On the integrability of Fourier—Jacobi transforms

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1. Introduction

For $\alpha \geqq \beta > -1/2$ and $x, \lambda \in [0, \infty)$, the Jacobi function $\varphi_\lambda^{(\alpha,\beta)}(x)$ of order (α, β) is defined by

$$\varphi_\lambda^{(\alpha,\beta)}(x) = {}_2F_1\big((\varrho + i\lambda)/2,\ (\varrho - i\lambda)/2;\ \alpha+1;\ -(\sinh x)^2\big),$$

where ${}_2F_1$ denotes the hypergeometric function (see [2, ch. 2]), and where $\varrho = \alpha + \beta + 1$ and $i = \sqrt{-1}$. It is known from [4] that

$$(1.1) \qquad \varphi_\lambda^{(\alpha,\beta)}(0) = 1, \quad \frac{d}{dx}\varphi_\lambda^{(\alpha,\beta)}(x)|_{x=0} = 0$$

and

$$\big(\Delta_{\alpha,\beta}(x)\big)^{-1}\frac{d}{dx}\Big(\Delta_{\alpha,\beta}(x)\frac{d}{dx}\varphi_\lambda^{(\alpha,\beta)}(x)\Big) = -(\lambda^2 + \varrho^2)\varphi_\lambda^{(\alpha,\beta)}(x),$$

where

$$\Delta_{\alpha,\beta}(x) = 2^{2\varrho}(\sinh x)^{2\alpha+1}(\cosh x)^{2\beta+1}.$$

Let $L^p(d\mu_{\alpha,\beta})$, $1 \leqq p < \infty$, be the class of all measurable functions $f(x)$ on $[0, \infty)$ such that

$$\|f\|_{p,\mu_{\alpha,\beta}} = \Big\{\int_0^\infty |f(x)|^p\, d\mu_{\alpha,\beta}(x)\Big\}^{1/p} < \infty,$$

where

$$d\mu_{\alpha,\beta}(x) = \frac{\sqrt{2}}{\Gamma(\alpha+1)}\Delta_{\alpha,\beta}(x)\, dx.$$

We denote $L(d\mu_{\alpha,\beta}) = L^1(d\mu_{\alpha,\beta})$ and $\|f\|_{\mu_{\alpha,\beta}} = \|f\|_{1,\mu_{\alpha,\beta}}$. Further, let $L^\infty(d\mu_{\alpha,\beta})$ be the class of all measurable functions $f(x)$ on $[0, \infty)$ such that

$$\|f\|_{\infty,\mu_{\alpha,\beta}} = \operatorname*{ess\,sup}_{0 \leqq x < \infty} |f(x)| < \infty.$$

Let $L^q(d\nu_{\alpha,\beta})$, $1 \leqq q < \infty$, be the class of all measurable functions $g(\lambda)$ on $[0, \infty)$ such that

$$\|g\|_{q,\nu_{\alpha,\beta}} = \Big\{\int_0^\infty |g(\lambda)|^q\, d\nu_{\alpha,\beta}(\lambda)\Big\}^{1/q} < \infty,$$

where

$$dv_{\alpha,\beta}(\lambda) = \frac{\sqrt{2}}{\Gamma(\alpha+1)} |C_{\alpha,\beta}(\lambda)|^{-2} d\lambda$$

and

$$C_{\alpha,\beta}(\lambda) = \frac{2^{\varrho}\Gamma((1/2)i\lambda)\Gamma((1+i\lambda)/2)}{\Gamma((\varrho+i\lambda)/2)\Gamma((\varrho+i\lambda)/2-\beta)} \tag{1.2}$$

(see [6]). For $f \in L(d\mu_{\alpha,\beta})$, the Fourier—Jacobi transform of f is defined by

$$f^{\wedge}_{\alpha,\beta}(\lambda) = \int_0^\infty f(x)\varphi_\lambda^{(\alpha,\beta)}(x)\, d\mu_{\alpha,\beta}(x),$$

and further the inverse transform is given formally by

$$\int_0^\infty f^{\wedge}_{\alpha,\beta}(\lambda)\varphi_\lambda^{(\alpha,\beta)}(x)\, dv_{\alpha,\beta}(\lambda)$$

(see [3]). Flensted—Jensen and Koornwinder [4] proved that the Fourier—Jacobi transform is injective on $L^p(d\mu_{\alpha,\beta})$ for $1 \leqq p \leqq 2$.

S. Bernstein proved that *if f is periodic with period 2π and satisfies a Lipschitz condition with exponent exceeding $1/2$ then the Fourier series of f converges absolutely* (see [9, p. 240—241]). The analogous theorems were obtained by C. Ganser [5] and H. Bavinck [1] for the Fourier—Jacobi series and by A. L. Schwartz [8] for the Hankel transforms. We give the analogous theorem for the absolute integrability of the Fourier—Jacobi transforms.

Throughout the paper, the letter M, with or without a suffix, denotes a positive constant, not necessarily the same on each appearance.

2. Preliminaries

There are the integral representations of $\varphi_\lambda^{(\alpha,\beta)}(x)$ as follows (see [3]):

$$\varphi_\lambda^{(\alpha,\beta)}(x) = \int_{r=0}^{1}\int_{\psi=0}^{\pi} |\cosh x + (\sinh x) re^{i\psi}|^{i\lambda-\varrho}\, dm_{\alpha,\beta}(r,\psi) \tag{2.1}$$

for $\alpha > \beta > -1/2$, where

$$dm_{\alpha,\beta}(r,\psi) = \frac{2\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha-\beta)\Gamma(\alpha+1/2)} (1-r^2)^{\alpha-\beta-1} r^{2\beta+1} (\sin\psi)^{2\beta}\, dr\, d\psi,$$

and

$$\varphi_\lambda^{(\alpha,\alpha)}(x) = \int_0^\pi \{\cosh 2x + (\sinh 2x)\cos\psi\}^{(i\lambda-2\alpha-1)/2}\, dm_{\alpha,\alpha}(\psi)$$

for $\alpha > -1/2$ $(\alpha=\beta)$, where

$$dm_{\alpha,\alpha}(\psi) = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha+1/2)} (\sin\psi)^{2\alpha}\, d\psi.$$

The following three equalities are known (see [6, p. 148 and p. 151]).

(2.2) $$\frac{d\varphi_\lambda^{(\alpha,\beta)}(x)}{dx} = -\frac{1}{4(\alpha+1)}(\varrho^2+\lambda^2)(\sinh 2x)\varphi_\lambda^{(\alpha+1,\beta+1)}(x),$$

(2.3) $$\frac{d^m\varphi_\lambda^{(\alpha,\beta)}(x)}{dx^m} = (\varrho^2+\lambda^2)\frac{d^{m-2}}{dx^{m-2}}\left(\frac{\varrho\cosh 2x+\alpha-\beta}{2(\alpha+1)}\varphi_\lambda^{(\alpha+1,\beta+1)}(x)-\varphi_\lambda^{(\alpha,\beta)}(x)\right)$$

for $m=2, 3, \ldots$

(put $d^0u/dx^0=u$), and

(2.4) $$\int_0^x \varphi_\lambda^{(\alpha,\beta)}(y)\,d\mu_{\alpha,\beta}(y) = 2^{-7/2}(\Gamma(\alpha+2))^{-1}(\sinh 2x)^{-1}\Delta_{\alpha+1,\beta+1}(x)\varphi_\lambda^{(\alpha+1,\beta+1)}(x).$$

We define the function $K_{\alpha,\beta}(x,y,z)$ for $x,y,z\geqq 0$ as follows:

(K1) for $\alpha\geqq\beta>-1/2$ and $z\notin(|x-y|, x+y)$,

$$K_{\alpha,\beta}(x,y,z)=0$$

(K2) for $\alpha>\beta>-1/2$ and $|x-y|<z<x+y$,

$$K_{\alpha,\beta}(x,y,z)=\frac{2^{(1/2)-2\varrho}(\Gamma(\alpha+1))^2}{\sqrt{\pi}\,\Gamma(\alpha-\beta)\Gamma(\beta+1/2)}(\sinh x\sinh y\sinh z)^{-2\alpha}\int_0^\pi\{1-(\cosh x)^2$$
$$-(\cosh y)^2-(\cosh z)^2+2\cosh x\cosh y\cosh z\cos\psi\}_+^{\alpha-\beta-1}(\sin\psi)^{2\beta}\,d\psi,$$

where

$$\{u\}_+=\begin{cases}u & \text{for } u>0\\ 0 & \text{for } u\leqq 0.\end{cases}$$

(K3) for $\alpha>-1/2(\alpha=\beta)$ and $|x-y|<z<x+y$,

$$K_{\alpha,\alpha}(x,y,z)=\frac{2^{-2\alpha-1/2}(\Gamma(\alpha+1))^2}{\sqrt{\pi}\,\Gamma(\alpha+1/2)}(\sinh 2x\sinh 2y\sinh 2z)^{-2\alpha}\{1-(\cosh 2x)^2$$
$$-(\cosh 2y)^2-(\cosh 2z)^2+2\cosh 2x\cosh 2y\cosh 2z\}_+^{\alpha-1/2}.$$

The function $K_{\alpha,\beta}(x,y,z)$ is symmetric in the three variables, and further it has the following four properties:

(2.5) $$K_{\alpha,\beta}(x,y,z)\geqq 0,$$

(2.6) $$\varphi_\lambda^{(\alpha,\beta)}(x)\varphi_\lambda^{(\alpha,\beta)}(y)=\int_0^\infty\varphi_\lambda^{(\alpha,\beta)}(z)K_{\alpha,\beta}(x,y,z)\,d\mu_{\alpha,\beta}(z),$$

(2.7) $$\int_0^\infty K_{\alpha,\beta}(x,y,z)\,d\mu_{\alpha,\beta}(z)=1,$$

$$\int_0^\infty f(z)K_{\alpha,\beta}(0,y,z)\,d\mu_{\alpha,\beta}(z)=f(y).$$

For $\alpha>\beta>-1/2$, Flensted—Jensen and Koornwinder [4] started with (2.1) and obtained (K1) and (K2). It is easy to see that (K1) and (K3) for $\alpha=\beta$ can be obtained by a limit.

Let f be a suitable function on $[0, \infty)$, and let $x \geqq 0$. The generalized translation operation $T_x^{(\alpha,\beta)}$ is defined by

$$T_x^{(\alpha,\beta)} f(y) = \int_0^\infty f(z) K_{\alpha,\beta}(x, y, z)\, d\mu_{\alpha,\beta}(z).$$

Obviously $T_x^{(\alpha,\beta)} f(y) = T_y^{(\alpha,\beta)} f(x)$, and

$$\|T_x^{(\alpha,\beta)} f\|_{p,\mu_{\alpha,\beta}} \leqq \|f\|_{p,\mu_{\alpha,\beta}} \tag{2.8}$$

for $f \in L^p(d\mu_{\alpha,\beta})$, $1 \leqq p \leqq \infty$, and $x \geqq 0$ (see [4]).

For suitable functions f and g on $[0, \infty)$, the convolution product $f * g$ is defined by

$$\begin{aligned}(f * g)_{\alpha,\beta}(x) &= \int_0^\infty f(y) T_x^{(\alpha,\beta)} g(y)\, d\mu_{\alpha,\beta}(y) \\ &= \int_0^\infty \int_0^\infty f(y) g(z) K_{\alpha,\beta}(x, y, z)\, d\mu_{\alpha,\beta}(z)\, d\mu_{\alpha,\beta}(y).\end{aligned}$$

Then the following three properties are obtained (see [4]).

$$\|(f * g)_{\alpha,\beta}\|_{\mu_{\alpha,\beta}} \leqq \|f\|_{\mu_{\alpha,\beta}} \|g\|_{\mu_{\alpha,\beta}} \quad \text{for} \quad f, g \in L(d\mu_{\alpha,\beta}), \tag{2.9}$$

$$\|(f * g)_{\alpha,\beta}\|_{\infty,\mu_{\alpha,\beta}} \leqq \|f\|_{\mu_{\alpha,\beta}} \|g\|_{\infty,\mu_{\alpha,\beta}} \tag{2.10}$$

for $f \in L(d\mu_{\alpha,\beta})$ and $g \in L^\infty(d\mu_{\alpha,\beta})$, and

$$(f * g)^\wedge_{\alpha,\beta}(\lambda) = f^\wedge_{\alpha,\beta}(\lambda) g^\wedge_{\alpha,\beta}(\lambda) \quad \text{for} \quad f, g \in L(d\mu_{\alpha,\beta}).$$

Flensted—Jensen [3] showed that the Fourier—Jacobi transform gives an isometric mapping from $L^2(d\mu_{\alpha,\beta})$ onto $L^2(d\nu_{\alpha,\beta})$ as follows:

For $f \in L^2(d\mu_{\alpha,\beta})$, the Fourier—Jacobi transform $f^\wedge_{\alpha,\beta}(\lambda)$ exists as a limit in $L^2(d\nu_{\alpha,\beta})$ of

$$\int_0^X f(x) \varphi_\lambda^{(\alpha,\beta)}(x)\, d\mu_{\alpha,\beta}(x)$$

as $X \to \infty$, and inversely $f(x)$ exists as a limit in $L^2(d\mu_{\alpha,\beta})$ of

$$\int_0^\Lambda f^\wedge_{\alpha,\beta}(\lambda) \varphi_\lambda^{(\alpha,\beta)}(x)\, d\nu_{\alpha,\beta}(\lambda)$$

as $\Lambda \to \infty$. Further the Parseval's formula

$$\|f^\wedge_{\alpha,\beta}\|_{2,\nu_{\alpha,\beta}} = \|f\|_{2,\mu_{\alpha,\beta}} \tag{2.11}$$

holds. (See also Koornwinder [6, Remark 3].)

For $f \in L(d\mu_{\alpha,\beta})$ or $f \in L^2(d\mu_{\alpha,\beta})$, we have easily, by the symmetric property of $K_{\alpha,\beta}(h, y, z)$, (2.6), (2.8) ($p=1$ or 2) and the above-mentioned results,

$$(T_h^{(\alpha,\beta)} f)^\wedge_{\alpha,\beta}(\lambda) = f^\wedge_{\alpha,\beta}(\lambda) \varphi_\lambda^{(\alpha,\beta)}(h) \quad \text{for} \quad h \geqq 0, \tag{2.12}$$

where the equality holds a.e. λ for $f \in L^2(d\mu_{\alpha,\beta})$.

We give the following two definitions.

Definition 1. We define $\mathcal{D}^0 f(x)=f(x)$ on $[0, \infty)$. Also, if $f(x)$ is n-times differentiable on $[0, \infty)$ for a positive integer n, then we define

$$\mathcal{D}^m f(x) = (\sinh 2x)^{-1} \frac{d\mathcal{D}^{m-1} f(x)}{dx} \quad \text{for} \quad m = 1, 2, \ldots, n.$$

Definition 2. Let n be a non-negative integer. If $f(x)$ is n-times differentiable on $[0, \infty)$, and if all its derivatives are uniformly bounded on $[0, \infty)$, then we define the integral moduli of continuity of $f^{(j)}$ for $j=0, 1, \ldots, n$ by

$$w_j(h; f, \alpha+n, \beta+n) = \|W_j(h, \cdot\,; f)\|_{2, \mu_{\alpha+n, \beta+n}} \quad \text{for} \quad h \geqq 0,$$

where

$$W_j(h, x; f) = \sup |f^{(j)}(x) - f^{(j)}(y)|,$$

taking the supremum over all y such that $|x-y| \leqq h$. (We put $f^{(0)}(x)=f(x)$.)

3. Results

First we give a theorem as follows:

Theorem 1. *Let* $\alpha \geqq \beta > -1/2$, $\delta > \alpha+1$ *and* $n=[\alpha+1]$, *where the symbol* $[\zeta]$ *denotes the greatest integer not exceeding* ζ. *Let* $f \in L^2(d\mu_{\alpha,\beta})$. *For positive integer* n, *let* $f(x)$ *be a* n-*times differentiable function on* $[0, \infty)$ *such that*

$$\text{(3.1)} \qquad \mathcal{D}^m f(x) = o(x^{-1} e^{-(\varrho+2m)x}) \quad \text{as} \quad x \to \infty \quad \text{for} \quad m = 0, 1, \ldots, n-1.$$

Moreover, suppose that, for non-negative integer n,

$$\text{(3.2)} \qquad \|T_h^{(\alpha+n, \beta+n)} \mathcal{D}^n f - \mathcal{D}^n f\|_{2, \mu_{\alpha+n, \beta+n}} = O(h^{\delta-n}) \quad \text{as} \quad h \to +0.$$

Then $\hat{f}_{\alpha,\beta} \in L(d\nu_{\alpha,\beta})$.

Remark. For a positive integer n in Theorem 1, we have easily $f(x) \in L^2(d\nu_{\alpha,\beta})$ from the n-times differentiability and (3.1) ($m=0$).

Since the result of Theorem 1 depends on the behavior of $T_h^{(\alpha+n, \beta+n)} \mathcal{D}^n f$ rather than only on f and its derivatives, we are not entirely satisfied with it. Now, in order to obtain $\hat{f}_{\alpha,\beta} \in L(d\nu_{\alpha,\beta})$, we give the following theorem depending on f, its derivatives and their integral moduli of continuity.

Theorem 2. *Let* $\alpha \geqq \beta > -1/2$, $\delta > \alpha+1$ *and* $n=[\alpha+1]$. *Suppose that* $f \in L^2(d\mu_{\alpha,\beta})$, *and that*

$$\text{(3.3)} \qquad w_j(h; f, \alpha+n, \beta+n) = O(h^{\delta-n}) \quad \text{as} \quad h \to +0 \quad \text{for} \quad j = 0, 1, \ldots, n.$$

For positive integer n, *let*

$$\text{(3.4)} \qquad f^{(n)}(x) = o(x^{-1} e^{-\varrho x}) \quad \text{as} \quad x \to \infty.$$

Then $\hat{f}_{\alpha,\beta} \in L(d\nu_{\alpha,\beta})$.

4. The proof of Theorem 1

We need the following three lemmas.

Lemma 1. *We have the three estimates as follows: For each* α, β *and for each non-negative integer* m *there exists a positive constant* Ω *such that*

$$\left|\frac{d^m \varphi_\lambda^{(\alpha,\beta)}(x)}{dx^m}\right| \leqq \Omega(1+\lambda)^m(1+x)^{-\varrho x} \quad \text{for all} \quad x, \lambda \in [0, \infty). \tag{4.1}$$

For each α, β *there exist two positive constants* Ω_1 *and* Ω_2 *respectively such that*

$$|C_{\alpha,\beta}(\lambda)| \leqq \Omega_1 \lambda^{-1}(1+\lambda)^{-\alpha+1/2} \quad \text{for all} \quad \lambda > 0, \tag{4.2}$$

and such that

$$|C_{\alpha,\beta}(\lambda)|^{-1} \leqq \Omega_2(1+\lambda)^{\alpha+1/2} \quad \text{for all} \quad \lambda \geqq 0. \tag{4.3}$$

The estimate (4.1) is due to Flensted—Jensen [3, Lemmas 13 and 15]. Since $\overline{C(\lambda)}=C(-\lambda)$ by (1.2), the estimates (4.2) and (4.3) are due to Flensted—Jensen [3, Corollary 9] (see also Koornwinder [6, Lemma 2.2 and Remark 2] for (4.3)).

Lemma 2. *For a positive integer* n, *we have the equality*

$$\mathscr{D}^n f(x) = \sum_{j=0}^{n_1} A_j(x) f^{(n-2j)}(x) + \sum_{j=0}^{n_2} B_j(x) f^{(n-2j-1)}(x), \tag{4.4}$$

where

$$n_1 = \frac{n-1}{2} \quad \text{and} \quad n_2 = \frac{n-3}{2} \geqq 0 \quad \text{for odd } n,$$

$$n_1 = n_2 = \frac{n-2}{2} \quad \text{for even } n,$$

and where

$$A_j(x) \sum_{k=0}^{j} a_k(n, j)(\sinh 2x)^{-n-2k},$$

$$B_j(x) = \sum_{k=0}^{j} b_k(n, j)(\cosh 2x)(\sinh 2x)^{-n-2k-1},$$

the constants $a_k(n, j)$ *and* $b_k(n, j)$ *depending only on* n, j *and* k. *Further we have*

$$A_j(x) = \begin{cases} O(x^{-n-2j}) & \text{as} \quad x \to +0 \\ O(e^{-2nx}) & \text{as} \quad x \to \infty \end{cases} \quad (j = 0, 1, \dots, n_1) \tag{4.5}$$

and

$$B_j(x) = \begin{cases} O(x^{-n-2j-1}) & \text{as} \quad x \to +0 \\ O(e^{-2nx}) & \text{as} \quad x \to \infty. \end{cases} \quad (j = 0, 1, \dots, n_2). \tag{4.6}$$

Proof. We can easily prove (4.4) by induction, and so we omit it. The estimates (4.5) and (4.6) are trivial from

$$\sinh 2x = O(x) \quad \text{and} \quad \cosh 2x = O(1) \quad \text{as} \quad x \to +0,$$
$$\sinh 2x = O(e^{2x}) \quad \text{and} \quad \cosh 2x = O(e^{2x}) \quad \text{as} \quad x \to \infty.$$

Thus Lemma 2 is proved.

Lemma 3. *Let n be a positive integer. Suppose that $f(x)$ is n-times differentiable on* $[0, \infty)$ *and satisfies* (3.1). *Then $f \in L^2(d\mu_{\alpha,\beta})$ and*

$$f^{\wedge}_{\alpha,\beta}(\lambda) = (-1)^n 2^{-4n} (\mathscr{D}^n f)^{\wedge}_{\alpha+n,\beta+n}(\lambda) \quad \text{a.e.}. \tag{4.7}$$

Proof. From Remark, we have $f \in L^2(d\mu_{\alpha,\beta})$. For $n \geqq 2$, since $f^{(m)}(x)$, $m = 1, 2, \ldots, n-1$, are uniformly bounded on $[0, \infty)$ by assumption, we obtain, from Lemma 2 (replace n by m),

$$\mathscr{D}^m f(x) = O(x^{-2m+1}) \quad \text{as} \quad x \to +0 \quad \text{for} \quad m = 1, 2, \ldots, n-1. \tag{4.8}$$

Integrating repeatedly by parts and using (2.4), we get, for $X > 0$,

$$\int_0^X f(x) \varphi_\lambda^{(\alpha,\beta)}(x)\, d\mu_{\alpha,\beta}(x) = \sum_{m=0}^{n-1} \frac{(-1)^m}{2^{4m+7/2}\Gamma(\alpha+m+2)}$$
$$\times \left[(\mathscr{D}^m f(x)) (\sinh 2x)^{-1} \varphi_\lambda^{(\alpha+m+1,\beta+m+1)}(x) \Delta_{\alpha+m+1,\beta+m+1}(x) \right]_0^X$$
$$+ (-1)^n 2^{-4n} \int_0^X (\mathscr{D}^n f(x)) \varphi_\lambda^{(\alpha+n,\beta+n)}(x)\, d\mu_{\alpha+n,\beta+n}(x).$$

From the continuity of $f(x)$, (4.1) (for $m=0$), (4.8) and (3.1), the finite series on the right-hand side is $o(1)$ as $X \to \infty$. Hence we have (4.7), since $f^{\wedge}_{\alpha,\beta} \in L^2(d\nu_{\alpha,\beta})$. Thus Lemma 3 is proved.

Proof of Theorem 1. Applying (2.11) and then (2.12) to the left-hand side of (3.2), we have

$$\int_0^\infty \left| (1 - \varphi_\lambda^{(\alpha+n,\beta+n)}(h)) (\mathscr{D}^n f)^{\wedge}_{\alpha+n,\beta+n}(\lambda) \right|^2 d\nu_{\alpha+n,\beta+n}(\lambda) = O(h^{2\delta-2n}) \quad \text{as} \quad h \to +0. \tag{4.9}$$

We get, by (1.1) and (2.3),

$$\left. \frac{d^2 \varphi_\lambda^{(\alpha+n,\beta+n)}(x)}{dx^2} \right|_{x=0} = -\frac{(\varrho+2n)^2 + \lambda^2}{2(\alpha+n+1)}, \quad \left. \frac{d^3 \varphi_\lambda^{(\alpha+n,\beta+n)}(x)}{dx^3} \right|_{x=0} = 0. \tag{4.10}$$

When we put $m=4$ in (4.1), there exists a positive constant Ω_0 depending only on α, β and n such that

$$\left| \frac{d^4 \varphi_\lambda^{(\alpha+n,\beta+n)}(x)}{dx^4} \right| \leqq \Omega_0 \lambda^4 \quad \text{for all} \quad x \geqq 0 \quad \text{and all} \quad \lambda \geqq 1. \tag{4.11}$$

By Maclaurin's theorem, (1.1) and (4.10), there exists a positive number $\theta<1$ such that

$$\varphi_\lambda^{(\alpha+n,\beta+n)}(h)=1-\frac{\{(\varrho+2n)^2+\lambda^2\}h^2}{4(\alpha+n+1)}+\frac{h^4}{24}\cdot\frac{d^4\varphi_\lambda^{(\alpha+n,\beta+n)}(u)}{du^4}\bigg|_{u=\theta h}.$$

Now we put, for $s=1, 2, \dots,$

$$h=2^{-s}\xi,\quad \xi=\frac{1}{2}\sqrt{\frac{3}{(\alpha+n+1)\Omega_0}}\quad\text{and}\quad 2^{s-1}\leqq\lambda\leqq 2^s.$$

Then, by (4.11),

$$1-\varphi_\lambda^{(\alpha+n,\beta+n)}(2^{-s}\xi)\geqq\frac{\{(\varrho+2n)^2+\lambda^2\}2^{-2s}\xi^2}{4(\alpha+n+1)}-\frac{2^{-4s}\xi^4}{24}\Omega_0\lambda^4$$

$$\geqq\left(\frac{1}{2(\alpha+n+1)}-\frac{\Omega_0\xi^2}{3}\right)\frac{\xi^2}{8}=\frac{3}{128(\alpha+n+1)^2\Omega_0}>0.$$

Hence, from (4.9),

$$\int_{2^{s-1}}^{2^s}|(\mathscr{D}^n f)\hat{}_{\alpha+n,\beta+n}(\lambda)|^2\,d\nu_{\alpha+n,\beta+n}(\lambda)=O(2^{2ns-2\delta s}).$$

Thus, by Schwarz's inequality, (4.2), (4.3) and Lemma 3, we have

$$\text{(4.12)}\qquad \int_{2^{s-1}}^{2^s}|\hat{f}_{\alpha,\beta}(\lambda)|\,d\nu_{\alpha,\beta}(\lambda)\leqq\frac{2^{1/4}(\Gamma(\alpha+n+1))^{1/2}}{\Gamma(\alpha+1)}$$

$$\times\Big\{\int_{2^{s-1}}^{2^s}|C_{\alpha+n,\beta+n}(\lambda)|^2|C_{\alpha,\beta}(\lambda)|^{-4}\,d\lambda\Big\}^{1/2}\Big\{\int_{2^{s-1}}^{2^s}|\hat{f}_{\alpha,\beta}(\lambda)|^2\,d\nu_{\alpha+n,\beta+n}(\lambda)\Big\}^{1/2}$$

$$\leqq M\Big\{\int_{2^{s-1}}^{2^s}\lambda^{-2(\alpha+n+1/2)+4(\alpha+1/2)}\,d\lambda\Big\}^{1/2}\Big\{\int_{2^{s-1}}^{2^s}|(\mathscr{D}^n f)\hat{}_{\alpha+n,\beta+n}(\lambda)|^2\,d\nu_{\alpha+n,\beta+n}(\lambda)\Big\}^{1/2}$$

$$\leqq M_1 2^{(\alpha-n+1)s+(n-\delta)s}=M_1 2^{-(\delta-\alpha-1)s}.$$

Since $f\in L^2(d\mu_{\alpha,\beta})$, we obtain, by Schwarz's inequality, (4.3) and (2.11),

$$\int_0^1|\hat{f}_{\alpha,\beta}(\lambda)|\,d\nu_{\alpha,\beta}(\lambda)\leqq\Big(\int_0^1|\hat{f}_{\alpha,\beta}(\lambda)|^2\,d\nu_{\alpha,\beta}(\lambda)\Big)^{1/2}\Big(\int_0^1 d\nu_{\alpha,\beta}(\lambda)\Big)^{1/2}$$

$$\leqq M\|\hat{f}_{\alpha,\beta}\|_{2,\nu_{\alpha,\beta}}=M\|f\|_{2,\mu_{\alpha,\beta}}.$$

Now, by this and (4.12), we have

$$\|\hat{f}_{\alpha,\beta}\|_{\nu_{\alpha,\beta}}=\Big(\int_0^1+\sum_{s=1}^\infty\int_{2^{s-1}}^{2^s}\Big)|\hat{f}_{\alpha,\beta}(\lambda)|\,d\nu_{\alpha,\beta}(\lambda)\leqq M_1+M_2\sum_{s=1}^\infty 2^{-(\delta-\alpha-1)s}<\infty.$$

Thus Theorem 1 is proved.

5. Proof of Theorem 2

We need the following two lemmas.

Lemma 4. *Suppose that $g(x)$ is an infinitely differentiable function on $[0, \infty)$ with compact support. Then, for each α and β, $g^{\wedge}_{\alpha,\beta}(\lambda)$ is an analytic function on $[0, \infty)$ and there exist positive constants N_s $(s=0, 1, 2, \ldots)$ such that*

$$|g^{\wedge}_{\alpha,\beta}(\lambda)| \leqq N_s(1+\lambda)^{-s}$$

for all $\lambda\in[0, \infty)$ and all $s=0, 1, \ldots$.

Lemma 4 is due to Flensted—Jensen [3] (or Koornwinder [6]).

Lemma 5. *Let $A_j(x)$ $(j=0, 1, \ldots, n_1)$ and $B_j(x)$ $(j=0, 1, \ldots, n_2)$ be defined as in Lemma 2, and let $0\leqq h\leqq 1/2$. Then we have the following estimates:*

(5.1) $$|A_j(x)-A_j(y)| \leqq Mh \sum_{k=0}^{j} \sum_{m=0}^{n+2k-1} x^{-m-1} y^{m-n-2k}$$

$$\text{for} \quad 0 < x,\ y \leqq 2 \quad \text{and} \quad |x-h| \leqq y \leqq x+h,$$

(5.2) $$|A_j(x)-A_j(y)| \leqq Mh$$

$$\text{for} \quad 1/2 \leqq x \leqq 1,\ 1 \leqq y \leqq 3/2 \quad \text{and} \quad x-h \leqq y \leqq x+h,$$

(5.3) $$|A_j(x)-A_j(y)| \leqq Mh \sum_{k=0}^{j} \sum_{m=0}^{n+2k-1} e^{-(2m+1)x+(2m-2n-4k+1)y}$$

$$\text{for} \quad x \geqq 1/2,\ y \geqq 1 \quad \text{and} \quad x-h \leqq y \leqq x+h,$$

and further

$$|B_j(x)-B_j(y)| \leqq Mh \sum_{k=0}^{j} \Big(\sum_{m=0}^{n+2k} x^{-m-1} y^{m-n-2k-1} + y^{-n-2k-1}\Big)$$

$$\text{for} \quad 0 < x,\ y \leqq 2 \quad \text{and} \quad |x-h| \leqq y \leqq x+h,$$

$$|B_j(x)-B_j(y)| \leqq Mh \quad \text{for} \quad 1/2 \leqq x \leqq 1, \quad 1 \leqq y \leqq 3/2 \quad \text{and} \quad x-h \leqq y \leqq x+h,$$

$$|B_j(x)-B_j(y)| \leqq Mh \sum_{k=0}^{j} \Big(\sum_{m=0}^{n+2k} e^{-(2m-1)x+(2m-2n-4k-1)y} + e^{x-(2n+4k+1)y}\Big)$$

$$\text{for} \quad x \geqq 1/2,\ y \geqq 1 \quad \text{and} \quad x-h \leqq y \leqq x+h,$$

where the constants M depend only on n.

Proof. The estimates (5.1)—(5.3) are easily obtained from

$$|A_j(x)-A_j(y)| \leqq M \sum_{k=0}^{j} |(\sinh 2x)^{-n-2k} - (\sinh 2y)^{-n-2k}|$$

$$= M|(\sinh 2x)^{-1} - (\sinh 2y)^{-1}| \sum_{k=0}^{j} \sum_{m=0}^{n+2k-1} (\sinh 2x)^{-m} (\sinh 2y)^{m-n-2k+1}$$

$$\leqq M_1 |\sinh (x-y)| \cosh (x+y) \sum_{k=0}^{j} \sum_{m=0}^{n+2k-1} (\sinh 2x)^{-m-1} (\sinh 2y)^{m-n-2k}.$$

Also, the three estimates for $|B_j(x)-B_j(y)|$ are clear from

$$|B_j(x)-B_j(y)| \leqq M \sum_{k=0}^{j} \{\cosh 2x\,|(\sinh 2x)^{-n-2k-1}-(\sinh 2y)^{-n-2k-1}| + |\cosh 2x-\cosh 2y|\,(\sinh 2y)^{-n-2k-1}\}$$

$$\leqq M_1 |\sinh(x-y)| \sum_{k=0}^{j} \Big\{\sum_{m=0}^{n+2k} \cosh(x+y)\cosh 2x(\sinh 2x)^{-m-1}(\sinh 2y)^{m-n-2k-1} + \sinh(x+y)(\sinh 2y)^{-n-2k-1}\Big\}.$$

Thus Lemma 5 is proved.

Proof of Theorem 2. It is sufficient to show that the assumptions of Theorem 2 satisfy those of Theorem 1 and especially (3.1) and (3.2).

For positive integer n, by integrating repeatedly (3.4) on $[x, \infty)$ for sufficiently large x, we have

(5.4) $$f^{(m)}(x) = o(x^{-1}e^{-\varrho x}) \quad \text{as} \quad x \to \infty \quad \text{for} \quad m = 0, 1, \dots, n-1.$$

Hence, from Lemma 2, we get (3.1).

Next, we show that (3.2) is obtained by the assumptions. From (3.3), we remark that $f^{(j)}(x)$, $j=0, 1, \dots, n$, are uniformly bounded on $[0, \infty)$. Let $g(x)$ be an infinitely differentiable function on $[0, \infty)$ with compact support such that

$$g^{(j)}(0) = f^{(j)}(0) \quad \text{for} \quad j = 0, 1, \dots, n.$$

If we put $s=[2\alpha+1]+2$ in Lemma 4, then $\hat{g}_{\alpha,\beta} \in L(dv_{\alpha,\beta})$ by (4.3). Now, $\hat{f}_{\alpha,\beta} \in L(dv_{\alpha,\beta})$ if and only if $(f-g)\hat{}_{\alpha,\beta} \in L(dv_{\alpha,\beta})$. Thus we may assume

$$f^{(j)}(0) = 0 \quad \text{for} \quad j = 0, 1, \dots, n.$$

Hence, applying Maclaurin's theorem to the case $n \neq 0$,

(5.5) $$|f^{(j)}(x)| \leqq Mx^{n-j} \quad \text{for} \quad x \geqq 0 \quad \text{and} \quad j = 0, 1, \dots, n \quad (n \geqq 0).$$

By Schwarz's inequality and (2.7), we have

(5.6) $$\|T_h^{(\alpha+n,\beta+n)}\mathscr{D}^n f - \mathscr{D}^n f\|_{2,\mu_{\alpha+n,\beta+n}}$$
$$\leqq \Big(\int_0^\infty \int_0^\infty |\mathscr{D}^n f(y) - \mathscr{D}^n f(x)|^2 K_{\alpha+n,\beta+n}(h, x, y)\, d\mu_{\alpha+n,\beta+n}(y)\, d\mu_{\alpha+n,\beta+n}(x)\Big)^{1/2}.$$

First we consider the case $-1/2<\alpha<0$. Then $n=0$. Now, from (5.6), (K1), (2.10) and (3.3),

$$\|T_h^{(\alpha,\beta)}f-f\|_{2,\mu_{\alpha,\beta}} \leqq \Big(\int_0^\infty \int_{|x-h|}^{x+h} |f(y)-f(x)|^2 K_{\alpha,\beta}(h, x, y)\, d\mu_{\alpha,\beta}(y)\, d\mu_{\alpha,\beta}(x)\Big)^{1/2}$$
$$\leqq \Big(\int_0^\infty \int_0^\infty W_0(h, x; f)^2 K_{\alpha,\beta}(h, x, y)\, d\mu_{\alpha,\beta}(y)\, d\mu_{\alpha,\beta}(x)\Big)^{1/2}$$
$$= w_0(h; f, \alpha, \beta) = O(h^\delta) \quad \text{as} \quad h \to +0.$$

Thus (3.2) is obtained for $-1/2<\alpha<0$.

Secondly, we consider the case $\alpha \geqq 0$. Then n is a positive integer. Without loss of generality, we may assume

$$\alpha+1<\delta \leqq n+1, \quad 0<h \leqq 1/2.$$

Hereafter, for the sake of simplicity, we write

$$\mu_{\alpha+n, \beta+n}=\mu, \quad K_{\alpha+n, \beta+n}(h, x, y)=K(h, x, y)$$

and

$$T_h^{(\alpha+n, \beta+n)}=T_h, \quad w_j(h ; f, \alpha+n, \beta+n)=w_j(h ; f).$$

We have, from (5.6), (4.4) and Minkowski's inequality,

$$\begin{aligned}
\text{(5.7)} \qquad \|T_h \mathscr{D}^n f-\mathscr{D}^n f\|_{2, \mu} \leqq \sum_{j=0}^{n_1}\Big(\int_0^{\infty} \int_0^{\infty} |A_j(x) f^{(n-2j)}(x) \\
-A_j(y) f^{(n-2j)}(y)|^2 K(h, x, y)\, d\mu(y)\, d\mu(x)\Big)^{1/2} \\
+\sum_{j=0}^{n_2}\Big(\int_0^{\infty} \int_0^{\infty} |B_j(x) f^{(n-2j-1)}(x)-B_j(y) f^{(n-2j-1)}(y)|^2 K(h, x, y)\, d\mu(y)\, d\mu(x)\Big)^{1/2} \\
=\sum_{j=0}^{n_1} G_j(h)+\sum_{j=0}^{n_2} H_j(h),
\end{aligned}$$

say. Further, from Minkowski's inequality,

$$\begin{aligned}
\text{(5.8)} \quad G_j(h)^2 \leqq \int_0^{\infty} \int_0^{\infty} |A_j(x)|^2 |f^{(n-2j)}(x)-f^{(n-2j)}(y)|^2 K(h, x, y)\, d\mu(y)\, d\mu(x) \\
+\int_0^{\infty} \int_0^{\infty} |A_j(x)-A_j(y)|^2 |f^{(n-2j)}(y)|^2 K(h, x, y)\, d\mu(y)\, d\mu(x)=P_j(h)+Q_j(h), \\
(j=0,1, \ldots, n_1),
\end{aligned}$$

say.

We put

$$\text{(5.9)} \qquad P_j(h)=\int_0^1 \int_0^{\infty}+\int_1^{\infty} \int_0^{\infty}=P_{j,1}(h)+P_{j,2}(h) \quad (j=0,1, \ldots, n_1).$$

We estimate $P_{j,1}(h)$. By (K1), (2.9), (4.5) and (3.3), we get, for $j=0,1, \ldots, n_1$,

$$\begin{aligned}
\text{(5.10)} \qquad P_{j,1}(h) \leqq \int_0^1 \int_0^{\infty} |A_j(x)|^2 W_{n-2j}(h, y ; f)^2 K(h, x, y)\, d\mu(y)\, d\mu(x) \\
\leqq w_{n-2j}(h ; f)^2 \int_0^1 |A_j(x)|^2\, d\mu(x) \\
=O(h^{2(\delta-n)}) \int_0^1 O(x^{-2n-4j}) O(x^{2(\alpha+n)+1})\, dx=O(h^{2(\delta-n)}) \quad \text{as} \quad h \to +0.
\end{aligned}$$

Secondly, we estimate $P_{j,2}(h)$. Since $A_j(x)$ is uniformly bounded on $[1, \infty)$ from Lemma 2, we have, by the symmetric property of $K(h, x, y)$, $(K1)$, (2.10) and (3.3),

$$\begin{aligned}
P_{j,2}(h) \leqq \int_1^{\infty} \int_0^{\infty} O(1) W_{n-2j}(h, x ; f)^2 K(h, x, y)\, d\mu(y)\, d\mu(x) \\
\leqq O(1) w_{n-2j}(h ; f)^2=O(h^{2(\delta-n)}) \quad \text{as} \quad h \to +0.
\end{aligned}$$

Thus, from this, (5.10) and (5.9),

$$\text{(5.11)} \qquad P_j(h)=O(h^{2(\delta-n)}) \quad \text{as} \quad h \to +0 \quad \text{for} \quad j=0,1, \ldots, n_1.$$

We set

(5.12) $$Q_j(h)=\int_0^1\int_0^1+\int_0^1\int_1^\infty+\int_1^\infty\int_0^1+\int_1^\infty\int_1^\infty=\sum_{s=1}^4 Q_{j,s}(h)\quad (j=0,1,\dots,n_1).$$

We estimate $Q_{j,1}(h)$. By Minkowski's inequality, (5.5), (5.1) and (2.9),

$$Q_{j,1}(h)\leqq Mh^2\sum_{k=0}^j\sum_{m=0}^{n+2k-1}\int_0^1\int_0^1 y^{4j}x^{-2m-2}y^{2m-2n-4k}K(h,x,y)\,d\mu(y)\,d\mu(x)$$

(5.13) $$\leqq Mh^2\sum_{k=0}^j\sum_{m=0}^{n+2k-1}\int_0^1 x^{-2m-2}\,d\mu(x)\int_0^1 y^{4j+2m-2n-4k}\,d\mu(y)$$
$$=O(h^2)=O(h^{2(\delta-n)})\quad\text{as}\quad h\to+0.$$

We estimate $Q_{j,2}(h)$. By the symmetric property of $K(h,x,y)$ and $(K1)$, we have $K(h,x,y)=0$ unless $y-h<x$ or $y<x+h$. Moreover $f^{(n-2j)}(y)$, $j=0,1,\dots,n_1$, are uniformly bounded on $[0,\infty)$. Hence, from (5.2) and (2.9),

(5.14) $$Q_{j,2}(h)\leqq Mh^2\int_{1/2}^1\int_1^{3/2}K(h,x,y)\,d\mu(y)\,d\mu(x)$$
$$\leqq Mh^2\int_{1/2}^1 d\mu(x)\int_1^{3/2}d\mu(y)=O(h^2)=O(h^{2(\delta-n)})\quad\text{as}\quad h\to+0.$$

Similarly we get

(5.15) $$Q_{j,3}(h)=O(h^{2(\delta-n)})\quad\text{as}\quad h\to+0.$$

Lastly, we estimate $Q_{j,4}(h)$. By Minkowski's inequality, (5.3), (5.4), (3.4), (K1) and (2.10), we have

(5.16) $$Q_{j,4}(h)\leqq Mh^2\sum_{k=0}^j\sum_{m=0}^{n+2k-1}\int_1^\infty\int_{y-h}^{y+h}O(y^{-2}e^{-2\varrho y})$$
$$\times e^{-2(2m+1)x+2(2m-2n-4k+1)y}K(h,x,y)\,d\mu(x)\,d\mu(y)$$
$$\leqq Mh^2\sum_{k=0}^j\int_1^\infty\int_{y-h}^{y+h}O(y^{-2}e^{-2(\varrho+2n+4k)y})K(h,x,y)\,d\mu(x)\,d\mu(y)$$
$$\leqq Mh^2\sum_{k=0}^j\int_1^\infty O(y^{-2}e^{-2(\varrho+2n+4k)y})\,d\mu(y)$$
$$=O(h^2)=O(h^{2(\delta-n)})\quad\text{as}\quad h\to+0.$$

Thus, by (5.12)—(5.16),

$$Q_j(h)=O(h^{2(\delta-n)})\quad\text{as}\quad h\to+0\quad\text{for}\quad j=0,1,\dots,n_1.$$

From this, (5.11) and (5.8), we have

(5.17) $$G_j(h)=O(h^{\delta-n})\quad\text{as}\quad h\to+0\quad\text{for}\quad j=0,1,\dots,n_1.$$

Using the three estimates of $|B_j(x)-B_j(y)|$ of Lemma 4 ($j=0,1,\dots,n_2$) and so on, we get similarly

$$H_j(h)=O(h^{\delta-n})\quad\text{as}\quad h\to+0\quad\text{for}\quad j=0,1,\dots,n_2.$$

Hence, combining this with (5.17) and (5.7), we obtain (3.2). Thus Theorem 2 is proved.

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