Convexity of means and growth of certain subharmonic functions

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1. Introduction

Consider a function $f(z)$, analytic in the closed right halfplane and satisfying $|f(z)|\leqq 1$ on the imaginary axis. We define a mean $m(r)$ along half-circles for such functions by the formula

$$m(r) = \int_{-\pi/2}^{\pi/2} \log^+ |f(re^{i\varphi})| \cos\varphi \, d\varphi.$$

This mean is convex with respect to the family of functions $Ar+Br^{-1}$, according to a classical result of Ahlfors [1]. The integral contains the subharmonic function $\log^+|f(z)|$, which in fact may be replaced by an arbitrary subharmonic function satisfying the corresponding boundary condition, without affecting the validity of the theorem. This was shown by Dinghas [2] for a class of means containing the one mentioned. The notion of convexity with respect to a family of functions has been treated by Heins [6].

Here we are going to generalize in another direction by substituting the condition of boundedness from above on the imaginary axis for a more general condition. It is convenient for applications to consider the open complex plane D cut along the negative real axis, which we get from a half-plane by a simple transformation. Let $u(z)$ be a subharmonic function in D and define its boundary values $u(-r)$, $r>0$, by $u_1(-r)=\limsup_{z\to -r+i0} u(z)$, $u_2(-r)=\limsup_{z\to -r-i0} u(z)$ and $u(-r)=\left[\frac{1}{2}(u_1^\alpha(-r)+u_2^\alpha(-r))\right]^{1/\alpha}$, where α is a constant to be specified later on. To make $u(-r)$ well defined, we assume the function to be non-negative when $\alpha\neq 1$ and $\alpha\neq\infty$. We are going to study functions $u(z)$ with the property

$$u(-r) \leqq \cos\pi\lambda\, u(r),$$

where λ is a constant. The odd form of the constant in the inequality appears to be handy in formulating the results. The following cases will be considered:

$$\alpha = 1, \quad 0 < \lambda \leqq 1, \quad u(-r) = \frac{1}{2}\big(u_1(-r)+u_2(-r)\big) \leqq \cos \pi\lambda u(r), \tag{1a}$$

$$\alpha = \infty, \quad 0 < \lambda \leqq \frac{1}{2}, \quad u(-r) = \max\,[u_1(-r), u_2(-r)] \leqq \cos \pi\lambda u(r), \tag{1b}$$

$$1 < \alpha < \infty, \quad 0 < \lambda \leqq \frac{1}{2}, \quad u(-r) = \left[\frac{1}{2}\big(u_1^\alpha(-r)+u_2^\alpha(-r)\big)\right]^{1/\alpha} \leqq \cos \pi\lambda u(r). \tag{1c}$$

As the inequality (1a) fails to secure boundedness from above locally on the boundary when $\frac{1}{2} < \lambda \leqq 1$, we make this boundedness a separate requirement in this case.

Next we define the mean $L_\alpha(r)$ and the generalized mean $J(r)$, which we are going to investigate, by putting

$$L_\alpha(r, u) = L_\alpha(r) = \left[\int_{-\pi}^{\pi} \big(u(re^{i\varphi})\big)^\alpha (\cos \lambda\varphi)^{1-\alpha} \sin \lambda(\pi - |\varphi|)\, d\varphi\right]^{1/\alpha}, \tag{2}$$

$$J(r, u) = J(r) = \sup_{|\varphi|<\pi} \frac{u(re^{i\varphi})}{\cos \lambda\varphi}. \tag{3}$$

Note that $L_\alpha(r)$, $\alpha \neq 1$, is defined for non-negative functions only, but $L_1(r)$ and $J(r)$ are defined for functions of any sign. For simplicity we write $L(r)$ for $L_1(r)$. The parameters α and λ are the constants in (1). Knowledge of the growth of $L(r)$ and $J(r)$ will provide information about the growth of $M(r) = \sup_{|\varphi|<\pi} u(re^{i\varphi})$.

2. Convexity and growth theorems

Theorem I: *Let $u(z)$ be subharmonic in the open complex plane, cut along the negative real axis. If λ is a constant, $0 < \lambda \leqq 1$, such that $u(z)$ satisfies the inequality* (1a) *then the mean $L(r)$ is a convex function with respect to the family of functions $Ar^\lambda + Br^{-\lambda}$ (A and B are constants). When $\frac{1}{2} < \lambda \leqq 1$, the boundary values of $u(z)$ are assumed to be locally bounded from above.*

The classical case corresponds after a square root transformation to $\lambda = \frac{1}{2}$ in the theorem. Furthermore, in the case $\frac{1}{2} \leqq \lambda \leqq 1$ one could replace (1a) by the inequality $u(-r) \leqq \cos \pi\lambda\, M^+(r)$, which implies (1a). It will be seen from the proof that the result is of a local character, i.e. if (1a) holds only in a finite interval (r_1, r_2), then the conclusion is valid for the corresponding "cut" annulus $D_1 = \{z : r_1 < |z| < r_2, |\arg z| < \pi\}$. The theorem has been announced previously in the survey by Kjellberg [9] and applications are found in Essén [3] and Essén and Shea [4].

Theorem II: *Let $u(z)$ be subharmonic in the open complex plane, cut along the negative real axis. If λ is a constant, $0<\lambda\leqq\frac{1}{2}$, such that $u(z)$ satisfies the inequality* (1b), *then the generalized mean $J(r)$ is a convex function with respect to the family of functions $Ar^{\lambda}+Br^{-\lambda}$ (A and B are constants).*

This is a local result in the same way as Theorem I. The convexity that is the conclusion of these two theorems implies a growth property of the functions $L(r)$ and $J(r)$ which we state as a corollary.

Corollary: *The functions $r^{-\lambda}L(r)$ and $r^{-\lambda}J(r)$ are non-decreasing, i.e. the limits $\lim_{r\to\infty} r^{-\lambda}L(r)$ and $\lim_{r\to\infty} r^{-\lambda}J(r)$ exist (possibly infinite) under the assumptions of Theorem I and Theorem II respectively.*

There is a close connection between the two functions $L(r)$ and $J(r)$, which may be used to obtain information about $M(r)$. In fact, for large values of r, $J(r)$ and $M(r)$ are almost the same thing although $J(r)$ has a more regular growth. From the definition of $J(r)$ we have $u(z)\leqq J(r)\cos\lambda\varphi$ and taking the mean L of both sides we get $L(r)\leqq\pi\sin\pi\lambda\, J(r)$. The precision of this inequality increases with r. This fact and the connection with $M(r)$ constitute the next theorem.

Theorem III: *Under the assumptions of Theorem II either*

$$\lim_{r\to\infty} r^{-\lambda}M(r)=\infty$$

or

$$\frac{1}{\pi\sin\pi\lambda}\lim_{r\to\infty} r^{-\lambda}L(r)=\lim_{r\to\infty} r^{-\lambda}J(r)=\lim_{r\to\infty} r^{-\lambda}M(r).$$

The fact that $\lim_{r\to\infty} r^{-\lambda}M(r)$ exists in the present situation is a well-known theorem of Kjellberg [8]. As an indication of the usefulness of Theorem I and II, we will show that $r^{-\lambda}M(r)=r^{-\lambda}J(r)+o(1)$ from which the existence of $\lim_{r\to\infty} r^{-\lambda}M(r)$ is obvious. Finally, if we restrict ourselves to non-negative functions, we find that the convexity of $L(r)$ and $J(r)$ is in fact the analogous case of the corresponding convexity of all means $L_{\alpha}(r)$ $(\alpha>1)$. This observation is the last theorem.

Theorem IV: *Let $u(z)$ be a non-negative subharmonic function in the open complex plane, cut along the negative real axis. If α and λ are constants, $\alpha>1$, $0<\lambda\leqq\frac{1}{2}$, such that $u(z)$ satisfies the inequality* (1c), *then the mean $L_{\alpha}(r)$ is a convex function with respect to the family of functions $Ar^{\lambda}+Br^{-\lambda}$ (A and B are constants).*

3. Some lemmas

Lemma 1. *The harmonic functions* h *in* D, *that*
a) are symmetric on the boundary, $h(re^{i\pi})=h(re^{-i\pi})$,
b) satisfy (1a) *with equality,* $0<\lambda<1$,
c) are $O(r^{\lambda})$ *when* $r\to\infty$,
are constant multiples of $r^{\lambda}\cos\lambda\varphi$.

Proof: Let $h(z)$ be a function with the stated properties. Poisson's formula applied to $h(z)$ in the right half-plane gives

$$h(r)=\frac{r}{\pi}\int_{-\infty}^{\infty}\frac{h(iy)\,dy}{r^2+y^2}=\frac{r}{\pi}\int_0^{\infty}\frac{\big(h(iy)+h(-iy)\big)\,dy}{r^2+y^2}.$$

Applications of the formula to $h(z)$ in the upper and lower half-planes yield

$$h(\pm iy)=\frac{y}{\pi}\int_{-\infty}^{\infty}\frac{h(x)\,dx}{x^2+y^2}=\frac{y}{\pi}\int_0^{\infty}\frac{\big(h(-x)+h(x)\big)\,dx}{x^2+y^2}.$$

Eliminate $h(\pm iy)$ between these three relations and we get

$$h(r)=\frac{2}{\pi^2}\int_0^{\infty}\frac{\big(h(-x)+h(x)\big)r\log r/x}{r^2-x^2}\,dx.$$

Here we substitute $\cos\pi\lambda h(x)$ for $h(-x)$ according to (1a), and so we end up with the integral equation

$$h(r)=\int_0^{\infty}K(t,r)\,h(t)\,dt,$$

where

$$K(t,r)=\frac{2(1+\cos\pi\lambda)}{\pi^2}\cdot\frac{r\log r/t}{r^2-t^2}.$$

By the change of variables $r=e^x$, $t=e^s$ and $\varphi(x)=e^{-\lambda x}h(e^x)$, we get

$$\varphi(x)=\int_{-\infty}^{\infty}K_0(x-s)\,\varphi(s)\,ds,$$

with $K_0(x)=\dfrac{1+\cos\pi\lambda}{\pi^2}\cdot\dfrac{xe^{-\lambda x}}{\sinh x}$. Since $\varphi(x)=1$ is a solution we have that $\int_{-\infty}^{\infty}K_0(t)\,dt=1$. Taking Fourier transforms we obtain $\hat{\varphi}(\xi)=\hat{K}_0(\xi)\hat{\varphi}(\xi)$ or $[1-\hat{K}_0(\xi)]\hat{\varphi}(\xi)=0$, where $1-\hat{K}_0(\xi)$ has a simple zero at the origin. This is seen from

$$\frac{\pi^2\hat{K}_0'(0)}{1+\cos\pi\lambda}=i\int_{-\infty}^{\infty}\frac{x^2e^{-\lambda x}\,dx}{\sinh x}=-2i\int_0^{\infty}\frac{x^2\sinh\lambda x}{\sinh x}\,dx\neq 0.$$

The general solution is $\hat{\varphi}(\xi)=C\delta(\xi)$ which means that $\varphi(x)$ is a constant or $h(r)=Cr^{\lambda}$. But $h(z)$ is uniquely defined by its values on the real axis. The lemma is proved.

Lemma 2. *Let $u(x)$ be a subharmonic and $h(z)$ a harmonic function in $D_1=\{z: r_1<|z|<r_2, |\arg z|<\pi\}$, where $u(z)$ is upper semicontinuous in $\bar{D}_1$ and $h(z)$ is continuous in $\{r_1\leqq|z|\leqq r_2\}$. Suppose that $u(z)$ and $h(z)$ satisfy (1b) ($h(z)$ with equality), $0<\lambda\leqq\frac{1}{2}$. If $h(z)$ majorizes $u(z)$ on $|z|=r_1$ and $|z|=r_2$, then it majorizes $u(z)$ in D_1.*

Proof: We define $v(z)=u(z)-h(z)$ which is a subharmonic function in D_1 satisfying (1b). The upper semi-continuous function $v(z)$ attains a maximum on $\bar{D}_1$. By the maximum principle for subharmonic functions this will happen on ∂D_1. But we know that $v(r_1e^{i\varphi})\leqq 0$ and $v(r_2e^{i\varphi})\leqq 0$ by assumption. When $\lambda=1/2$ we also have that $v(-r)\leqq 0$ by (1b), i.e. $v(z)\leqq 0$ on ∂D_1 which implies $v(z)\leqq 0$ in D_1 which was to be proved. In the remaining case $0<\lambda<1/2$ we assume that $v(z)$ reaches its maximum at the point $-r_0$ on the negative real axis. From (1b) we get $v(-r_0)\leqq\cos\pi\lambda\, v(r_0)$, which is incompatible with $v(-r_0)$ being a maximum of $v(z)$ unless the maximum is non-positive. This conclusion is immediate in case the maximum is reached on $|z|=r_1$ or $|z|=r_2$. Hence, we have found that the maximum of $v(z)$ on ∂D_1, which is the maximum in D_1, is non-positive. The proof is complete.

Lemma 3. *Let $f(x)$ be a positive function in $C_2(-\pi,\pi)$. If there are constants α and λ, $\alpha\geqq 1$, $0<\lambda\leqq\frac{1}{2}$, such that $[\frac{1}{2}(f^{\alpha}(\pi)+f^{\alpha}(-\pi))]^{1/\alpha}\leqq\cos\pi\lambda f(0)$, then*

$$I(f)=\int_{-\pi}^{\pi}\left(\frac{f(x)}{\cos\lambda x}\right)^{\alpha-1}(\lambda^2 f(x)+f''(x))\sin\lambda(\pi-|x|)\,dx\leqq 0.$$

Proof: Let $0<\lambda<1/2$. We define the new function $g(x)$ by $f(x)=g(x)\cos\lambda x$. Differentiation yields

$$f'(x)=g'(x)\cos\lambda x-\lambda g(x)\sin\lambda x,$$

$$f''(x)=g''(x)\cos\lambda x-2\lambda g'(x)\sin\lambda x-\lambda^2 g(x)\cos\lambda x,$$

$$\lambda^2 f(x)+f''(x)=g''(x)\cos\lambda x-2\lambda g'(x)\sin\lambda x.$$

The condition $[\frac{1}{2}(f^{\alpha}(\pi)+f^{\alpha}(-\pi))]^{1/\alpha}\leqq\cos\pi\lambda f(0)$ implies $g^{\alpha}(\pi)+g^{\alpha}(-\pi)\leqq 2g^{\alpha}(0)$. We divide $I(f)$ in two parts according to the formula $I(f)=\int_{-\pi}^{\pi}=\int_0^{\pi}+\int_{-\pi}^{0}=I_1(f)+I_2(f)$ and substitute g for f. Then we get

$$I_1(f)=\int_0^{\pi}g(x)^{\alpha-1}(g''(x)\cos\lambda x-2\lambda g'(x)\sin\lambda x)\sin\lambda(\pi-x)\,dx.$$

The first term is integrated by parts and the remaining terms are rearranged to yield

$$I_1(f) = -(g(0))^{\alpha-1}g'(0)\sin\lambda\pi$$
$$-\int_0^{\pi}(\alpha-1)(g(x))^{\alpha-2}(g'(x))^2\cos\lambda x\sin\lambda(\pi-x)\,dx$$
$$+\int_0^{\pi}\lambda\cos\pi\lambda(g(x))^{\alpha-1}g'(x)\,dx.$$

But $I_2(f)=I_1(f^*)$ if $f^*(x)=f(-x)$ and so

$$I_2(f) = (g(0))^{\alpha-1}g'(0)\sin\lambda\pi$$
$$-\int_{-\pi}^{0}(\alpha-1)(g(x))^{\alpha-2}(g'(x))^2\cos\lambda x\sin\lambda(\pi+x)\,dx$$
$$-\int_{-\pi}^{0}\lambda\cos\pi\lambda(g(x))^{\alpha-1}g'(x)\,dx.$$

Adding these two expressions and performing the integration in the last terms we finally get

$$I(f) = -\int_{-\pi}^{\pi}(\alpha-1)(g(x))^{\alpha-2}(g'(x))^2\cos\lambda x\sin\lambda(\pi-|x|)\,dx$$
$$+\frac{\lambda\cos\pi\lambda}{\alpha}\left(g^{\alpha}(\pi)+g^{\alpha}(-\pi)-2g^{\alpha}(0)\right).$$

The first term is obviously non-positive and the second one is non-positive by assumption. In the case $\lambda=1/2$ we choose $\varepsilon>0$ and consider the interval $[-\pi+\varepsilon, \pi-\varepsilon]$ and obtain analogously (without the splitting in I_1 and I_2)

$$I_\varepsilon(f) = (g(x))^{\alpha-1}g'(x)\cos^2\frac{x}{2}\Big|_{-\pi+\varepsilon}^{\pi-\varepsilon} - \int_{-\pi+\varepsilon}^{\pi-\varepsilon}(\alpha-1)(g(x))^{\alpha-2}(g'(x))^2\cos^2\frac{x}{2}\,dx.$$

Since g and $g'\cos\frac{x}{2}$ are bounded to the right of $x=-\pi$ and to the left of $x=\pi$, we find that the first term vanishes as $\varepsilon\to 0$ because of the factor $\cos\frac{x}{2}$. Hence $I(f)\leqq 0$ for $0<\lambda\leqq\frac{1}{2}$ as asserted.

4. Proof of Theorem I

First we notice that the mean (2) and the condition (1a) in the assumption are both invariant with respect to reflections in the real axis. Hence it is no restriction to assume that $u(z)$ is symmetric with respect to this axis because otherwise we could consider $\frac{1}{2}(u(z)+u(\bar{z}))$ instead. Thus we suppose in the sequel that $u(z)=u(\bar{z})$.

Next we observe that $u(z)$, which may take the value $-\infty$, is the limit of the sequence $\max(u(z), -n)$ of finite subharmonic functions. The members all satisfy

(1a) if $u(z)$ does and $0<\lambda<\frac{1}{2}$. Consequently we may assume that $u(z)$ is finite when $0<\lambda<\frac{1}{2}$. With the function $u(z)$ we associate an auxiliary function $u_\varepsilon(z)$, satisfying (1a) and subharmonic in a closed annulus, cut along the negative real axis. Fix a closed interval $0<r_1\leqq r\leqq r_2<\infty$ and choose $\varepsilon>0$. Then there exists $\delta>0$ (independent of r), such that $u(r^{1-\delta}e^{i\pi(1-\delta)})\leqq u(r^{1-\delta})\cos\pi\lambda+\varepsilon$ because of the semicontinuity of the function $u(z)$ (finite when $0<\lambda<\frac{1}{2}$). Now put $u_\varepsilon(z)=u(z^{1-\delta})-2\varepsilon/(1-\cos\pi\lambda)$, which gives

$$u_\varepsilon(-r)-u_\varepsilon(r)\cos\pi\lambda \leqq -\varepsilon < 0.$$

Next we make use of the known fact (Tsuji chap. II:5 [10]), that it is possible to construct a non-increasing sequence $u_n(z)$ of twice continuously differentiable subharmonic functions converging to $u_\varepsilon(z)$. Since $u_\varepsilon(z)$ satisfies (1a) with some margin, we find that $u_n(z)$ satisfies (1a) for n sufficiently large. To see this, choose $0<\varepsilon_1<\varepsilon/2$ and note that we may increase $u_\varepsilon(z)$ by ε_1 without violating (1a). A further application of the uniform semicontinuity on the intervals $[-r_2, -r_1]$ and $[r_1, r_2]$ shows the existence of a $\delta_1>0$ (independent of r) such that $u_\varepsilon(z)<u_\varepsilon(-r)+\varepsilon_1$ for $|z+r|<\delta_1$ and $u_\varepsilon(z)<u_\varepsilon(r)+\varepsilon_1$ for $|z-r|<\delta_1$. As $u_n(z)$ is constructed as a repeated mean of $u_\varepsilon(z)$ over a disc, the radius of which tends to zero as n goes to infinity, we get $u_n(-r)<u_\varepsilon(-r)+\varepsilon_1$ and $u_n(r)<u_\varepsilon(r)+\varepsilon_1$ for large n. On the other hand, we always have $u_\varepsilon(r)\leqq u_n(r)$ and so, when $0<\lambda<\frac{1}{2}$ we get

$$u_n(-r)-\cos\pi\lambda u_n(r) < u_\varepsilon(-r)+\varepsilon_1-\cos\pi\lambda u_\varepsilon(r) \leqq \varepsilon_1-\varepsilon < 0.$$

In case $\frac{1}{2}\leqq\lambda\leqq 1$ we obtain

$$u_n(-r)-\cos\pi\lambda u_n(r) < u_\varepsilon(-r)+\varepsilon_1-\cos\pi\lambda u_\varepsilon(r)-\varepsilon_1\cos\pi\lambda < \varepsilon_1(1-\cos\pi\lambda)-\varepsilon < 0.$$

This discussion shows that regardless of the sign of $\cos\pi\lambda$ it is sufficient to consider twice continuously differentiable functions that fulfill the requirements of the theorem in a closed annulus cut along the negative real axis. In this case we have the inequality

$$\Delta u = u_{rr}+\frac{1}{r}u_r+\frac{1}{r^2}u_{\varphi\varphi} \geqq 0,$$

which yields the following relation for the mean $L(r)$:

$$L''(r)+\frac{1}{r}L'(r)+\frac{1}{r^2}\int_{-\pi}^{\pi} u_{\varphi\varphi}\sin\lambda(\pi-|\varphi|)\,d\varphi \geqq 0. \tag{4}$$

To the last term we apply integration by parts twice, remembering that $u_\varphi(r)=0$. Hence

$$2\int_0^\pi u_{\varphi\varphi}\sin\lambda(\pi-\varphi)\,d\varphi = 2\lambda\big(u(-r)-u(r)\cos\pi\lambda\big)$$
$$-2\lambda^2\int_0^\pi u\sin\lambda(\pi-\varphi)\,d\varphi \leqq -\lambda^2 L(r)$$

because of (1a). Inserting this estimate into (4) yields

$$r^2L''(r)+rL'(r)-\lambda^2L(r) \geqq 0. \tag{5}$$

When we take the sign of equality in (5) we obtain the solution $Ar^\lambda+Br^{-\lambda}$. Therefore, the difference

$$F(r) = L(r)-Ar^\lambda-Br^{-\lambda}$$

also satisfies (5). For any two numbers R_1 and R_2 in $[r_1, r_2]$ we choose A and B such that $F(R_1)=F(R_2)=0$. Then it is obvious from

$$r^2F''(r)+rF'(r)-\lambda^2F(r) \geqq 0$$

that $F(r)$ cannot have a positive maximum in (R_1, R_2). For $F(r)>0$, $F'(r)=0$ implies $F''(r)>0$ which corresponds to a minimum!

The conclusion is that the mean $L(r)$ is a convex function with respect to the family $Ar^\lambda+Br^{-\lambda}$.

Remark. The result is not true when condition (1a) is replaced by $u(-r)\leqq \cos\pi\lambda M^+(r)$ in the case $0<\lambda<\frac{1}{2}$. The function $u(z)=r\sin(\varphi-\varphi_0)$, where φ_0 is a constant, $0<\varphi_0<\pi$, furnishes a counterexample. Here $u(-r)>0$ and $u(r)<0$, so (1a) is certainly not satisfied. On the other hand $M(r)\cos\pi\lambda-u(-r)= r(\cos\pi\lambda-\sin\varphi_0)>0$ for φ_0 sufficiently small. Furthermore $L(r)=kr$, where $k<0$. Put $g(r)=a(r^\lambda-r^{-\lambda})+kr^{-\lambda}$, so that $g(1)=L(1)$. By attributing a negative value of large modulus to the constant a, we can make $g(r)$ and $L(r)$ coincide for a large value of r. Clearly $g(r)$ cannot possibly dominate the linear function $L(r)$ to the left of such a point.

5. Proof of Theorem II and the Corollary

In order to obtain a convenient description of the convexity property we introduce the function $e_\lambda(t)=t^\lambda-t^{-\lambda}$. Let $F(r)$ denote a function which is convex with respect to $Ar^\lambda+Br^{-\lambda}$ and consider its values on three different radii $0<r_1<r<r_2$. We define A and B by $F(r_1)=Ar_1^\lambda+Br_1^{-\lambda}$ and $F(r_2)=Ar_2^\lambda+Br_2^{-\lambda}$. Eliminating A and B in $F(r)\leqq Ar^\lambda+Br^{-\lambda}$, we find that

$$e_\lambda(r_2/r_1)F(r) \leqq F(r_1)e_\lambda(r_2/r)+F(r_2)e_\lambda(r/r_1).$$

Assuming that $F(r)$ is bounded from above at the origin (which is the case when $F(r)=L(r)$ and $F(r)=J(r)$) and letting $r_1\to 0$ we obtain $r^{-\lambda}F(r)\leqq r_2^{-\lambda}F(r_2)$ which proves the Corollary.

To prove the theorem we form a harmonic majorant $H(z)$ of $u(z)$ in D_1. To this end put

$$H(z) = [J(r_1)e_\lambda(r_2/r)+J(r_2)e_\lambda(r/r_1)]\frac{\cos\lambda\varphi}{e_\lambda(r_2/r_1)}.$$

This function is obviously harmonic in D_1, being a linear combination of $\operatorname{Re} z^{\lambda}$ and $\operatorname{Re} z^{-\lambda}$. Furthermore we have that $H(r_k e^{i\varphi}) = J(r_k)\cos\lambda\varphi$, $k=1,2$ and that $H(z)$ satisfies (1b) with equality. As $u(z) \leqq J(r)\cos\lambda\varphi$ by definition, it is evident that $H(z)$ majorizes $u(z)$ on $|z|=r_1$ and $|z|=r_2$. An application of Lemma 2 yields that $u(z) \leqq H(z)$ in D_1, that is

$$u(z) \leqq [J(r_1)e_\lambda(r_2/r) + J(r_2)e_\lambda(r/r_1)]\frac{\cos\lambda\varphi}{e_\lambda(r_2/r_1)}$$

or

$$e_\lambda(r_2/r_1)J(r) \leqq J(r_1)e_\lambda(r_2/r) + J(r_2)e_\lambda(r/r_1)$$

by the definition of $J(r)$. This inequality is the desired conclusion.

6. Proof of Theorem III

We construct a harmonic majorant $H_R(z)$ of $u(z)$ in $D_2 = \{z : |z| \leqq R\} \cap D$ in the following way. The values on the positive real axis of a harmonic majorant in D_2 may be expressed by means of Poisson's formula as

$$h(r) = \int_0^R Q(r,t)h(-t)\,dt + \int_{-\pi}^{\pi} T(r,\varphi)h(Re^{i\varphi})\,d\varphi,$$

where

$$Q(r,t) = \frac{\sqrt{r}}{\pi\sqrt{t}}\left\{\frac{1}{t+r} - \frac{R}{R^2+rt}\right\}$$

and

$$T(r,\varphi) = \frac{\sqrt{Rr}(R-r)\cos(\varphi/2)}{\pi(R^2+r^2-2Rr\cos\varphi)}.$$

Here we have assumed that the values on the upper and lower edge of the cut are equal. This formula is derived in [7] p. 187. Now put $H_R(Re^{i\varphi}) = u(Re^{i\varphi})$ and $H_R(-r) = \cos\pi\lambda\, H_R(r)$. Inserting this in the relation above, Poisson's formula turns into an integral equation for $H_R(r)$ (or $H_R(-r)$). We get

$$H_R(r) = \cos\pi\lambda \int_0^R Q(r,t)H_R(t)\,dt + \int_{-\pi}^{\pi} T(r,\varphi)u(Re^{i\varphi})\,d\varphi.$$

This equation has a unique solution (see [7] p. 189) which defines $H_R(-r)$. Let $H_R(z)$ be the harmonic function with boundary values $H_R(-r)$ and $u(Re^{i\varphi})$ in D_2. By construction $H_R(-r) = \cos\pi\lambda\, H_R(r)$ and an appeal to Lemma 2 with $r_1 = 0$ and $r_2 = R$ shows that $H_R(z)$ majorizes $u(z)$. But $H_R(z)$ is majorized by the harmonic function $U(z)$, where

$$U(z) = \operatorname{Re}\frac{2M(R)}{\pi}\tan\frac{\pi\lambda}{2}\int_0^{z/R}\frac{t^{\lambda-1}-t^{1-\lambda}}{1-t^2}\,dt$$

and $M(R)=\sup_{|z|=R} u(z)$. This is so because $U(-r)=\cos\pi\lambda\, U(r)$ and $U(Re^{i\varphi})=M(R)$, i.e. Lemma 2 is applicable. If we assume $\liminf_{r\to\infty} r^{-\lambda} M(r)<\infty$, which is the only case of interest here, then $U(z)$ converges to a constant multiple of $r^{\lambda}\cos\lambda\varphi$ when R tends to infinity through a suitable sequence of values. This implies that $\lim_{r\to\infty} r^{-\lambda}J(r)=A<\infty$. From $H_R(Re^{i\varphi})=u(Re^{i\varphi})\leqq J(R)\cos\lambda\varphi\leqq AR^{\lambda}\cos\lambda\varphi$ we see by means of Lemma 2 that $H_R(z)$ has an upper bound $Ar^{\lambda}\cos\lambda\varphi$ in D_2 that is independent of R.

The same lemma also shows that $H_{R_1}(z)\leqq H_{R_2}(z)$ when $R_1<R_2$. By the Harnack convergence principle we conclude that $H_R(z)$ has a harmonic limit function $H_1(z)$ in D. But from Lemma 1 we find that such a function must be of the type $r^{\lambda}\cos\lambda\varphi$, and so $H_1(z)=Ar^{\lambda}\cos\lambda\varphi$ since a smaller $H_1(z)$ would contradict the extremal nature of A. From the construction of $H_1(z)$ it is clear that

$$\lim_{R\to\infty} R^{-\lambda}L(R,u) = \lim_{R\to\infty} R^{-\lambda}L(R,H_R) = R^{-\lambda}L(R,H_1) = A\pi\sin\pi\lambda,$$

which is the first part of the statement in Theorem III.

To prove the remaining part we use an idea of Heins [5]. Recall that $u(z)\leqq J(r)\cos\lambda\varphi$. From

$$u(z)\leqq \min\,[M(r),\, J(r)\cos\lambda\varphi]$$

we get

$$L(r,u)\leqq L\big(r,\min\,[M(r),\, J(r)\cos\lambda\varphi]\big)$$

or

$$b\pi\sin\pi\lambda-d\leqq L\big(r,\min\,(a,\, b\cos\lambda\varphi)\big)$$

where $a=r^{-\lambda}M(r)$, $b=r^{-\lambda}J(r)$ and $d=b\pi\sin\pi\lambda-r^{-\lambda}L(r)$ for convenience. But

$$L\big(r,\min\,(a,b\cos\lambda\varphi)\big) = b\pi\sin\pi\lambda-2\int_0^{\varphi_0}(b\cos\lambda\varphi-a)\sin\lambda(\pi-\varphi)\,d\varphi$$

where $a=b\cos\lambda\varphi_0$, and so

$$d\geqq 2\int_0^{\varphi_0}(b\cos\lambda\varphi-a)\sin\lambda(\pi-\varphi)\,d\varphi.$$

The convex factors in the integrand are estimated by $b\cos\lambda\varphi-a\geqq(b-a)(1-\varphi/\varphi_0)$ and $\sin\lambda(\pi-\varphi)\geqq(1-\varphi/\varphi_0)\sin\pi\lambda$, which yields

$$d\geqq\frac{2}{3}\varphi_0(b-a)\sin\pi\lambda\geqq\frac{4}{3}(b-a)\arccos\frac{a}{b}>\frac{4}{3}b\left(1-\frac{a}{b}\right)^{3/2},$$

or

$$a>b-0{,}9b^{1/3}d^{2/3}>b-0{,}9A^{1/3}d^{2/3}.$$

Here we have also used that $0<\lambda\leqq 1/2$ implies $\dfrac{\sin\pi\lambda}{\lambda}\geqq 2$, and in the last step we made use of the inequality $b=r^{-\lambda}J(r)\leqq\lim_{r\to\infty} r^{-\lambda}J(r)=A$, which follows from the corollary. We have shown that

$$r^{-\lambda}J(r)-0{,}9A^{1/3}[r^{-\lambda}J(r)\pi\sin\pi\lambda-r^{-\lambda}L(r)]^{2/3}<r^{-\lambda}M(r)\leqq r^{-\lambda}J(r)$$

from which the remaining part of the statement in Theorem III follows.

7. Proof of Theorem IV

A similar discussion, as in the beginning of the proof of Theorem I, shows that we need only consider symmetric, twice continuously differentiable functions in a closed annulus $\bar{D}_1$, cut along the negative real axis. As a result of this regularization process, the function may now have a negative greatest lower bound in $\bar{D}_1$. We make the function positive in D_1 by adding $\varepsilon k r^{\lambda}\cos\lambda\varphi$, which has no influence on the validity of (1c). Here $k>0$ is a suitably chosen constant (independent of ε).

Clearly, this resulting new function, which we also denote $u(z)$, has a mean that is arbitrarily close to the mean of the original function. Put $G(r)=L_\alpha^\alpha(r)$, $u(z)=q(z)\cos\lambda\varphi$ and note that $q(z)\geqq\varepsilon k r_1^{\lambda}$ in $\bar{D}_1$. Differentiating twice, we obtain

$$G(r)=\int_{-\pi}^{\pi} q^\alpha(z)\cos\lambda\varphi\sin\lambda(\pi-|\varphi|)\,d\varphi.$$

$$G'(r)=\alpha\int_{-\pi}^{\pi} q^{\alpha-1}(z)u_r(z)\sin\lambda(\pi-|\varphi|)\,d\varphi.$$

$$G''(r)=\alpha(\alpha-1)\int_{-\pi}^{\pi}\frac{q^{\alpha-2}(z)u_r^2(z)\sin\lambda(\pi-|\varphi|)\,d\varphi}{\cos\lambda\varphi}$$

$$+\alpha\int_{-\pi}^{\pi} q^{\alpha-1}(z)u_{rr}(z)\sin\lambda(\pi-|\varphi|)\,d\varphi.$$

Next we apply Cauchy—Schwarz inequality to $G'(r)$ yielding

$$(G'(r))^2\leqq\alpha^2G(r)\int_{-\pi}^{\pi}\frac{q^{\alpha-2}(z)u_r^2(z)\sin\lambda(\pi-|\varphi|)}{\cos\lambda\varphi}\,d\varphi.$$

Since $u(z)$ is subharmonic we have

$$u_{rr}+\frac{1}{r}u_r+\frac{1}{r^2}u_{\varphi\varphi}\geqq 0,$$

which together with the previous estimate is inserted in the expression for $G''(r)$. There results

$$G''(r)\geqq\frac{\alpha-1}{\alpha}\frac{(G'(r))^2}{G(r)}-\frac{1}{r}G'(r)-\frac{\alpha}{r^2}\int_{-\pi}^{\pi} q^{\alpha-1}(z)u_{\varphi\varphi}(z)\sin\lambda(\pi-|\varphi|)\,d\varphi.$$

The last term is estimated by means of Lemma 3 and we get

$$G''(r) \geqq \frac{\alpha-1}{\alpha}\frac{(G'(r))^2}{G(r)} - \frac{1}{r}G'(r) + \frac{\alpha\lambda^2}{r^2}G(r).$$

Going back to $L_\alpha(r)$ this is

$$L''_\alpha(r) + \frac{1}{r}L'_\alpha(r) - \frac{\lambda^2}{r^2}L_\alpha(r) \geqq 0.$$

The remaining arguments are the same as in the proof of Theorem I.

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