=\sum g_{jk}(y)\,dy_j\,dy_k.$$

By Lemma 2.2. we can then choose linear forms L_t, $t=0,\ldots,s_n$, such that

$$g(0)=\textstyle\sum_0^{s_n}dL_t(y)^2/(s_n+1),$$

and every quadratic form h in dy can be written uniquely

$$h=\textstyle\sum_0^{s_n}\lambda_t(h)\,dL_t(y)^2,\quad \sum\lambda_t(h)=1.$$

Let ω_x' be a neighborhood of x which is relatively compact in ω_x such that

$$\lambda_t(g(z))>1/2(s_n+1) \quad \text{if} \quad z\in\omega_x'.$$

Then $\{\omega_x'\}_{x\in X}$ is an open covering of X. Since X is compact there is a finite subcovering and it can be refined to a covering $\{\Omega_j\}_{j\in J}$ such that no point in X belongs to more than $n+1$ different Ω_j. This implies (i). Choose χ_j satisfying (ii). If $\Omega_j \subset \omega'_x$ we set with the notations used above

$$\varphi_t^j(z) = L_t(y(z)), \quad b_t^j(m) = \lambda_t(m)^{1/2}$$

if $z \in \Omega_j$ and $m \in S^2(T^*\Omega_j)$ is so close to $g(x)$ that $\lambda_t(m) > 1/2(s_n+1)$. We can extend φ_t^j and b_t^j to C^∞ functions on X and a neighborhood of $g(x)$ in $S^2(T^*X)$ respectively, and have then proved (2.8).

Now define

$$\varphi(x, \{M_t^j\}, \{a_t^j\}) = \sum \left((\chi_j(x)a_t^j)^2(d\varphi_t^j)^2 + \chi_j(x)a_t^j M_t^j\right) \in S^2(T^*X)_x$$

for $a_t^j \in \mathbf{R}$ and $M_t^j \in S^2(T^*X)_x$. Let Q be the number of indices j and t. This induces a fiber preserving C^∞ map

$$\Phi\colon S^2(T^*X)^Q \oplus (X \times \mathbf{R}^Q) \to S^2(T^*X)^Q \oplus S^2(T^*X)$$

defined by taking

$$\Phi(x, \{M_t^j\}, \{a_t^j\}) = \left(x, \{M_t^j\}, \varphi(x, \{M_t^j\}, \{a_t^j\})\right)$$

in the fiber over x. Here $\oplus$ denotes the Whitney fiber sum.

Now (2.8) means that there is a neighborhood V of $g(X)$ in $S^2(T^*X)$ such that the restricted map

$$0 \oplus (X \times \mathbf{R}^Q) \to 0 \oplus S^2(T^*X) \quad \text{(0 means the zero section)}$$

has a right inverse $0 \oplus V \to 0 \oplus (X \times \mathbf{R}^Q)$. This can be trivially continued to a map

$$\psi\colon S^2(T^*X)^Q \oplus V \to S^2(T^*X)^Q \oplus (X \times \mathbf{R}^Q)$$

by defining $\psi(x, \{M_t^j\}, m) = \left(x, \{M_t^j\}, \{b_t^j(m)\}\right)$ where $b_t^j(m)$ are defined by (2.8).

Now $\Phi \circ \psi | 0 \oplus V =$ identity on $0 \oplus V$, and this implies that the differential of $\Phi \circ \psi$ at $(0, V) \in 0 \oplus S^2(T_x^*X) \subset 0 \oplus V$ has the triangular form

$$\begin{pmatrix} \mathrm{id} & 0 \\ * & \mathrm{id} \end{pmatrix}\colon S^2(T_x^*X)^Q \oplus T_v S^2(T^*X) \to S^2(T_x^*X)^Q \oplus T_v S^2(T^*X).$$

(If E is a vector bundle over M, then TE can at the zero section be identified with $E \oplus TM$; regard $S^2(T^*X)^Q \oplus S^2(T^*X)$ as a vector bundle over $S^2(T^*X)$.) It is therefore invertible. If we take a relatively compact subset V_1 of V the inverse function theorem then gives a neighborhood U of the zero section of $S^2(T^*X)^Q$ and a C^∞ map ψ_1 from $U \oplus V_1$ into $S^2(T^*X)^Q \oplus V$ such that $\Phi \circ \psi \circ \psi_1 =$ identity on $U \oplus V_1$. Thus

$$\psi \circ \psi_1\colon U \oplus V_1 \to S^2(T^*X)^Q \oplus (X \times \mathbf{R}^Q)$$

is a right inverse to φ. It is clear that we can assume U and V_1 so small that $\psi \circ \psi_1(U \oplus V_1) \subseteq \left\{(x, \{M_t^j\}, \{a_t^j\}) \in S^2(T^*X)^Q \oplus (X \times \mathbf{R}^Q);\ 0 \leqq a_t^j \leqq 1\right\}$. This implies (iv), (2.4) and (2.6).

Finally we shall construct normal vector fields for embeddings of X.

Lemma 2.4. *Let u_0 be a given C^∞ embedding in $\mathbf{R}^N$ of the n-dimensional compact C^∞ manifold X, where $N \geqq p+2n$. We can then find an orthonormal family $\{\zeta_i\}_1^p$ of C^∞ normals to $u_0(X)$. This means that ζ_i is a C^∞ function from X to $\mathbf{R}^N$ such that $(\zeta_i, du_0)=0$ and $(\zeta_i, \zeta_j)=\delta_{ij}$. Moreover we can find a first order differential operator $v \mapsto \zeta_i(v)$ defined in a H^1-neighborhood W of u_0 such that $\{\zeta_i(v)\}_1^p$ is an orthonormal family of normals to $v(X)$ and $\zeta_i(u_0)=\zeta_i$.*

According to (i) in **H 4** we then have the estimate

$$\|\zeta_i(v)\| \leqq C_a(1+\|v\|_{a+1}), \quad i = 1, \dots, p, \quad a \geqq 0. \tag{2.9}$$

For the proof we need the following well-known

Lemma 2.5. *Let X be an n-dimensional compact C^∞ manifold, E a sub-bundle of $X \times \mathbf{R}^N$ and $E^\perp$ its orthogonal bundle. If the fiber dimension of $E^\perp$ is at least $n+1$ there is a C^∞ section over X of the unit sphere bundle of $E^\perp$.*

Proof of Lemma 2.5. If k is the fiber dimension of $E^\perp$ then $\dim E = n+N-k < N$ and according to the Morse—Sard theorem the image of the projection of E on $\mathbf{R}^N$ is not all of $\mathbf{R}^N$. Now take an element of $\mathbf{R}^N$ not in this image and project it orthogonally on $E_x^\perp$ for every x in X. The wanted section is then obtained from a normalization.

Proof of Lemma 2.4. We identify X and $u_0(X)$ and define ζ_1 by taking $E=TX$ in Lemma 2.5. Then define successively ζ_k by taking $E = TX \oplus F_1 \oplus \dots \oplus F_{k-1}$ with $F_i = \{(x, t\zeta_i(x)) \colon x \in X, t \in \mathbf{R}\}$, noting that $N-(n+k-1) > n$ if $k \leqq p$.

Now take a tubular neighborhood Ω_X of X in $\mathbf{R}^N$ with projection map $q\colon \Omega_X \to X$. We can then continue these vector fields to a full neighborhood of X in $\mathbf{R}^N$ by defining $Z_i(y) = \zeta_i(q(y))$, $y \in \Omega_X$. If v is another C^∞ embedding of X in $\mathbf{R}^N$ close enough to u_0 in the H^1-topology, we can recursively define $\zeta_i(v)(x)$ by subtracting from $Z_i(v(x))$ its projection on the space spanned by the tangent plane at x of $v(X)$ and $\zeta_1(v)(x), \dots, \zeta_{i-1}(v)(x)$ and then normalizing. The lemma will be proved if we show that there is a neighborhood of u_0 in the H^1-topology where the procedure above defines a differential operator $v \mapsto \zeta_i(v)$. To do so let B be a compact subset of a coordinate patch in X and let ∂_j denote differentiation in the local coordinates. In order to compute $\zeta_i(v)$ with respect to these coordinates let

$$\eta_i(v) = Z_i \circ v - \sum_{j=1}^n r_{ij}(v)\, \partial_j v$$

where r_{ij} are given by

$$0 = (\eta_i(v), \partial_k v) = (Z_i \circ v, \partial_k v) - \sum_{j=1}^n r_{ij}(v)(\partial_j v, \partial_k v) \quad k = 1, \dots, n.$$

The matrix $((\partial_j v, \partial_k v))$ is invertible if v is close to u_0, and since $Z_1, \dots, Z_p$ are given C^∞ functions it follows that $r_{ij}(v)$ is a C^∞ function of v and $\partial_1 v, \dots, \partial_n v$, that is, a first order differential operator for v in a neighbourhood of u_0 in $H^1(B, \mathbf{R}^N)$ with $r_{ij}(u_0)=0$. The estimate (ii) in **H4** then shows that if v is close enough to u_0 in $H^1(B, \mathbf{R}^N)$ then $\zeta_1(v)=\eta_1(v)/|\eta_1(v)|$ is well defined as a first order differential operator in v. Now $\zeta_i(v)$ is defined recursively. Suppose that $\zeta_1(v), \dots, \zeta_{k-1}(v)$ are already defined. Then put

$$\theta_k(v) = \eta_k(v) - \sum_{j=1}^{k-1} (Z_k \circ v, \zeta_j(v))\zeta_j(v).$$

Since $\theta_k(u_0)=\zeta_k$, **H4** again shows that we can normalize, and thus define $\zeta_k(v)=\theta_k(v)/|\theta_k(v)|$, if v is close enough to u_0 in $H^1(B, \mathbf{R}^N)$.

3. The Embedding Theorem

In this section we shall prove the first half of Theorem 1.1, with an estimate of N. Suppose that $G \in H^\beta(X, S^2(T^*X))$ is positive definite, $0<\beta<2$. If $1<\alpha<1+\beta/2$ and $N \geqq 3(n+1)(n^2+n+2)+2n$, then we shall prove that there is an embedding $U \in H^\alpha(X, \mathbf{R}^N)$ such that

$$(dU, dU) = G. \tag{3.1}$$

We can of course assume that $\alpha > \max(1, \beta)$ and that $N=N_1+N_2+N_3$ where $N_1=2(n+1)(s_n+1)+2n$ and $N_2=N_3=2(n+1)(s_n+1)$. In order to construct an embedding that solves (3.1) we first take, in the terminology of Nash [7], a short embedding, that is a C^∞ embedding u_0 of X in $\mathbf{R}^{N_1}$ such that $g=G-(du_0, du_0)$ is positive definite. Such an embedding can be constructed from any C^∞ embedding of X in $\mathbf{R}^{N_1}$ by a change of scale in $\mathbf{R}^{N_1}$. The embedding u_0 defines an embedding U_0 of X in $\mathbf{R}^N$ by $U_0(x)=(u_0(x), 0)$. By successively constructing functions $U_k \in C^\infty(X, \mathbf{R}^N)$ we shall increase the C^∞ metric (dU_0, dU_0) to the metric G. To do so we introduce the notation $\gamma=2(\alpha-1)$ and decompose the metric g so that, with a large parameter θ,

$$g = \sum_0^\infty g_i \quad \text{where} \quad g_i = \theta^{-i\gamma}(1-\theta^{-\gamma})g.$$

The aim of the iteration scheme is to make

$$e_k = (dU_k, dU_k) - (dU_0, dU_0) - \sum_0^{k-1} g_i \tag{3.2}$$

much smaller than g_k and the difference

$$v_k = U_k - U_{k-1} \tag{3.3}$$

so small that U_k has a limit $U \in H^\alpha(X, \mathbf{R}^N)$.

First we choose $\varepsilon>0$ with the following three properties:

P1. If $u \in C^1(X, \mathbf{R}^{N_1})$ and $\|u-u_0\|_1<\varepsilon$ then u is an embedding.

P2. If $m, M_t^j, j\in J, t=0, \ldots, s_n$ are sections of $S^2(T^*X)$ such that $\|m-g\|_0<\varepsilon$ and $\|M_t^j\|_0<\varepsilon$ for all j and t then we can decompose m according to (2.4).

P3. When $\|u-u_0\|_1<\varepsilon$, $u\in C^\infty(X, \mathbf{R}^{N_1})$, then there exist $2(n+1)(s_n+1)$ orthonormal vector fields $\zeta_{l,s}(u)$ $l=1, \ldots, n+1$, $s=0, \ldots, 2s_n+1$, normal to $u(X)$, which are first order differential operators in u. We set $\zeta_s^j(u)=\zeta_{l,s}(u)$ if $j\in J_l$ (see Lemma 2.3).

The existence of this $\varepsilon>0$ follows from Lemma 2.3 and Lemma 2.4 together with the fact that the set of embeddings is open in $C^1(X, \mathbf{R}^{N_1})$ (see Golubitsky—Guillemin [2], Ch 2, Prop 5.8). Let $\theta_k=\theta_0\theta^k$ where θ_0 is a large parameter. We assume that $\theta\geqq 2$ and $\theta_0\geqq 2$.

Lemma 3.1. *It is possible to find a constant $K\geqq 1$ such that whenever θ and θ_0/θ are large enough there exist embeddings $U_i\in C^\infty(X, \mathbf{R}^N)$ such that with e_i and v_i defined by* (3.2) *and* (3.3) *and* $i=1, 2, \ldots,$

$$\|v_i\|_a \leqq K\theta_0^{\alpha-1}\theta_{i-1}^{a-\alpha}, \quad 0\leqq a\leqq 4; \tag{3.4}$$

$$\|e_i\|_a \leqq \frac{1}{2}\varepsilon\theta_{i-1}^a\theta^{-i\gamma}, \quad 0\leqq a\leqq\beta. \tag{3.5}$$

The proof of Lemma 3.1 occupies the major part of this section and it gives easily the statement made at the beginning of the section.

Lemma 3.2. *From* (3.4) *and* (3.5) *it follows that*

$$U = U_0+\textstyle\sum_1^\infty v_j\in H^\alpha(X, \mathbf{R}^N) \tag{3.6}$$

and that U satisfies (3.1). *Moreover, if we suppose* (3.4) *to hold only for* $i=1, \ldots, k$ *and set* $b^+=\max(b, 0)$ *then we have the estimate*

$$\|U_k\|_a \leqq CK\theta_0^{L(a)-1}\theta_{k-1}^{(a-\alpha)^+}, \quad 0\leqq a\leqq 4, \tag{3.7}$$

where $L(a)=\max(1, \min(a, \alpha))$.

Here and elsewhere constants C are independent of θ_0, θ and K.

Proof. First we fix a $\delta>0$ so that $1<\alpha-\delta<\alpha+\delta<2$. Then (3.6) is an immediate consequence of property **H6**, and this property also shows that

$$\|U_k-U_0\|_\alpha \leqq CK\theta_0^{\alpha-1}/(1-\theta^{-\delta}).$$

From (3.4) we get the estimates

$$\|U_k-U_0\|_4 \leqq K\theta_0^{\alpha-1}\textstyle\sum_{0}^{k-1}\theta_j^{4-\alpha} \leqq K\theta_0^{\alpha-1}\theta_{k-1}^{4-\alpha}/(1-\theta^{\alpha-4})$$

$$\|U_k-U_0\|_a \leqq K\theta_0^{a-1}/(1-\theta^{-\delta}) \quad \text{if} \quad 0\leqq a\leqq \alpha-\delta.$$

In view of the logarithmic convexity in **H1** we then get

$$\|U_k-U_0\|_a \leqq CK\theta_0^{\min(a,\alpha)-1}\theta_{k-1}^{(a-\alpha)_+}, \quad 0 \leqq a \leqq 4.$$

This implies (3.7). Moreover, using (3.4), (3.7) and (2.1)

$$\|(dU_k, dU_k)-(dU, dU)\|_0 \leqq C(\|U_k\|_1+\|U\|_1)(\|U_k-U\|_1)$$

$$\leqq CK^2\theta_0^{\alpha-1}\sum_k^\infty \theta_j^{1-\alpha} = CK^2\theta_0^{\alpha-1}\theta_k^{1-\alpha}/(1-\theta^{1-\alpha}) \to 0 \quad \text{as} \quad k\to\infty.$$

Since $\|e_k\|_0\to 0$ as $k\to\infty$ this implies (3.1). The lemma is proved.

The first two steps, that is, the definition of U_1 and U_2 respectively, will differ slightly from the others. The reason for a separate first step is that we want to be able to apply Lemma 2.4, and in it we are going to alter U_0 in $\mathbf{R}^{N_2}$, that is, in the coordinate directions in which U_0 vanishes. The reason for the second step, in which we are going to alter U_1 in $\mathbf{R}^{N_3}$, that is, in the coordinate directions in which U_1 vanishes, we will return to. In the remaining steps we are only going to modify the first N_1 coordinates. Let u_k denote the projection of U_k on the first N_1 coordinates. Then (3.4) gives us the estimate

$$(3.8) \qquad \|u_k-u_0\|_1 = \|U_k-U_2\|_1 \leqq K\sum_2^{k-1}\theta^{j(1-\alpha)} \leqq K\theta^{2(1-\alpha)}/(1-\theta^{1-\alpha}) < \frac{1}{2}\varepsilon$$

if we only take θ sufficiently large. Hence U must be an embedding. The fact that U is an immersion follows of course also directly from (3.1).

The first step. Define $m_0=S_{\theta_0}g_0$. From the definition of g_0 and **H5** (iii) we get the estimate

$$\|m_0-g\|_0 \leqq C\theta_0^{-\beta}\|g\|_\beta+\theta^{-\gamma}\|g\|_0.$$

If we take θ and θ_0 large enough, this will be less than ε. Lemma 2.3 then gives us functions $a_{0,t}^j$ such that

$$(3.9) \qquad m_0 = \sum_{j,t}(\chi_j a_{0,t}^j)^2(d\varphi_t^j)^2.$$

Using **H5** (i) and (ii) we get

$$\|m_0\|_a \leqq C\theta_0^{(a-\beta)_+}\|g\|_\beta, \quad 0 \leqq a \leqq 4.$$

Then (2.5) and **H3** implies

$$(3.10) \qquad \|a_{0,t}^j\|_a \leqq C\theta_0^{(a-\beta)_+}, \quad 0 \leqq a \leqq 4.$$

We can now define

$$v_1 = \sum_{j,t}\chi_j a_{0,t}^j(\cos(\theta_0\varphi_t^j)\zeta_{0,t}^j+\sin(\theta_0\varphi_t^j)\eta_{0,t}^j)/\theta_0.$$

Here the normals are defined so that if $j\in J_k$ then $\zeta_{0,t}^j=e_\nu$ and $\eta_{0,t}^j=e_{\nu+N_2/2}$, $\nu=N_1+(k-1)(s_n+1)+t+1$, where e_i is the i^{th} basis vector in $\mathbf{R}^N$. In order to prove (3.4) we observe that (3.10) implies

$$\|a_{0,t}^j\cos(\theta_0\varphi_t^j)\|_a \leqq C(\theta_0^{(a-\beta)_+}+\theta_0^a) \leqq 2C\theta_0^a, \quad 0 \leqq a \leqq 4,$$

if we also use **H 2** and **H 3**. Of course we get the same estimate if we substitute sin for cos. Summing up we get the estimate

$$\|v_1\|_a \leqq C_0\theta_0^{a-1} = C_0\theta_0^{\alpha-1}\theta_0^{a-\alpha}, \quad 0 \leqq a \leqq 4. \tag{3.11}$$

Here C_0 is a constant independent of θ_0, so this will give (3.4) for $i=1$ if we take K larger than C_0. We will not fix the value of K before having considered the general step.

In order to get (3.5) for $i=1$ we compute

$$e_1 = (dU_1, dU_1)-(dU_0, dU_0)-g_0 = \big((dv_1, dv_1)-m_0\big)+(m_0-g_0).$$

Then **H 5** (iii) gives

$$\|m_0-g_0\|_a \leqq C\theta_0^{a-\beta}\|g\|_\beta, \quad 0 \leqq a \leqq \beta. \tag{3.12}$$

If we differentiate v_1 and use (3.9) and the fact that the normals are orthonormal and constant we get

$$(dv_1, dv_1)-m_0 = \sum\nolimits_{j,t}\big(d(\chi_j a_{0,t}^j)\big)^2/\theta_0^2.$$

Using **H 2**, **H 3** and (3.10) we get the estimate

$$\|(dv_1, dv_1)-m_0\|_a \leqq \sum\nolimits_{j,t}\theta_0^{-2}\left\|\big(d(\chi_j a_{0,t}^j)\big)^2\right\|_a \tag{3.13}$$

$$\leqq C\sum\nolimits_{j,t}\theta_0^{-2}\|\chi_j a_{0,t}^j\|_{a+1}\|\chi_j a_{0,t}^j\|_1 \leqq C\theta_0^{(a+1-\beta)^+ + (1-\beta)^+ -2}, \quad 0 \leqq a \leqq \beta.$$

Here the exponent is less than $a-\beta$ which is obvious if $\beta \leqq 1$; if $1<\beta<2$ it is easy to see that $(a+1-\beta)^+ -2 \leqq a-\beta$. Combining (3.12) and (3.13) we obtain the estimate

$$\|e_1\|_a \leqq C\theta_0^{a-\beta}, \quad 0 \leqq a \leqq \beta,$$

which implies (3.5) with $i=1$ if we take θ_0/θ so large that

$$C\theta^\gamma\theta_0^{-\beta} \leqq C(\theta/\theta_0)^\beta \leqq \frac{1}{2}\varepsilon.$$

The second step. Define $m_1 = S_{\theta_1}(g_1-e_1)$. Since $g_1=\theta^{-\gamma}g_0$ we get, using (3.5) and **H 5** (iii)

$$\|\theta^\gamma m_1-g\|_0 \leqq \theta^\gamma\|m_1-g_1\|_0+\|\theta^\gamma g_1-g\|_0$$

$$\leqq \theta^\gamma(C\theta_1^{-\beta}\|g\|_\beta\theta^{-\gamma}+\|S_{\theta_1}e_1-e_1\|_0+\|e_1\|_0)+\theta^{-\gamma}\|g\|_0$$

$$\leqq C\big(\theta_1^{-\beta}+\frac{1}{2}\varepsilon\theta^{-\beta}+\theta^{-\gamma}\big)+\frac{1}{2}\varepsilon.$$

If we take θ so large that the first term is less than $\frac{1}{2}\varepsilon$ this implies

$$\|\theta^\gamma m_1-g\|_0 < \varepsilon.$$

Thus **P 2** is fulfilled and according to Lemma 2.3 we can find functions $a_{1,t}^j$ such that

$$\theta^\gamma m_1 = \sum\nolimits_{j,t}(\chi_j\theta^{\gamma/2}a_{1,t}^j)^2(d\varphi_t^j)^2. \tag{3.14}$$

From the definition of m_1 and (3.5) with $i=1$ if follows that

$$\|\theta^\gamma m_1\|_a \leqq C\theta^\gamma(\|g\|_\beta \theta^{-\gamma}+\|e_1\|_a) \leqq C\theta_0^a, \quad 0 \leqq a \leqq \beta;$$

$$\|\theta^\gamma m_1\|_a \leqq C\theta_0^\beta \theta_1^{a-\beta} = C\theta^{-\beta}\theta_1^a, \quad \beta \leqq a \leqq 4,$$

using **H5**. According to (2.5) and **H3** this implies that

$$\|a_{1,t}^j\|_a \leqq C(1+C\theta^{-\min(a,\beta)}\theta_1^a)\theta^{-\gamma/2}, \quad 0 \leqq a \leqq 4.$$

In particular we have the estimates

$$\|a_{1,t}^j\|_a \leqq C\theta_1^a\theta^{-\gamma/2}, \quad 0 \leqq a \leqq 4; \tag{3.15}$$

$$\|a_{1,t}^j\|_a \leqq C(\theta^{-1}+\theta^{-\beta})\theta_1^a\theta^{-\gamma/2}, \quad 1 \leqq a \leqq 4, \tag{3.16}$$

when θ_0/θ is large. The second estimate is crucial when we estimate e_2.

We now define

$$v_2 = \sum_{j,t}\chi_j a_{1,t}^j(\cos(\theta_1\varphi_t^j)\zeta_{1,t}^j+\sin(\theta_1\varphi_t^j)\eta_{1,t}^j)/\theta_1.$$

Here the normals are defined so that if $j\in J_k$ then $\zeta_{1,t}^j=e_\mu$, $\eta_{1,t}^j=e_{\mu+N_3/2}$, $\mu=N_1+N_2+(k-1)(s_n+1)+t+1$, where e_i is the i^{th} basis vector in $\mathbf{R}^N$. In order to prove (3.4) for $i=2$ we observe that (3.15) implies

$$\|a_{1,t}^j\cos(\theta_1\varphi_t^j)\|_a \leqq C\theta_1^a\theta^{-\gamma/2}, \quad 0 \leqq a \leqq 4,$$

and that we have the same estimate if we substitute sin for cos. This implies the estimate

$$\|v_2\|_a \leqq C_1\theta_1^{a-1}\theta^{-\gamma/2} = C_1\theta_0^{\alpha-1}\theta_1^{a-\alpha}, \quad 0 \leqq a \leqq 4. \tag{3.17}$$

Here C_1 is independent of θ_0 and θ, so this will give (3.4) for $i=2$ if we take K larger than C_1.

In order to get (3.5) for $i=2$ we compute

$$e_2 = (dU_2, dU_2)-(dU_0, dU_0)-(g_0+g_1) = ((dv_2, dv_2)-m_1)+(m_1-g_1+e_1).$$

Using **H5** (iii) and (3.5) for $i=1$ we obtain the following two estimates

$$\|S_{\theta_1}g_1-g_1\|_a \leqq C\theta_1^{-\beta}\|g\|_\beta\theta_1^a\theta^{-\gamma} \quad 0 \leqq a \leqq \beta;$$

$$\|S_{\theta_1}e_1-e_1\|_a \leqq C\theta_1^{a-\beta}\|e_1\|_\beta \leqq C\theta_1^{a-\beta}\theta_0^\beta\theta^{-\gamma} = C\theta^{\gamma-\beta}\theta_1^a\theta^{-2\gamma}, \quad 0 \leqq a \leqq \beta.$$

In view of the definition of m_1 it follows that

$$\|m_1-g_1+e_1\|_a \leqq C\theta^{\gamma-\beta}\theta_1^a\theta^{-2\gamma}, \quad 0 \leqq a \leqq \beta. \tag{3.18}$$

As in the first step, we get

$$(dv_2, dv_2)-m_1 = \sum_{j,t}(d(\chi_j a_{1,t}^j))^2/\theta_1^2$$

and using (3.16) this shows that

(3.19) $$\|(dv_2, dv_2)-m_1\|_a \leqq C\theta_1^{-2}(\theta^{-2}+\theta^{-2\beta})\theta_1^{a+2}\theta^{-\gamma} \leqq C\theta^{\gamma-\beta}\theta_1^a\theta^{-2\gamma}, \quad 0 \leqq a \leqq \beta.$$

Combining (3.18) and (3.19) we obtain the estimate

$$\|e_2\|_a \leqq C\theta^{\gamma-\beta}\theta_1^a\theta^{-2\gamma}, \quad 0 \leqq a \leqq \beta,$$

which implies (3.5) with $i=2$ if we take θ sufficiently large.

The general step. We shall construct U_{k+1} from U_k, $k \geqq 2$. Since we shall only work in the N_1 first coordinate directions this means the construction of u_{k+1} from u_k. Let

$$\tilde{u}_k = S_{\theta_k} u_k.$$

We start with defining some vector fields. If θ is sufficiently large then (3.7) with $a=\alpha$ implies in view of **H5** (iii) that $\|\tilde{u}_k-u_k\|_1<\frac{1}{2}\varepsilon$. Together with (3.8) this shows that $\|\tilde{u}_k-u_0\|_1<\varepsilon$. According to property **P3** of ε we can then define the normal vector fields

$$\zeta_{k,s}^j = \zeta_s^j(\tilde{u}_k), \quad j\in J, \quad s = 0, \ldots, 2s_n+1,$$

and (2.8) gives the estimate

$$\|\zeta_{k,s}^j\|_a \leqq C_K(1+\|\tilde{u}_k\|_{a+1}), \quad 0 \leqq a \leqq 4.$$

In view of **H5** and (3.7) this gives the estimates

(3.20) $$\|\zeta_{k,s}^j\|_a \leqq C_K\theta_0^{\min(a+1,\alpha)-1}\theta_{k-1}^{(a+1-\alpha)^+}, \quad 0 \leqq a \leqq 3;$$

(3.21) $$\|\zeta_{k,s}^j\|_a \leqq C_K\theta^{\alpha-4}\theta_0^{\alpha-1}\theta_k^{a+1-\alpha}, \quad 3 \leqq a \leqq 4,$$

for the unit vectors $\zeta_{k,s}^j$. Here and elsewhere constants C_K depend on K.

Now define

(3.22) $$m_k = S_{\theta_k}(g_k-e_k).$$

Lemma 3.3. *We can find real valued C^∞ functions $c_{k,s}^j$ with support in Ω_j,*

(3.23) $$\|c_{k,s}^j\|_a \leqq C\theta_0^{\alpha-1}\theta_k^{a-\alpha}, \quad 0 \leqq a \leqq 4,$$

such that

(3.24) $$R_k = m_k-\sum\nolimits_{j,s}\left((dc_{k,s}^j)^2+2c_{k,s}^j(d\tilde{u}_k, d\zeta_{k,s}^j)\right)$$

has the estimate

(3.25) $$\|R_k\|_a \leqq C_K\theta^{\gamma-\beta}\theta_k^a\theta^{-(k+1)\gamma}, \quad 0 \leqq a \leqq \beta.$$

We postpone the proof of Lemma 3.3 in order to prove that it allows us to complete the proof of Lemma 3.1. Define

$$v_{k+1} = \sum\nolimits_{j,s} c_{k,s}^j\zeta_{k,s}^j.$$

In order to prove (3.4) for $i=k+1$ we estimate the a-norm of v_{k+1} for $a=0$ and $a=4$:

$$\|v_{k+1}\|_0 \leqq \sum_{j,s} \|c^j_{k,s}\|_0 \|\zeta^j_{k,s}\|_0 \leqq C\theta_0^{\alpha-1}\theta_k^{-\alpha},$$

$$\|v_{k+1}\|_4 \leqq C \sum_{j,t} (\|c^j_{k,s}\|_4 \|\zeta^j_{k,s}\|_0 + \|c^j_{k,s}\|_0 \|\zeta^j_{k,s}\|_4)$$

$$\leqq C(\theta_k^{4-\alpha}\theta_0^{\alpha-1} + C_K\theta_0^{\alpha-1}\theta_k^{-\alpha}\theta^{\alpha-4}\theta_0^{\alpha-1}\theta_k^{5-\alpha}) = C\theta_0^{\alpha-1}\theta_k^{4-\alpha}(1+C_K\theta^{\alpha-4}\theta^{k(1-\alpha)}).$$

Here we have used **H 3**, (3.21) and (3.23). If we take θ so large that

$$C_K\theta^{\alpha-4} \leqq 1,$$

the logarithmic convexity of **H 1** implies that

$$\|v_{k+1}\|_a \leqq C_2\theta_0^{\alpha-1}\theta_k^{a-\alpha}, \quad 0 \leqq a \leqq 4. \tag{3.26}$$

Now choose K equal to the maximum of this constant C_2 and of the constants C_0 in (3.11) and C_1 in (3.17). Then (3.4) with $i=k+1$ follows when θ and θ_0/θ are large enough.

In order to prove (3.5) for $i=k+1$ we compute e_{k+1},

$$e_{k+1} = (dU_{k+1}, dU_{k+1}) - (dU_0, dU_0) - \textstyle\sum_0^k g_i$$

$$= (du_{k+1}, du_{k+1}) - (du_k, du_k) + e_k - g_k = [(du_{k+1}, du_{k+1}) - (du_k, du_k) - m_k] + [m_k - g_k + e_k].$$

The term in the first bracket we call the iteration error and the term in the second bracket the smoothing error.

The smoothing error. Using **H 5** (iii) and (3.5) for $i=k$ we obtain the following two estimates

$$\|S_{\theta_k} g_k - g_k\|_a \leqq C\|g\|_\beta \theta_k^{a-\beta}\theta^{-k\gamma}, \quad 0 \leqq a \leqq \beta;$$

$$\|S_{\theta_k} e_k - e_k\|_a \leqq C\theta_k^{a-\beta}\|e_k\|_\beta \leqq C\theta_k^{a-\beta}\theta_{k-1}^{\beta}\theta^{-k\gamma}$$

$$= C\theta^{\gamma-\beta}\theta_k^a\theta^{-(k+1)\gamma}, \quad 0 \leqq a \leqq \beta.$$

In view of the definition (3.22) of m_k it follows that

$$\|m_k - g_k + e_k\|_a \leqq C\theta^{\gamma-\beta}\theta_k^a\theta^{-(k+1)\gamma}, \quad 0 \leqq a \leqq \beta, \tag{3.27}$$

since $k \geqq 2$.

The iteration error. A direct computation gives

$$(du_{k+1}, du_{k+1}) - (du_k, du_k) - m_k = 2(du_k, dv_{k+1}) + (dv_{k+1}, dv_{k+1}) - m_k$$

$$= 2\big((du_k, dv_{k+1}) - \textstyle\sum_{j,s} c^j_{k,s}(d\tilde{u}_k, d\zeta^j_{k,s})\big) + (dv_{k+1}, dv_{k+1}) - \sum_{j,s}(dc^j_{k,s})^2 - R_k$$

if we use (3.24). The linear term can be estimated by

$$\big\|(du_k, dv_{k+1}) - \textstyle\sum_{j,s} c^j_{k,s}(d\tilde{u}_k, d\zeta^j_{k,s})\big\|_a \tag{3.28}$$

$$\leqq C_K\theta^{\alpha-4+\gamma}\theta_k^a\theta^{-(k+1)\gamma}, \quad 0 \leqq a \leqq \beta.$$

In fact, we have

$$(du_k, dv_{k+1}) = \big(d(u_k-\tilde{u}_k), dv_{k+1}\big)+\textstyle\sum_{j,s} c^j_{k,s}(d\tilde{u}_k, d\zeta^j_{k,s})$$

since $\zeta^j_{k,s}$ are orthogonal to $d\tilde{u}_k$. Moreover, by **H2**, **H5**, (3.26) and (3.7)

$$\begin{aligned}\big\|\big(d(u_k-\tilde{u}_k), dv_{k+1}\big)\big\|_a &\leqq C(\|u_k-\tilde{u}_k\|_{a+1}\|v_{k+1}\|_1+\|u_k-\tilde{u}_k\|_1\|v_{k+1}\|_{a+1}) \\ &\leqq C(\theta_k^{a+1-4}\|u_k\|_4\|v_{k+1}\|_1+\theta_k^{1-4}\|u_k\|_4\|v_{k+1}\|_{a+1}) \\ &\leqq C_K\theta_0^{\alpha-1}\theta_k^{a-\alpha}\theta_{k-1}^{4-\alpha}\theta_k^{-2}\theta_0^{\alpha-1} = C_K\theta^{\alpha-4+\gamma}\theta_k^a\theta^{-(k+1)\gamma}, \quad 0\leqq a\leqq\beta.\end{aligned}$$

This proves (3.28).

The term $(dv_{k+1}, dv_{k+1})-\sum_{j,s}(dc^j_{k,s})^2$ is a sum of terms

$$(c^j_{k,s}d\zeta^j_{k,s'}, w^{j'}_{k,s'}) \quad \text{where} \quad w^j_{k,s} = (dc^j_{k,s})\zeta^j_{k,s} \quad \text{or} \quad w^j_{k,s} = c^j_{k,s}d\zeta^j_{k,s}.$$

A factor $c^j_{k,s}d\zeta^j_{k,s}$ has an estimate

$$\begin{aligned}&\|c^j_{k,s}d\zeta^j_{k,s}\|_a \leqq C(\|c^j_{k,s}\|_a\|\zeta^j_{k,s}\|_1+\|c^j_{k,s}\|_0\|\zeta^j_{k,s}\|_{a+1}) \\ &\leqq C_K\theta_0^{\alpha-1}\theta_k^a\theta_k^{-\alpha}\theta_0^{\alpha-1}\theta_{k-1}^{2-\alpha} = C_K\theta^{\alpha-2}\theta_k^a(\theta_0/\theta_k)^{2(\alpha-1)}, \quad 0\leqq a\leqq\beta.\end{aligned}$$

Here we have used **H2**, **H3**, (3.20), (3.21) and (3.23). It is an immediate consequence of the derivation of (3.26) that we have the estimate

$$\|w^j_{k,s}\|_a \leqq C_K\theta_k^a(\theta_0/\theta_k)^{\alpha-1}, \quad 0\leqq a\leqq\beta.$$

Now we have the estimate

$$\begin{aligned}&\|(c^j_{k,s}d\zeta^j_{k,s}, w^{j'}_{k,s'})\|_a \leqq C(\|c^j_{k,s}d\zeta^j_{k,s}\|_a\|w^{j'}_{k,s'}\|_0 \\ &+\|c^j_{k,s}d\zeta^j_{k,s}\|_0\|w^{j'}_{k,s'}\|_a) \leqq C_K\theta^{\alpha-2}\theta_k^a(\theta_0/\theta_k)^{3(\alpha-1)} \\ &\leqq C_K\theta^{\alpha-2}\theta_k^a\theta^{-(k+1)\gamma}, \quad 0\leqq a\leqq\beta.\end{aligned}$$

Here we have used $3k/2\geqq k+1$, that is, $k\geqq 2$. The fact that this estimate is not true for $k=1$ is the reason why the second step above could not be covered by the general step.

Combining this estimate with (3.25) and (3.28) we obtain

$$\|(du_{k+1}, du_{k+1})-(du_k, du_k)-m_k\| \leqq C_K\theta^{\gamma-\beta}\theta_k^a\theta^{-(k+1)\gamma}, \quad 0\leqq a\leqq\beta,$$

since $\alpha\geqq\beta$. In view of (3.27) we then get the estimate

$$\|e_{k+1}\|_a \leqq C_K\theta^{\gamma-\beta}\theta_k^a\theta^{-(k+1)\gamma}, \quad 0\leqq a\leqq\beta.$$

If θ is large enough the coefficient will be less than $\frac{1}{2}\varepsilon$, which means that (3.5) is fulfilled for $i=k+1$.

To prove Lemma 3.1 it is therefore sufficient to prove Lemma 3.3.

Proof of Lemma 3.3. Write formally half of the functions $c^j_{k,s}$ as

$$\chi_j a^j_{k,t}\cos(\theta_k\varphi^j_t)/\theta_k, \quad j\in J, \quad t = 0, \ldots, s_n,$$

and the other half as

$$\chi_j a^j_{k,t} \sin(\theta_k \varphi^j_t)/\theta_k, \quad j \in J, \quad t = 0, \dots, s_n;$$

where φ^j_t are defined in Lemma 2.3. Then for (3.23) and (3.24) to be fulfilled it is sufficient to find real valued functions $a^j_{k,t}$ with

$$\|a^j_{k,t}\|_a \leqq C\theta^a_k(\theta_0/\theta_k)^{\alpha-1}, \quad 0 \leqq a \leqq 4; \tag{3.29}$$

$$\sum_{j,t} \big((\chi_j a^j_{k,t})^2 (d\varphi^j_t)^2 + \chi_j a^j_{k,t} M^j_{k,t} + (d(\chi_j a^j_{k,t})/\theta_k)^2\big) = m_k - R_k, \tag{3.30}$$

where we have set

$$M^j_{k,t} = 2\big(\cos(\theta_k\varphi^j_t)(d\tilde{u}_k, d\xi^j_{k,t}) + \sin(\theta_k\varphi^j_t)(d\tilde{u}_k, d\eta^j_{k,t})\big)/\theta_k. \tag{3.31}$$

Here $\{\xi^j_{k,t}, \eta^j_{k,t}\}$ is a partition of $\{\zeta^j_{k,s}\}$ corresponding to the partition of $\{c^j_{k,s}\}$ made above.

The construction of $a^j_{k,t}$ is made by a heavy use of Lemma 2.3. Let I be an integer so large that $\alpha < 2-\beta/(I+1)$. We then want to show that we can define functions $a^{j,i}_{k,t}$ by the formula

$$a^{j,i+1}_{k,t} = F^j_t\big(\theta^{k\gamma/2}\mathbf{M}_k, \theta^{k\gamma}(m_k - \textstyle\sum_{j,t}(d(\chi_j a^{j,i}_{k,t})/\theta_k)^2)\big)\theta^{-k\gamma/2}, \quad i = 0, \dots, I-1, \tag{3.32}$$

$$a^{j,0}_{k,t} = 0,$$

where $\mathbf{M}_k = \{M^j_{k,t}\}$, so that

$$\|a^{j,i}_{k,t}\|_a \leqq C_i\theta^a_k(\theta_0/\theta_k)^{\alpha-1}, \quad 0 \leqq a \leqq 4+I-i; \tag{3.33}$$

$$\|a^{j,i}_{k,t}\|_a \leqq C_i(\theta^{\alpha-2}+\theta^{-\beta})\theta^a_k\theta^{-k\gamma/2}, \quad 1 \leqq a \leqq 4+I-i; \tag{3.34}$$

$$\theta_k^{-2}\big\|(d(\chi_j a^{j,i}_{k,t}))^2 - (d(\chi_j a^{j,i-1}_{k,t}))^2\big\|_a \tag{3.35}$$
$$\leqq C_i(\theta^{\alpha-2}+\theta^{-\beta})^{i+1}\theta^a_k\theta^{-k\gamma}, \quad 0 \leqq a \leqq 3+I-i.$$

Here the constants C_i depend on i and K and F^j_t are the C^∞ functions of (2.5). Since $(I+1)(\alpha-2) < -\beta$ and

$$R_k = -\sum_{j,t}\theta_k^{-2}\big((d\chi_j a^{j,I}_{k,t})^2 - (d\chi_j a^{j,I-1}_{k,t})^2\big),$$

if we choose $a^j_{k,t} = a^{j,I}_{k,t}$, this will prove Lemma 3.3.

It suffices to prove that the function $a^{j,i+1}_{k,t}$ fullfills (3.33)—(3.35) if $a^{j,i}_{k,t}$ does. First we prove that the right hand side of (3.32) is well defined. If we take θ large enough then the computation at the beginning of the second step gives

$$\|\theta^{k\gamma}m_k - g\|_0 \leqq C\left(\theta_k^{-\beta} + \frac{1}{2}\varepsilon\theta^{-\beta} + \theta^{-\gamma}\right) + \frac{1}{2}\varepsilon < 3\varepsilon/4.$$

Using (3.31) and (3.20) we obtain

$$\|\theta^{k\gamma/2}M^j_{k,t}\|_0 \leqq C_K\theta^{k\gamma/2}\theta_k^{-1}\theta_0^{\alpha-1}\theta_{k-1}^{2-\alpha} = C_K\theta^{\alpha-2} < \varepsilon,$$

and finally (3.34) gives

$$\|\theta^{k\gamma}(da_{k,t}^{j,i}/\theta_k)^2\|_0 \leqq C_i(\theta^{\alpha-2}+\theta^{-\beta})^2 < \varepsilon/4.$$

In view of the property **P 2** this shows that (3.32) is well-defined if θ is large enough. To obtain an estimate for $a_{k,t}^{j,i+1}$ we first have to estimate the H^a-norms of $\theta^{k\gamma}m_k$ and $\theta^{k\gamma/2}M_{k,t}^j$. Repeating the estimate of $\theta^\gamma m_1$ in the second step we get

$$\|\theta^{k\gamma}m_k\|_a \leqq C\theta^{-\min(\alpha,\beta)}\theta_k^a, \quad 0 \leqq a \leqq 4+I,$$

and in view of **H 5** (i) the estimate of $\|\theta^{k\gamma/2}M_{k,t}^j\|_0$ above can be extended to

$$\|\theta^{k\gamma/2}M_{k,t}^j\|_a \leqq C_K\theta^{\alpha-2}\theta_k^a, \quad 0 \leqq a \leqq 4+I.$$

Using (3.34) this gives

$$\|a_{k,t}^{j,i+1}\|_a \leqq C\big(1+(C_K\theta^{\alpha-2}+C\theta^{-\min(\alpha,\beta)})\theta_k^a\big)\theta^{-k\gamma/2}, \quad 0 \leqq a \leqq 3+I-i,$$

from which we deduce (3.33) and (3.34) for $i+1$ if θ is larger than some number depending on K and θ_0/θ is large. This also gives (3.35) for $i=1$.

It remains only to prove (3.35) for $i+1$. First we note that if F is a C^∞ function of M_t^j and f, and $\|f_0\|_0 \leqq C$ then **H 4** (ii) implies

$$\|F(\mathbf{M}, f_0+f)-F(\mathbf{M}, f_0)\|_a \leqq C_a\big((\|\mathbf{M}\|_a+\|f_0\|_a+\|f\|_a+1)\|f\|_0+\|f\|_a\big).$$

By definition

$$a_{k,t}^{j,i+1}-a_{k,t}^{j,i} = \big(F_t^j(\mathbf{M}, f_0+f)+F_t^j(\mathbf{M}, f_0)\big)\theta^{-k\gamma/2}, \quad i>0,$$

where

$$\mathbf{M} = \theta^{k\gamma/2}\mathbf{M}_k,\ f_0 = \theta^{k\gamma}\big(m_k-\textstyle\sum_{j,t}(d(\chi_j a_{k,t}^{j,i-1})/\theta_k)^2\big)$$

and

$$f = \theta^{k\gamma}\textstyle\sum_{j,t}\theta_k^{-2}\big((d(\chi_j a_{k,t}^{j,i-1}))^2-(d(\chi_j a_{k,t}^{j,i}))^2\big).$$

In view of the estimates above this implies that

$$\|a_{k,t}^{j,i+1}-a_{k,t}^{j,i}\|_a \leqq C_i(\theta^{\alpha-2}+\theta^{-\beta})^{i+1}\theta_k^a\theta^{-k\gamma/2}, \quad 0 \leqq a \leqq 3+I-i.$$

Using this together with (3.33) for i and $i+1$ we get the estimate

$$\begin{aligned}
&\theta_k^{-2}\big\|(d(\chi_j a_{k,t}^{j,i+1}))^2-(d(\chi_j a_{k,t}^{j,i}))^2\big\|_a \\
&\leqq C\theta_k^{-2}\big((\|a_{k,t}^{j,i+1}\|_{a+1}+\|a_{k,t}^{j,i}\|_{a+1})\|a_{k,t}^{j,i+1}-a_{k,t}^{j,i}\|_1 \\
&\quad +(\|a_{k,t}^{j,i+1}\|_1+\|a_{k,t}^{j,i}\|_1)\|a_{k,t}^{j,i+1}-a_{k,t}^{j,i}\|_{a+1}\big) \\
&\leqq C_{i+1}(\theta_k^{\alpha-2}+\theta^{-\beta})^{i+2}\theta_k^a\theta^{-k\gamma}, \quad 0 \leqq a \leqq 3+I-i.
\end{aligned}$$

This proves (3.34) for $i+1$, and therefore the first half of Theorem 1.1.

4. A necessary condition on the regularity

In this section we shall prove the second half of Theorem 1.1. Since regularity is a local property we can assume that X is a ball in $\mathbf{R}^n$ with center at 0. The equation $(du, du)=g$ is then equivalent to $n(n+1)/2$ equations $(\partial_i u, \partial_j u)=g_{ij}$. Let X_h denote the set of points in X whose distance to the boundary is at least h.

Now fix $\varphi\in C_0^\infty$ with support in the unit ball with $\int \varphi\, dx=1$ and define

$$d_k v(x) = \int v(x-hy)\,\partial_k\varphi(y)\,dy \quad \text{for} \quad v\in C^0(X, \mathbf{R}), \quad x\in X_h.$$

Then d_k will be an operator depending on h and we make it a convention that formulas involving d_k are valid in X_h and formulas involving $d_s d_k$ are valid in X_{2h}. Then we have the following properties:

$$d_s d_k v = d_k d_s v, \quad v\in C^0; \tag{4.1}$$

$$d_k \partial_i v = d_i \partial_k v, \quad v\in C^1; \tag{4.2}$$

$$d_k v = O(h^a), \quad v\in H^a, \quad 0\leqq a\leqq 1; \tag{4.3}$$

$$d_k(uv) = (d_k u)v+u d_k v+O(h^{a+b}) \quad \text{if} \quad u\in H^a \quad \text{or} \quad u = O(h^a), \tag{4.4}$$

$$v\in H^b \quad \text{or} \quad v = O(h^b), \quad 0\leqq a,\ b\leqq 1.$$

Here $O(h)$ represents any function $v(x, h)$ such that, for h in a neighborhood of 0, $|v(x, h)|/|h|$ is bounded by a constant independent of x and h. (4.1)—(4.3) are immediate consequences of the definition and the fact that

$$\int \partial_k\varphi\, dx = 0, \quad k = 1, \dots, n,$$

and (4.4) follows from the formula

$$\big(d_k(vw)-(d_k v)w-v d_k w\big)(x) = \int \big(v(x-hy)-v(x)\big)\big(w(x-hy)-w(x)\big)\,\partial_k\varphi(y)\,dy.$$

Lemma 4.1. *Let* $g=(du, du)$, $u\in H^\alpha$, $\alpha>1$, and set

$$L_h(g) = \frac{1}{2}(d_s d_i g_{jk}+d_k d_j g_{si}-d_s d_j g_{ik}-d_k d_i g_{js}).$$

Then we have the estimate

$$\sup_{X_{2h}} |L_h(g)| \leqq C h^{2(\alpha-1)} \tag{4.5}$$

for h small. Here C is independent of h.

Proof. First note that $g\in H^{\alpha-1}$. Set $u_i=\partial_i u$. From $(u_i, u_j)=g_{ij}$ we get

$$d_k g_{ij} = (d_k u_i, u_j)+(u_i, d_k u_j)+O(h^{2(\alpha-1)})$$

using (4.4). Permuting the indices and using (4.2) we obtain

$$(d_k u_i, u_j) = T_{ijk}+O(h^{2(\alpha-1)})$$

where

$$T_{ijk} = \frac{1}{2}(d_k g_{ij} + d_i g_{jk} - d_j g_{ik}) = O(h^{\alpha-1}).$$

We can then write

(4.6) $\quad d_k u_i = \sum_m (T_{ki}^m + O(h^{2(\alpha-1)}))u_m + F_{ki}, \quad (F_{ki}, u_j) = 0, \quad j = 1, \dots, n,$

with

$$T_{ki}^m = \sum_j g^{mj} T_{ijk} = O(h^{\alpha-1}).$$

Here (g^{ij}) is the inverse of (g_{ij}) and belongs to $H^{\alpha-1}$. Using (4.6) and (4.4) we then obtain

$$d_s d_k u_i = \sum_m (d_s T_{ki}^m) u_m + \sum_m T_{ki}^m d_s u_m + d_s F_{ki} + O(h^{2(\alpha-1)}).$$

But $T_{ki}^m d_s u_m = O(h^{2(\alpha-1)})$ since $d_s u_m = O(h^{\alpha-1})$, and using (4.4) we get

$$d_s T_{ki}^m = \sum_l g^{ml}(d_s T_{ilk}) + \sum_l (d_s g^{ml}) T_{ilk} + O(h^{2(\alpha-1)})$$
$$= \sum_l g^{ml} d_s T_{ilk} + O(h^{2(\alpha-1)}).$$

From this we deduce that

$$(d_s d_k u_i, u_j) = \sum_{m,l} g_{mj} g^{ml} d_s T_{ilk} + (d_s F_{ki}, u_j) + O(h^{2(\alpha-1)}).$$

Moreover, $|F_{ki}| \leqq |d_k u_i|$ implies that $F_{ki} = O(h^{\alpha-1})$ so

$$(d_s F_{ki}, u_j) = -(F_{ki}, d_s u_j) + O(h^{2(\alpha-1)}) = O(h^{2(\alpha-1)}).$$

Since $\sum_m g_{mj} g^{ml} = \delta_{jl}$ (Kronecker delta) this shows that

$$(d_s d_k u_i, u_j) = d_s T_{ijk} + O(h^{2(\alpha-1)}).$$

The equation

$$(d_s d_k u_i - d_k d_s u_i, u_j) = 0$$

then implies (4.5).

Lemma 4.2. *Let E be the set of all $g \in H^\beta$ with $\|g\|_\beta \leqq C$ and*

$$\sup_{X_{2h}} |L_h(g)| \leqq C' h^{\beta+\varepsilon}, \quad 0 < h < 1$$

for some $\varepsilon > 0$ and some constant C'. Then E is of the first category.

Proof. Let $\varphi^h(x) = h^{-n}\varphi(x/h)$. Then

$$d_j g(x) = \int g(x-hy)\,\partial_j \varphi(y)\,dy = h(\partial_j g * \varphi^h)(x),$$

from which we obtain that $d_i d_j g = h^2(\partial_i \partial_j g * \tilde{\varphi}^h)$ with $\tilde{\varphi} = \varphi * \varphi$. If we set

$$L(g) = \frac{1}{2}(\partial_s \partial_i g_{jk} + \partial_k \partial_j g_{si} - \partial_s \partial_j g_{ik} - \partial_k \partial_i g_{js}),$$

this shows that $L_h(g) = h^2(L(g) * \tilde{\varphi}^h)$. Here all derivatives are taken in the sense of distribution theory. Now define

$$E_\varepsilon = \{g \in H^\beta \colon \|g\|_\beta \leqq C, \ \sup_{X_{2h}} |L_h(g)| \leqq h^{\beta+\varepsilon}/\varepsilon, \ 0 < h < 1\}.$$

It is clear that E_ε is closed, symmetric and convex. To show that E_ε has no interior points it is sufficient to show that 0 is not an interior point. For this we take $\psi\in C_0^\infty$ with $L(\psi)*\tilde{\varphi}\neq 0$ and define $\psi_h(x)=h^b\psi(x/h)$, $\beta<b<\beta+\varepsilon$. Note that if $i=s=1$, $j=k=2$ then

$$L(g) = \frac{1}{2}(\partial_1\partial_1 g_{22}+\partial_2\partial_2 g_{11}-2\partial_1\partial_2 g_{12}) \not\equiv 0.$$

Then we have $L_h(\psi_h)(x)=h^b(L(\psi)*\tilde{\varphi})(x/h)$ which shows that

$$h^{-(\beta+\varepsilon)}\sup|L_h(\psi_h)| = h^{b-(\beta+\varepsilon)}\sup|L(\psi)*\tilde{\varphi}| \to \infty \quad \text{as} \quad h\to 0.$$

Moreover, $\|\psi_h\|_a\leqq C_a h^{b-a}$ since this is true when a is an integer. Hence $\|\psi_h\|_\beta\to 0$ as $h\to 0$ which proves the lemma.

Suppose $g\in H^\beta$ and that there is some $u\in H^\alpha$, $2(\alpha-1)>\beta$ with $(du, du)=g$. Then (4.5) implies that

$$h^{-2(\alpha-1)}\sup_{X_{2h}}|L_h(g)| \leqq C, \quad 0<h<1,$$

with C independent of h. But Lemma 4.2 then implies that g must belong to a set of the first category in H^β. This completes the proof of Theorem 1.1.

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