Differential equations and the Bergman—Šilov boundary on the Siegel upper half plane

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1. Preliminaries

Let I_n be the $n\times n$ identity matrix and set

$$J=\begin{pmatrix}0 & I_n\\ -I_n & 0\end{pmatrix}.$$

The symplectic group $Sp(n, \mathbf{R})$ is defined as the group of $2n\times 2n$-matrices M with real entries for which $MJ=JM'$ where M' denotes the transpose of M. The group $K=0(2n)\cap Sp(n,\mathbf{R})$ is easily seen to be a maximal compact subgroup of $Sp(n,\mathbf{R})$ and the space $Sp(n,\mathbf{R})/K=H_n$ is a (hermitian) symmetric space. The space H_n is called the Siegel upper half plane of rank n and has a geometric realization as the space of all complex $n\times n$-matrices Z for which $Z=Z'$ and $\operatorname{Im} Z>0$.

If

$$M=\begin{pmatrix}A & B\\ C & D\end{pmatrix}\in \mathrm{Sp}\,(n,\mathbf{R})$$

where A, B, C, D are $n\times n$-matrices then for $Z\in H_n$ we define $M(Z)=(AZ+B)(CZ+D)^{-1}\in H_n$. Then K is the subgroup of $Sp(n,\mathbf{R})$ which fixes iI_n. The $Sp(n,\mathbf{R})$-invariant metric on H_n is

$$ds^2=-tr\big((Z-\bar{Z})^{-1}d\bar{Z}(Z-\bar{Z})^{-1}dZ\big)$$

and the corresponding $Sp(n,\mathbf{R})$-invariant Laplacian is given by

$$\Delta=-tr\big((Z-\bar{Z})\,\partial_Z(Z-\bar{Z})\,\partial_Z\big)$$

where for $A=A'=(a_{ij})$ $\partial_A=(\partial_{ij})$ and $\partial_{ij}=\frac{1}{2}(1+\delta_{ij})\dfrac{\partial}{\partial a_{ij}}$ and the ∂_Z does not differentiate the $Z-\bar{Z}$ matrix (see Hua [6]).

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Of particular interest to us in this paper are the concepts of boundaries of H_n. Of the many boundaries the most crucial to us are the Furstenberg boundary and the Bergman—Šilov boundary. The Furstenberg boundary is the space $B=K/M$ where M is the group of diagonal matrices in K and the Bergman—Šilov boundary is the space $B_0=K/K_0$ where

$$K_0 = \left\{\begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} : A \in O(n)\right\}.$$

To each of these boundaries there is associated a Poisson kernel and if $P\colon H_n \times B \to \mathbf{R}^+$ (resp. $P_0\colon H_n \times B_0 \to \mathbf{R}^+$) is the Poisson kernel of H_n associated to the Furstenberg (resp. Bergman—Šilov) boundary then

$$P_0(Z, kK_0) = \int_{K_0} P(Z, kk_0 M)\, dk_0.$$

Using the fact that the space of all real symmetric $n\times n$-matrices, S_n, may be imbedded in B_0 so that $B_0 \sim S_n$ has measure 0 we have that on $H_n \times S_n$ (almost everywhere in $H_n \times B_0$)

$$P_0(Z, U) = c\,\frac{\det(Z-\bar{Z})^{\frac{n+1}{2}}}{|\det(Z-U)|^{n+1}} = c\det\big(Y+(X-U)Y^{-1}(X-U)\big)^{-(n+1)}.$$

It was first observed by Hua [5] that every entry of the matrix

$$D = (Z-\bar{Z})\,\partial_Z (Z-\bar{Z})\,\partial_Z$$

annihilates P_0, and it was conjectured by Stein in a more general setting that the functions annihilated by D characterize the Bergman—Šilov boundary. That is, if $Df=0$

$$f(Z) = \int_{B_0} P_0(Z, b)\, F(b)\, db$$

for some functional on B_0. In [7] A. Koranyi and P. Malliavin gave a partial affirmative answer to this conjecture by showing that if $f \in \mathscr{L}^\infty(H_2)$ $(n=2)$ and $\Delta f = \Delta' f = 0$ where

$$\Delta' f = c\,tr\big(\partial_Z (Z-\bar{Z})\,\partial_Z\big) f$$

then

$$f(Z) = \int_{B_0} P_0(Z, U)\, F(U)\, dU$$

for some $F \in \mathscr{L}^\infty(B_0)$.

In this paper we again give a partial affirmative answer to this conjecture by showing that if $f \in \mathscr{L}^\infty(H_n)$ (f is harmonic in the sense of [1]) and $Df=0$ that

$$f(Z) = \int_{B_0} P_0(Z, U)\, F(U)\, dU$$

for some $F \in \mathscr{L}^\infty(B_0)$ (F a functional on B_0).

Our techniques differ substantially from those of [7] in that rather than using compound diffusion processes we push our differential equations to the boundary B. To do this we use the result of H. Furstenberg [1] that

$$f(Z) = \int_B P(Z, b) F(b)\, db$$

for some $F \in \mathscr{L}^\infty(B)$. Under the initial assumption that $F \in C^\infty(B)$, we show using asymptotic growth that F is in fact in $C^\infty(B_0)$. Our result then follows using the fact that if $Df=0$

$$DLgf = 0 \quad (g \in \mathrm{Sp}(n, \mathbf{R}))$$

where $Lgf(Z)=f(g^{-1}(Z))$.

In section 2 we give a brief discussion of the Poisson kernel on a general symmetric space cross its Furstenberg boundary. In section 3 we prove our result under the assumption that F is C^∞, and in sections 4 and 5 we complete our proof.

In another forthcoming paper we shall give a new proof of the result of Malliavin and Koranyi which follows techniques similar to those discussed here.

2. Convergence properties of Poisson kernels

Let G be a non-compact semisimple Lie group with finite center. Fix $G=KAN$ an Iwasawa decomposition of G. That is, K is a maximal compact subgroup of G, A is a maximal vector subgroup of G consisting of semisimple elements normalizing N, a maximal simply connected nilpotent subgroup of G. Let M be the centralizer of A in K, and if θ is the Cartan involution of G which is the identity on K, set $\theta N=\overline{N}$. Let also $\mathfrak{G}, \mathfrak{K}, \mathfrak{A}, \mathfrak{M}, \mathfrak{N}$ and $\overline{\mathfrak{N}}$ respectively denote the Lie algebras of G, K, A, M, N and $\overline{N}$ respectively.

If $g \in G, g$ may be written uniquely as $g=k(g)\exp H(g) n(g)$ $(k(g)\in K, H(g) \in \mathfrak{A}, n(g) \in N)$ and we now define the general Poisson kernel of G/K as a function $P: G/K \times K/M \to \mathbf{R}^+$ by setting

$$P(gK, kM) = e^{-2\varrho H(g^{-1}k)}$$

where ϱ is the linear functional on $\mathfrak{A}$ given by $2\varrho(H)=\mathrm{tr}\,(\mathrm{ad}\, H|_{\mathfrak{N}})$ for $H \in \mathfrak{A}$. (Note that $G/MAN=K/M=B$.) It is a simple matter to show that P is well-defined and P is called a Piosson kernel because of the following result.

Theorem (Furstenberg [1]). *Suppose Δ is the Laplacian of the G-invariant Riemannian metric on G/K. If f is a bounded function on G/K for which $\Delta f=0$ then there exists $F \in \mathscr{L}^\infty(B)$ for which*

$$f(gK) = \int_B P(gK, b) F(b)\, db$$

where db is the K-invariant measure on B normalized so $\int_B db=1$.

Harish—Chandra has shown that for $F\in\mathscr{L}^1(K)$

$$\int_K F(k)\,dk = \int_{\bar N} d\bar n \int_M dm e^{-2\varrho(H(\bar n))} F(k(\bar n)m).$$

Thus we obtain for $F\in\mathscr{L}^1(K/M)$

$$\int_{K/M} e^{-2\varrho(H(g^{-1}k))} F(kM)\,dkM = \int_{\bar N} e^{-2\varrho H(g^{-1}\bar n)} F(k(\bar n)M)\,d\bar n.$$

Let $\overline{\mathfrak{A}^+}=\{H\in\mathfrak{A}:\operatorname{ad} H|_{\mathfrak{N}}\}$ has all its eigenvalues $\geqq 0\}$. If $H\in\overline{\mathfrak{A}^+}$ let $\mathfrak{G}_1$ be the kernel of ad H and $\mathfrak{N}_2$ be the subspace of $\mathfrak{N}$ which is the span of the positive eigenspaces of ad H. Then $\mathfrak{G}=\mathfrak{N}_2+\mathfrak{G}_1+\overline{\mathfrak{N}}_2$.

Let $\mathfrak{N}_1=\mathfrak{G}_1\cap\mathfrak{N}$, $\overline{\mathfrak{N}}_1=\theta\mathfrak{N}_1$, $N_2=\exp\mathfrak{N}_2$, $\bar N_2=\exp\overline{\mathfrak{N}}_2$ and let G_1 be the centralizer of H in G and $K_1=K\cap G_1$.

Lemma 2.1. *Suppose $F\in C^\infty(B)$ and*

$$f(gK) = \int_B P(gK, b)F(b)\,db.$$

Then for all $g\in G$, $f_H(g)=\lim_{t\to\infty} f(g\exp tHK)$ exists and is C^∞ on G/P where P is the group $K_1Z_1N_2$ where Z_1 is the center of G_1. In fact, if $g\in G$

$$g = k_0 g_1 n_2 \quad (k_0\in K,\ g_1\in G_1,\ n_2\in N_2)$$

$$f_H(g) = c\int_{K_1/M} e^{-2\varrho_1(H(g_1^{-1}k_1))} F(k_0k, M)\,d(k_1M)$$

where $c=\int_{\bar N_2} e^{-2\varrho(H(\bar n))}\,d\bar n$ and for $H'\in A$

$$2\varrho_1(H') = tr(\operatorname{ad} H'|_{N_1}).$$

Proof. An easy calculation yields

$$f(g\exp tHK)$$

$$=\int_{K_1/M} dk_1M \int_{\bar N_2} d\bar n_2 \exp -2\varrho\big(H(n_2^{-1})^{a_t^{-1}} k(g_1^{-1}\bar n_2k_1)\big) e^{-2\varrho(H(g_1^{-1}\bar n_2k_1))} F\big(k_0k(\bar n_2^{a_t})k_1M\big)$$

where $a_t=\exp tH$ and $x^y=yxy^{-1}$. Since the product of the first and third terms on the right handside is C^∞ and converges to $F(k_0k_1M)$ uniformly in k_1 as $t\to\infty$ and the second term is integrable we obtain

$$f_H(g) = \int_{K_1/M} dk_1M \int_{\bar N_2} d\bar n_2 e^{-2\varrho H(g_1^{-1}k_1\bar n_2)} F(k_0k_1M)$$

$$= \int_{K_1/M} dk_1M \int_{\bar N_2} d\bar n_2 e^{-2\varrho H(a(g_1^{-1}k_1)\bar n_2 a(g_1^{-1}k_1)^{-1}} e^{-2\varrho H(g_1^{-1}k_1)} F(k_0k_1M)$$

where $a(g^{-1}k_1)=\exp H(g_1^{-1}k_1)$ and thus our result.

This result guarantees that f_H restricted to G_1/K_1 is harmonic. The next lemma although trivial will be useful.

Lemma 2.2. *Suppose* $H_1, H_2 \in \overline{\mathfrak{A}^+}$ *and* f *as in lemma 2.1. Then*

$$(f_{H_1})_{H_2}(g) = \lim_{t\to\infty} f_{H_1}(g \exp tH_2) = f_{H_0}(g)$$

where H_0 *is any element of* $\overline{\mathfrak{A}^+}$ *with the property that*

$$\operatorname{Ker} \operatorname{ad} H_0 = (\operatorname{Ker} \operatorname{ad} H_1) \cap (\operatorname{Ker} \operatorname{ad} H_2).$$

We shall also make use of the following result of Harish—Chandra [2].

Lemma 2.3. *There is a representation* π *of* G *with highest weight* 2ϱ *(i.e.* $\pi(an)v_{2\varrho} = e^{2\varrho(H)}v_{2\varrho}$ *where* $a = \exp H \in A$ *and* $n \in N$*) and a* K*-fixed vector* v_0 *such that* $P(gk, kM) = (\pi(g^{-1}k)v_{2\varrho}, v_0)^{-1}$ *where (,) is a positive definite inner product invariant under* $G_0 = \exp(\mathfrak{K} + i\mathfrak{P})$ *where* $\mathfrak{P} = -1$ *eigenspace of* θ *acting on* $\mathfrak{G}$.

Lemma 2.4. *For* $x \in G$

$$\frac{1}{t}(P(g_1 \exp tXg_2K, kM) - P(g_1g_2K, kM))$$

converges uniformly to

$$-1(\pi(g_2^{-1}g_1^{-1}k)v_{2\varrho}, v_0)^{-2}(\pi(g_2^{-1}Xg_1^{-1}k)v_{2\varrho}, v_0)$$

as $t \to 0$ *for* $k \in K$ *and* g_2, g_1 *in a fixed compact set.*

The proof follows immediately from the fact that $P(gK, kM) > 0$ for all g and k and the simple lemma

Lemma 2.5. *Let* A, B *be* $n\times n$*-matrices in a fixed compact set and* X *and arbitrary* $n\times n$*-matrix. Then as* $t\to 0, \frac{1}{t}(Ae^{tX}B - AB)$ *converges uniformly to* AXB.

For $X \in G$ and $f \in C^\infty(G/L)$, L a subgroup of G, set $Xf(gL) = \frac{d}{dt} f(\exp - tXgL)|_{t=0}$.

From Lemma 2.4 it follows that if $F \in \mathscr{L}^p(K/M)$ $(1 \leqq p)$ $X \in \mathfrak{G}$ and

$$f(gK) = \int_{K/M} P(gK, kM)F(kM)\,dkM$$

that

$$Xf(gK) = \int_{K/M} \frac{d}{dt} P(\exp - tXgK, kM)|_{t=0} F(kM)\,dkM.$$

Unlike the problems of harmonic functions on Euclidean spaces the problem of taking radial limits on symmetric spaces causes expansions and contractions in some variables and for this reason we shall be forced to differentiate F (when possible) instead of P. For this purpose we state

Lemma 2.6. *If* $F \in C^\infty(B)$ *and* $X_1, \ldots, X_r \in \mathfrak{G}$

$$\sup_{b \in B} |X_1 \ldots X_r L_{k_1} F(b)| < \infty.$$

Proof. This is obvious by definition.

Lemma 2.7. *Suppose* $F \in C^\infty(K/M)$, $H \in \overline{\mathfrak{A}^+}$, $\mathfrak{N}_2 \subseteqq \mathfrak{N}$ *the linear span of the negative eigenspaces of* $\operatorname{ad} H$, $X_1, \ldots, X_r \in \mathfrak{N}_2$, *and* $\bar{n} \in \overline{\mathfrak{N}}_2$. *Then, if*

$$f(gK) = \int_B P(gK, b) F(b)\, db$$

$$(X_1 \ldots X_r L_{\bar{n}} f)_H(g) = c \int_{K_1/M} e^{-2\varrho_1(H(g_1^{-1} k_1))} (X_1 \ldots X_r L_{\bar{n}} F)(k_0 k_1 M)\, dk_1 M$$

where F is thought of as a function on B and c, K_1, ϱ_1, g_1 and k_0 are as in lemma 2.1.

Proof. This follows immediately by direct calculation from lemmas 2.1, 2.6 and the fact that

$$F(\exp -tXg) - F(g) = \int_0^t (XF)(\exp -sXg)\, ds.$$

Lemma 2.8. *Suppose* $F \in C^\infty(B)$, $H \in \overline{\mathfrak{A}^+}$, $X_1, \ldots, X_r \in \mathfrak{G}_1 + \mathfrak{N}_2$. *Then if*

$$f(gK) = \int_B P(gK, b) F(b)\, db$$

and $X_i = Y_i + Z_i$ *for* $Y_i \in \mathfrak{G}_1$ *and* $Z_i \in \mathfrak{N}_2$

$$(X_1 \ldots X_r f)_H(g_1) = Y_1 \ldots Y_r (f_H)(g_1) \quad (g_1 \in G_1).$$

Proof. This again follows by direct calculation from lemmas 2.1 and 2.4, the boundedness of F and the fact that for $\varepsilon > 0$

$$\int_{\bar{N}} e^{-(1+\varepsilon)\varrho(H(\bar{n}))}\, d\bar{n} < \infty.$$

Finally combining lemmas 2.7 and 2.8 we obtain the following.

Lemma 2.9. *Suppose* $F \in C^\infty(B)$, $H \in \overline{\mathfrak{A}^+}$, $X_1 \ldots X_r \in \mathfrak{N}_2$, $Y_1, \ldots, Y_l \in \mathfrak{G}_1 + \mathfrak{N}_2$ *and* $\bar{n} \in \bar{N}_2$. *Then, if*

$$f(gK) = \int_B P(gK, b) F(b)\, db$$

and

$$\tilde{f}(gK) = \int_B P(gK, b)(X_1 \ldots X_r L_{\bar{n}} F)(b)\, db$$

we have

$$(Y_1 \ldots Y_l X_1 \ldots X_r L_{\bar{n}} f)_H(g_1) = Y_1^0 \ldots Y_l^0 \tilde{f}_H(g_1) \quad (g_1 \in G_1)$$

for $Y_i = Y_i^0 + Z_i$ *with* $Y_i^0 \in \mathfrak{G}_1$, $Z_i \in \mathfrak{N}_2$.

We are now in a position to return to our study of the differential equations $Df = 0$ on H_n.

3. A special case

We now return to consider the Siegel upper half plane. We suppose throughout this section that f is a bounded function on H_n for which

$$Df = (Z-\bar{Z})\,\partial_Z(Z-\bar{Z})\,\partial_Z f = 0$$

and that

$$f(gK) = \int_{K/M} P(gK, kM)\,F(kM)\,dkM$$

where $F \in C^\infty(K/M)$. For this purpose we use the results and general structure given in section 2. We leave it as an exercise for the reader to verify the following facts.

1) An Iwasawa decomposition $\mathrm{Sp}(n, \mathbf{R}) = KAN$ of $\mathrm{Sp}(n, \mathbf{R})$ is given by

$$K = \left\{\begin{pmatrix} U & B \\ -V & A \end{pmatrix} \in \mathrm{Sp}(n, \mathbf{R})\right\}$$

$$A = \left\{\left(\begin{array}{c|c} D & 0 \\ \hline 0 & D^{-1} \end{array}\right) : D \text{ is an } n\times n\text{-diagonal matrix with positive entries on the diagonal}\right\}$$

$$N = \left\{\left(\begin{array}{c|c} T & 0 \\ \hline U & T'^{-1} \end{array}\right) \in \mathrm{Sp}(n, \mathbf{R}) : T \text{ is lower triangular with 1's on the diagonal}\right\}.$$

2) The Cartan involution θ which is the identity on K is given by $\theta(X) = X'^{-1}$ for $X \in \mathrm{Sp}(n, \mathbf{R})$.

3) M is the set of diagonal matrices in K.

4) $$\overline{\mathfrak{A}^+} = \left\{\left(\begin{array}{c|c} \begin{matrix} -a_1 & & 0 \\ & \ddots & \\ 0 & & -a_n \end{matrix} & 0 \\ \hline 0 & \begin{matrix} a_1 & & \\ & \ddots & \\ & & a_n \end{matrix} \end{array}\right) : a_i \geqq a_{i+1}\right\}.$$

Observe also that $\mathrm{Sp}(n, \mathbf{R})$ acts transitively on H_n since if $Z \in H_n$, $Z = X + iY$ and then $g_X g_Y(i) = Z$ where

$$g_X = \begin{pmatrix} I_n & X \\ 0 & I_n \end{pmatrix} \quad \text{and} \quad g_Y = \begin{pmatrix} Y^{1/2} & 0 \\ 0 & Y^{-1/2} \end{pmatrix}.$$

Observe also by direct computation or from Hua [6] that if

$$g = \begin{pmatrix} A_0 & B_0 \\ C_0 & D_0 \end{pmatrix} \in \mathrm{Sp}(n, \mathbf{R})$$

and $g(Z) = W$

$$Df(W) = (ZC_0' + D_0')^{-1} D (ZC_0' + D_0') L_{g^{-1}} f(Z)$$

where as before $ZC_0' + D_0'$ is not differentiated by D.

Now let

$$H = \begin{pmatrix} -I_n & 0 \\ 0 & I_n \end{pmatrix} \in \overline{\mathfrak{A}^+}.$$

In keeping with the notation of section 2 we have

$$\mathfrak{N}_2 = \left\{ \begin{pmatrix} 0 & 0 \\ U & 0 \end{pmatrix} : U = U' \right\} = \theta \overline{\mathfrak{N}}_2$$

$$\mathfrak{G}_1 = \left\{ \begin{pmatrix} V & 0 \\ 0 & -V' \end{pmatrix} \right\}$$

and

$$\mathfrak{N}_1 = \left\{ \begin{pmatrix} W & 0 \\ 0 & -W' \end{pmatrix} : W \text{ is lower triangular with 0's on the diagonal} \right\}.$$

Lemma 3.1. *Set* $f_H(g_X g_Y) = f_H(X, Y)$. *Then*

$$0 = (Df)_H = -Y\partial_Y Y \partial_Y f_H(X, Y).$$

Proof. Now $f(g_X g_Y \exp tHi) = f(X + iY(t))$ where $Y(t) = e^{-2t} Y$ and $\partial_{Y(t)} = e^{2t}\partial_Y$ and since each entry of ∂_X is in $\overline{\mathfrak{N}}_2$ and each entry of ∂_Y is a linear combination of elements of $\mathfrak{G}_1$ our result follows from lemma 2.9.

We now have a bounded function $f_H(X, Y)$ satisfying the system of differential equations

$$Y\partial_Y Y\partial_Y f_H = 0.$$

We wish to show that f_H is independent of Y. To do this we shall again apply our results of section 2. First note that

$$f_H(X, Y) = c \int_{K_0/M} e^{-2\varrho_1(H(g_Y^{-1}k))} (L_{g_X} F)(kM)\, dkM$$

for some constant c where ϱ_1 is as in lemma 2.1.

Let $H_1 \in \overline{\mathfrak{A}^+}$ be the matrix (h_{ij}) where $-h_{11} = h_{n-1,n-1} = 1$ and all other entries are 0 and consider the function $(f_H)_{H_1}(g_X g_Y) = \tilde{f}(X, Y)$.

In order to examine this function in more detail it is useful to reparameterize the matrix $Y > 0$ as

$$Y = \begin{pmatrix} 1 & \vec{t}' \\ 0 & I \end{pmatrix} \begin{pmatrix} y & 0 \\ 0 & Y^0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \vec{t} & I \end{pmatrix} = T\tilde{Y}T'$$

where $\vec{t}'$ ($\vec{t}$ resp.) is a vector in $\mathbf{R}^{n-1}$ written as a row (column resp.) and

$$T = \begin{pmatrix} 1 & \vec{t}' \\ 0 & I \end{pmatrix} \quad \text{and} \quad \tilde{Y} = \begin{pmatrix} y & 0 \\ 0 & Y_0 \end{pmatrix}.$$

Observe that $\begin{pmatrix} T & 0 \\ 0 & T'^{-1} \end{pmatrix} \in \exp \overline{\mathfrak{N}}_2$ where now $\overline{\mathfrak{N}}_2$ is the span of the negative eigenvectors of ad H_1.

A simple calculation of Jacobians yields

$$\partial_Y = \left(\begin{array}{c|c} \dfrac{\partial}{\partial y} & \partial_{\vec{t}'} Y_0^{-1} - \vec{t}' \dfrac{\partial}{\partial y} \\ \hline -\vec{t}\dfrac{\partial}{\partial y} + Y_0^{-1}\partial_{\vec{t}} & \partial_{Y^0} - \vec{t}\otimes\vec{t}'\dfrac{\partial}{\partial y} - \vec{t}\otimes\partial_{\vec{t}'} Y_0^{-1} - Y_0^{-1}\partial_{\vec{t}}\otimes\vec{t}' \end{array}\right)$$

where $\partial_{\vec{t}'} = \left(\frac{1}{2}\frac{\partial}{\partial t_1}, \dots, \frac{1}{2}\frac{\partial}{\partial t_{n-1}}\right)$ and $\vec{u}\otimes\vec{v}$ is the matrix which sends $\vec{e}$ to $(\vec{e}\cdot\vec{v})\vec{u}$.

Now since $Y\partial_Y Y\partial_Y f_H = 0$ we of course obtain

$$D_1 f_H = \begin{pmatrix} 1 & -\vec{t}' \\ 0 & 1 \end{pmatrix} Y\partial_Y Y\partial_Y f_H = 0$$

and

$$D_1 = \left(\begin{array}{c|c} y^2\dfrac{\partial}{\partial y} + y\vec{t}\cdot\partial_{\vec{t}} & y\partial_{\vec{t}'} \\ \hline y\partial_{\vec{t}} + Y\circ\partial_{Y_0} Y\vec{t} & Y_0\,\partial_{Y_0} Y_0 \end{array}\right)\circ\partial_Y$$

where ∂_{Y_0} does not differentiate Y_0.

By lemma 2.9 we see that

$$\left(y\frac{\partial}{\partial y} f_H\right)_{H_1} = 0 \quad\text{and}\quad (y f_H)_{H_1} = 0.$$

Thus, we see by direct computation that the $n-1\times 1$-column operator d_1 in the lower left hand corner of D_1 yields

$$(d_1 f_H)_{H_1} = Y_0\,\partial_{Y_0} Y_0 Y_0^{-1}\partial_{\vec{t}} \tilde{f} = 0$$

and the $n-1\times n-1$-matrix D_2 in the lower right hand corner yields

$$(D_2 f_H)_{H_1} = Y_0\,\partial_{Y_0} Y_0\,\partial_{Y_0}\tilde{f} - Y_0\,\partial_{Y_0} Y_0^0 Y_0^{1-1}\,\partial_{\vec{t}}\tilde{f}\otimes\vec{t}' = 0.$$

Thus we obtain

Lemma 3.2. $Y\circ\partial_{Y_0} Y\circ\partial_{Y_0}\tilde{f}=0$,

$Y\circ\partial_{Y_0} Y_0\circ Y_0^{-1}\partial_{\vec{t}}\tilde{f}=0$, and $\tilde{f}$ is harmonic on the space of $n-1\times n-1$-matrices Y_0.

Lemma 3.3. *Suppose φ is a harmonic function on the space of positive definite $n\times n$-matrices Y for which $Y\partial_Y Y\partial_{Y_0}\varphi=0$. Suppose also that*

$$\varphi(Y) = \int_B P(Y, b)\Phi(b)\,db$$

where B is the Furstenberg boundary of the positive matrices and $\Phi\in C^\infty(B)$. Then φ is a constant.

Proof. We prove our result by induction on the dimension n. If $n=1$ the result is obvious. So assume the result for $n \leqq r-1$. If $H=(h_{ij})$ where $h_{11}=-1$ and all $h_{ij}=0$ we write $\tilde{\varphi}(Y)=\tilde{\varphi}(Y_0, y, \mathfrak{t})$ where

$$Y = \begin{pmatrix} 1 & \mathfrak{t}' \\ 0 & I \end{pmatrix} \begin{pmatrix} y & 0 \\ 0 & Y_0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \mathfrak{t} & I \end{pmatrix}.$$

By lemmas 2.1, 3.2 and induction we see that $\tilde{\varphi}$ depends only on $\mathfrak{t}$ and satisfies the system of differential equations

$$Y \circ \partial_{Y_0} Y_0 \circ Y_0^{-1} \partial_i \tilde{\varphi} = 0 = -\left(\frac{r+1}{2}\right) \partial_i \tilde{\varphi} = 0$$

and hence $\tilde{\varphi}$ is also independent of $\mathfrak{t}$. Thus $\tilde{\varphi}$ is a constant and so φ is a constant.

4. Bounded case

Suppose now that f is a function on H^n and $Df=0$. Throughout this section we shall suppose that

$$f(Z) = \int_{K/M} P(Z, kM) F(kM)\, dkM$$

for $F \in \mathscr{L}^2(K/M)$. We shall show that F is in fact an $\mathscr{L}^2$-function on the Bergman—Šilov boundary of H_n.

Observe first that if $Df=0$, $D(L_g f)=0$ since if

$$g = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}$$

$$0 = Df(gZ) = (ZC_1' + D_1')^{-1} D(ZC_1' + D_1')(L_{g^{-1}} f)(Z).$$

So if $D=(D_{ij})$ we have that D_{ij} commutes with $(ZC_1' + D_1')$ and hence $D_{ij}(L_{g^{-1}} f)'=0$. Thus we see that if α is a continuous function on K

$$\alpha *_K f(Z) = \int_K \alpha(k) f(k^{-1}Z)\, dk$$

is annihilated by D.

Let $\hat{K}$ be the set of equivalence classes of finite dimensional representations of K and for $\tau \in \hat{K}$ we shall abuse notation and identify an element of τ with τ itself. Now if $\tau \in \hat{K}$ let $\mathfrak{X}_\tau(k) = (\deg \tau)\, tr\, \tau(k)$. We now have that following facts

(i) $(\mathfrak{X}_\tau *_K f)(Z) = \int_{K/M} P(Z, kM)(\mathfrak{X}_\tau *_K F)(kM)\, dkM$;

(ii) $F = \sum_{\tau \in \hat{K}} (\mathfrak{X}_\tau *_K F)$ in $\mathfrak{L}^2(K/M)$;

and

(iii) $f = \sum_{\tau \in \hat{K}} \mathfrak{X}_\tau *_K f$ in $C^\infty(H_n)$.

(i) is immediate, (ii) is the standard Fourier—Peter—Weyl decomposition of F, and (iii) is proved in Harish—Chandra [3].

Theorem 4.1. $F \in \mathcal{L}^2(B_0)$.

Proof. By our remarks preceeding the theorem we see that $Df=0$ only if $D(\mathfrak{X}_\tau *_K f)=0$ for every $\tau \in \hat{K}$. By (i), the fact that $\mathfrak{X}_\tau *_K F \in C^\infty(K/M)$ and theorem 3.1 we have that $\mathfrak{X}_\tau *_K F \in C^\infty(B_0)$. Now since $F \in \mathcal{L}^2(K)$ and each term is its Fourier expansion is a function in the Bergman—Šilov boundary $F \in \mathcal{L}^2(B_0)$.

As a corollary to this result and Furstenberg's theorem we have

Theorem 4.2. *If* $f \in \mathcal{L}^\infty(H_n)$ *and* $Df=0$

$$f = \int_{B_0} P(Z, kM) F(kM)\, dkM$$

for some $F \in \mathcal{L}^\infty(B_0)$.

5. Harmonic function

Suppose now that f is harmonic in the sense of Furstenberg [1] and $Df=0$. Then $\Delta f=0$ if $\Delta 1=0$ and Δ commutes with characters of G. Set

$$f = \sum_{\tau \in \hat{K}} \mathfrak{X}_\tau *_K f.$$

In this section we show that

$$f_\tau(Z) = \mathfrak{X}_\tau *_K f(Z) = \int_{B_0} P_0(Z, b) T_\tau(b)\, db$$

where $T_\tau = \mathfrak{X}_\tau *_K T_\tau$ and if we define the functional T on B_0 by setting for F a function on B_0

$$\int_{B_0} F(b) T(b)\, db = \sum_{\tau \in \hat{K}} \int_{B_0} F(b) T_\tau(b)\, db$$

then

$$f(Z) = \int_{B_0} P_0(Z, b) T(b)\, db.$$

Since $tr\, Df_\tau = \ldots = tr\, D^n f_\tau = 0$ we have that

$$f_\tau(Z) = \int_B P(Z, b) S_\tau(b)\, db \quad \text{(see appendix)}$$

for some $S_\tau = \mathfrak{X}_\tau *_K S_\tau$, but now S_τ is a function on B_0 by our results in sections 2 and 3. Thus we obtain our main result.

Theorem 5.1. *If f is a function on H_n, $Df=0$, and $\Delta f=0$ for all G-invariant Δ which kill constants then there is a functional T on the Bergman—Šilov boundary B_0 for which*

$$f(Z) = \int_{B_0} P_0(Z, b) T(b)\, db.$$

6. Appendix

Discussion of harmonic functions

Let G, K, A, M and N be as in section 2 and set $X=G/K$. Let M' be the normalizer of A in K and $W=M'/M$ be the restricted Weyl group which operates on A. Let $D_0(G/K)$ denote the ring of all differential operators on the space G/K which commute with left translation by elements of G. For $\tau\in\hat{K}$ let $E_\tau=\{f\in\mathcal{L}^2(K/M)\colon \mathfrak{X}_\tau *_K f=f\}$.

We are now in a position to state the main result of this section. Although this result is "known" in more generality, a precise reference is difficult to give.

Theorem 6.1. *Suppose* $f\in C^\infty(G/K), \mathfrak{X}_\tau *_K f=f$ *and* $Df=0$ *for all* $D\in D_0(G/K)$ *which annihilate constants. Then there is a* $T\in E_\tau$ *for which*

$$f(gK)=\int_{K/M} P(gK, kM)T(kM)\,dkM$$

Proof. For $\sigma\in\hat{K}$ let $C_\sigma(G/K)$ be the set of continuous functions F on G/K for which $\mathfrak{X}_\sigma *_K F=F$ and set $C_\sigma^\infty(G/K)=C^\infty(G/K)\cap C_\sigma(G/K)$ and $C_{c,\sigma}^\infty(G/K)=C_c^\infty(G/K)\cap C_\sigma(G/K)$. Consider the following sesqui-linear form

$$C_{c,\tau}^\infty(G/K)\otimes C_\tau^\infty(G,K)\to \mathbf{C}$$

defined by

$$g\otimes f\to\langle g,f\rangle=\int_{G/K}\overline{g(x)}f(x)\,dx.$$

Now for $D\in D_0(G/K)$, $\langle g, Df\rangle=\langle D*g, f\rangle$ where D^* is the formal adjoint of D. It is well known that $D_0(G/K)=\mathbf{C}[D_1,\dots,D_l]$ where $l=\dim A$ and each $D_i\in D_0(G/K)$ annihilates constants.

We shall now complete our proof with the aid of a few simple lemmas.

Lemma 6.2. *Let* $T\in\mathcal{D}'(G/K)$ *and* $\mathfrak{X}_\tau *T=T$. *Then* $T\in C^\infty(G/K)$ *and* $D_iT=0$ *for all* $i\leq l$ *if and only if* $T(\overline{D_i^*h})=0$ *for all* $h\in C_c^\infty(G/K)$.

Proof. This is immediate since $D_0(G/K)$ contains elliptic operators.

Thus our solutions to $D_if=0$ $(i\leq l)$ for $f\in C_\tau^\infty(G/K)$ are in one to one correspondence with the conjugate linear functionals on $C_{c,\tau}^\infty(G/K)$ which annihilate

$$\sum_{i=1}^{l} D_i^* C_{c,\tau}^\infty(G/K).$$

If $h\in C_{c,\tau}^\infty(G/K)$ we define

$$\hat{h}(g)=e^{\varrho(H(g))}\int_N h(gnK)\,dn$$

$\hat{h}$ is now a function on G/MN or equivalently on $K/M\times A$ and in fact we have that $\hat{h}\in E_\tau\otimes C_c^\infty(A)$. Furthermore, if $\hat{h}=\sum_{j=1}^{n} v_j\otimes h_j$ $(v_j\in E_\tau,\ h_j\in C_c^\infty(A))$ and $D\in$

$D_0(G/K)$ there is a differential operator with constant coefficients Δ on A for which

$$\widehat{Dh} = \sum_{j=1}^{n} v_j \otimes \Delta h_j = \Delta\left(\sum_{j=1}^{n} v_j \otimes h_j\right).$$

Thus we obtain the following lemma

Lemma 6.2.

$$\sum_{i=1}^{l} \widehat{D_i^* C_{c,\tau}^{\infty}(G/K)} \subset E_\tau \otimes \sum_{i=1}^{l} \Delta_i^* C_c^{\infty}(A).$$

From Theorem 8.5 of Helgason [5] we have:

Lemma 6.4.

$$\widehat{C_{c,\tau}^{\infty}(G/K)} \cap E_\tau \otimes \sum_{i=1}^{l} \Delta_i^* C_c^{\infty}(A) = \sum_{i=1}^{l} D_i^* \widehat{C_{c,\tau}^{\infty}(G/K)}.$$

Corollary. *Let* $L\colon E_\tau \otimes C_0^{\infty}(A) \to \mathbf{C}$ *be a conjugate linear functional and* $L=0$ *on* $E_\tau \otimes \sum_{i=1}^{l} \Delta_i^* C_c^{\infty}(A)$. *Then* L *restricted to* $\widehat{C_{c,\tau}^{\infty}(G/K)}$ *defines a function* $L \in C_\tau^{\infty}(G/K)$ *for which* $D_i L = 0$ $(1 \leq i \leq l)$.

If $\varrho \in A^*$ is defined as in section 2 and $s \in W$ then $s\varrho(H) = \varrho(s^{-1}H)$. The conjugate linear functionals L on $E_\tau \otimes C_c^{\infty}(A)$ which are 0 on $E_\tau \otimes \sum_{i=1}^{l} \Delta_i^* C_c^{\infty}(A)$ are given as follows:

For $s \in W$, $v \in E_\tau$

$$L(u \otimes \eta) = \int_K \overline{u(k)}\, v(k)\, dk \int_A e^{-s\varrho(\log a)} \eta(a)\, da$$

for $u \in E_\tau$ and $\eta \in C_c^{\infty}(A)$.

For $h \in C_{c,\tau}^{\infty}(G/K)$ we obtain

$$\begin{aligned} L(\hat{h}) &= \int_K dk \int_A da \overline{\hat{h}(ka)}\, v(k)\, e^{s\varrho(\log a)} \\ &= \int_K dk \int_A da \int_N dn \overline{h(kan)}\, v(k)\, e^{(\varrho + s\varrho)(\log a)} \\ &= \int_K dk \int_A da \int_N dn \overline{h(kn^{-1}a^{-1})}\, v(k)\, e^{(\varrho - s\varrho)(\log a)} \\ &= \int_K dk \int_G dx \overline{h(kx^{-1})}\, v(k)\, e^{-(\varrho + s\varrho)H(x)}\, dx \\ &= \int_G dx \int_K dk \overline{h(x)}\, e^{-(\varrho + s\varrho)H(x^{-1}k)}\, v(k). \end{aligned}$$

Thus as a function on G/K

$$L(x) = \int e^{-(\varrho + s\varrho)H(x^{-1}k)}\, v(k)\, dk.$$

Now as L is harmonic and

$$|L(x)| \leqq \sup_{k \in K} |v(k)| < \infty.$$

Thus we have from Furstenberg [1] that

$$L(gK) = \int_{K/M} P(gK, kM) T(kM)\, dkM$$

for some $T \in E_\tau$. This completes the proof of theorem 6.1.

Added in proof. It has been brought to my attention that lemmas 2.1. thru 2.5. may be found or easily derived from F. I. Karpelevič "The geometry of geodesics..." (Translations of the Moscow Math. Soc. (1965), 51—199). For a more extensive treatment see A. Koranyi "Poisson integrals and boundary components of symmetric spaces" (Inventiones Math. *34* (1976), 19—35).

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