An independence structure on indecomposable modules

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Introduction

A module A is called an *Le-module* provided its endomorphism ring is local, and a module M has an *Le-decomposition* if it is isomorphic to a direct sum of Le-modules.

In this paper we first give a simple proof of Azumaya's theorem concerning Le-decompositions. If $M=\bigoplus_I A_i$ is an Le-decomposition and A is any Le-module, we show that

$$\# \{i | A_i \cong A\} = \dim_{\Delta_A} F_A(M),$$

where $F_A(M)$ is a vector space over the divison ring Δ_A. Since $F_A(M)$ depends only on A and M the uniqueness of an Le-decomposition follows.

In the second section we show that the family $\mathfrak{L}(M)$ of direct summands of M which are Le-modules can be considered as an independence structure. This independence structure decomposes into simpler structures $\mathfrak{L}_A(M)$ which are closely related to the classical independence structure on the vector space $F_A(M)$. In particular, $\dim_{\Delta_A} F_A(M)=\dim \mathfrak{L}_A(M)$.

Any independence structure has a basis, and in theorem 2.10 we show that the Le-decomposition $M=\bigoplus_I A_i$ complements direct summands iff for any basis $\{B_j\}_J$ for $\mathfrak{L}(M)$ we have that $\bigoplus_J B_j=M$.

Terminology: Our notation follows Anderson and Fuller [1] and we refer the reader to this book. There is some overlap with results in [3] where some of the ideas in this paper are more developed.

We are considering left modules over an associative ring R with an identity. All the maps between modules are R-homomorphisms and map$=R$-homo-

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morphism. Furthermore, mono=monomorphism, epi=epimorhpism, and iso=isomorphism.

If $f: X \to \bigoplus_I Y_i$ we let $f_i = \Pi_i f$ where Π_i is the projection on Y_i. Furthermore, if $g: \oplus Y_i \to X$, we let $g_i = g\varepsilon_i$, where ε_i is the injection of Y_i into $\bigoplus_I Y_i$.

We emphasize the use of $\oplus$ in this paper. It is used exclusively on submodules of a given module *(internal sum)*.

The expression $M = N_1 \oplus N_2$ signifies that N_1, N_2 are submodules of M, $N_1 \cap N_2 = (0)$ and $N_1 + N_2 = M$. An expression $\bigoplus_I N_i \lhd M$ expresses that the sum $\sum_I N_i$ is direct and that $\sum_I N_i$ is a *direct summand* of M. Therefore, $\bigoplus_I N_i \not\!\lhd M$ means that the sum $\sum_I N_i$ is *not* direct *or* $\sum_I N_i$ is *not* a direct summand of M.

If M is a module, we let J_M denote the Jacobson radical of End M.

A decomposition $M = \bigoplus_I M_i$ *complements direct summands* if for any direct summand K of M, $M = K \oplus (\bigoplus_J M_j)$ for some subset $J \subset I$.

A module M has the *exchange property* if for any module Ω, if

$$\Omega = M' \oplus L = \bigoplus_I N_i$$

with $M' \cong M$ then there are submodules $N_i' \subset N_i$ such that $\Omega = M' \oplus (\bigoplus_I N_i')$.

A finite sum of Le-modules has the exchange property. A simple proof of this is given in [3] (Remark after Theorem 4). Another proof can be found in [1, Lemma 26.4].

1.

In this section we give a simple proof of Azumaya's theorem. Our proof is a combination of a few elementary observations.

Lemma 1.1. *Let* $f: M \to \bigoplus_I M_i$ *and* $g: \bigoplus_I M_i \to M$ *be maps such that* $gf(x) = x$ *for some* $x \in M$, $x \neq 0$. *Then* $g_i f_i \notin J_M$ *for some* $i \in I$.

Proof. For some finite set $\{i_1, \dots, i_k\} \subset I$ we have that

$$x = (g_{i_1} f_{i_1} + \dots + g_{i_k} f_{i_k})(x). \quad \text{Hence} \quad g_{i_1} f_{i_1} + \dots + g_{i_k} f_{i_k} \notin J_M.$$

Lemma 1.2. *Let* A *be an Le-module and* $f: A \to \bigoplus_I M_i$. *Then* f *is a split mono iff* f_i *is a split mono for some* $i \in I$.

Proof. If f_i is a split mono then trivially f is a split mono. Conversely assume that f is a split mono and let $g: \bigoplus_I M_i \to A$ have the property that $gf = I_A$. Hence $g_i f_i \notin J_A$ for some $i \in I$. This implies that $g_i f_i$ is an isomorphism, so f_i is a split mono.

For any Le-module A and arbitrary module M we let $J(A, M)$ denote the subset of $\text{Hom}_R(A, M)$ consisting of all maps $f: A \to M$ which are *not* split mono. One easily verifies that $J(A, M)$ is a submodule of the right End A-module $\text{Hom}_R(A, M)$, and furthermore $\text{Hom}_R(A, M) \cdot J_A \subset J(A, M)$. Hence $F_A(M) =$

$\mathrm{Hom}_R(A, M)/J(A, M)$ is a right Δ_A-module where Δ_A denotes the division ring $\mathrm{End}\, A/J_A$. We note that $F_A(A)=\Delta_A$.

Lemma 1.3. $F_A(\amalg_I M_i)\cong \amalg_I F_A(M_i)$ (Δ_A-isomorphism).

Proof. We have a canonical map $\varphi\colon F_A(\amalg_I M_i)\to \Pi_I F_A(M_i)$ defined by $\varphi(\bar f)=(\bar f_i)$ for any $f\colon A\to \amalg_I M_i$. Lemma 1.2 implies that θ is $1-1$. Trivially im $\varphi\supset \amalg_I F_A(M_i)$. Moreover, for any $f\colon A\to \amalg_I M_i$, f_i is a (split) monomorphism for at most a finite set of $i\in I$. Hence $\bar f_i=0$ for almost all $i\in I$. Hence im $\varphi=\amalg_I F_A(M_i)$.

The uniqueness of an Le-decomposition follows from these lemmas.

Theorem 1.4. Let $M=\bigoplus_I A_i$ be an Le-decomposition. For any Le-module A, $[F_A(M):\Delta_A]=\#\{i\,|\,A_i\cong A\}$.

Proof. For any indecomposable module B, $F_A(B)\cong\Delta_A$ if $B\cong A$ and zero otherwise. The result follows now from Lemma 1.3.

Theorem 1.4 implies that if $M=\bigoplus_I A_i=\bigoplus_J B_j$ are two Le-decompositions of M, then we have a bijection $\sigma\colon I\to J$ such that $A_i\cong B_{\sigma(i)}$ for all $i\in I$. To complete the proof of Azumaya's theorem we need two more simple lemmas.

Lemma 1.5. *Let $f\colon M\to N$ and $g\colon N\to M$ be R-homomorphisms. Then 1_N+fg is an isomorphism iff 1_M+gf is an isomorphism.*

Proof. $(1_N+fg)^{-1}=1_N-f(1_M+gf)^{-1}g$ whenever 1_M+gf is an isomorphism.

Lemma 1.6. *Let $f\colon A\to M$ and $g\colon M\to A$ be R-homomorphisms. Suppose that A is an Le-module and $fg\notin J_M$. Then f is a split mono and g a split epi.*

Proof. If $fg\notin J_M$ then 1_M+hfg is not an isomorphism for some $h\in\mathrm{End}\, M$. Hence 1_A+ghf is not an isomorphism (lemma 1.5), so ghf is an isomorphism. This proves lemma 1.6.

We are ready to prove the remaining part of Azumaya's theorem.

Theorem 1.7. *(Azumaya) Let $M=\bigoplus_I A_i$ be an Le-decomposition and let $X\oplus L=M$, $X\neq(0)$. Then*

i) *X contains a direct summand A which is isomorphic to A_i for some $i\in I$.*

ii) *If X is indecomposable, then for some $i_0\in I$, $M=A_{i_0}\oplus L=X\oplus(\bigoplus_{i\neq i_0} A_i)$.*

Proof. Let $f\colon \bigoplus_I A_i\to X$ be the projection on X along L, and let $g\colon X\to\bigoplus_I A_i$ be the injection of X into M. Then $f_ig_i\notin J_x$ for some $i\in I$, hence (lemma 1.6) f_i is a split mono and g_i a split epi. This proves i).

Assume now that X is indecomposable and that f_{i_0} is a split mono and g_{i_0} a split epi. Since both X and A_{i_0} are indecomposable it follows that f_{i_0} and g_{i_0} are isomorphisms, and this proves the second statement in Theorem 1.7.

Remark. Lemma 1.6 has several other interesting corollaries, (see [3]).

2.

Let M be an R-module. We define $\mathscr{L}(M)$ to be the set of direct summands of M which are Le-modules. For any Le-module A we let $\mathscr{L}_A(M)$ be the subset of $\mathscr{L}(M)$ consisting of those elements which are isomorphic to A. We let $\operatorname{supp} M$ be a set containing one representative from each isomorphism class in $\mathscr{L}(M)$. Hence $\mathscr{L}(M)=\bigcup_{\operatorname{supp} M}\mathscr{L}_A(M)$.

Independence structures arise when one considers abstract properties of linear independence of vectors in a vector space.

Definition. Let E be a non-empty set. A non-empty collection $\mathscr{E}$ of subsets of E is called an *independence structure* if it satisfies the following three axioms.

I1. If $X\in\mathscr{E}$ and $Y\subset X$ then $Y\in\mathscr{E}$.

I2. If X, Y are finite members of $\mathscr{E}$ and $|X|=|Y|+1$, then there exists an element $x\in X$ such that $Y\cup\{x\}\in\mathscr{E}$.

I3. A set $X\in\mathscr{E}$ iff any finite subset of X is in $\mathscr{E}$.

The members of $\mathscr{E}$ are called the *independent subsets*. A maximal independent subset is called a *basis* for E. Any two bases have the same cardinal number (R. Rado), and we let $\dim E$ denote this cardinal number.

For each R-module M we shall define an independence structure on $\mathscr{L}(M)$ and $\mathscr{L}_A(M)$. We let $\mathscr{E}=\mathscr{E}(M)$ be the following collection of subsets of $\mathscr{L}(M)$: A set $\mathscr{A}\in\mathscr{E}$ iff for any finite subset $\{A_1, \dots, A_k\}\subset\mathscr{A}$, $A_1\oplus\dots\oplus A_k\lhd M$. For any Le-module A we similarly define $\mathscr{E}_A=\mathscr{E}_A(M)=\{\mathscr{A}\cap\mathscr{L}_A(M)\}_{\mathscr{A}\in\mathscr{E}(M)}$.

Proposition 2.1. a) *$\mathscr{E}$ is an independence structure on $\mathscr{L}(M)$*

b) *$\mathscr{E}_A$ is an independence structure on $\mathscr{L}_A(M)$ ($A\in\operatorname{supp} M$).*

c) *Let $E\subset\mathscr{L}(M)$. Then $E\in\mathscr{E}$ iff $E\cap\mathscr{L}_A(M)\in\mathscr{E}_A$ for all $A\in\operatorname{supp} M$.*

d) *A set $E\subset\mathscr{L}(M)$ is a basis iff $E\cap\mathscr{L}_A(M)$ is a basis for $\mathscr{L}_A(M)$ for all $A\in\operatorname{supp} M$.*

Proof. a) I1 and I3 are trivially satisfied. We need to show I2. Let $X=\{A_1, \dots, A_{n+1}\}\in\mathscr{E}$ and $Y=\{B_1, \dots, B_n\}\in\mathscr{E}$. Let $A_1\oplus\dots\oplus A_{n+1}\oplus C=M$. Since $B_1\oplus\dots\oplus B_n\lhd M$ and $B_1\oplus\dots\oplus B_n$ has the exchange property, we have

$$(B_1\oplus\dots\oplus B_n)\oplus(A_1'\oplus\dots\oplus A_{n+1}'\oplus C') = M.$$

If $A_1'=A_2'=\dots=A_{n+1}'=(0)$ we would get that $B_1\oplus\dots\oplus B_n\cong A_1\oplus\dots\oplus A_{n+1}\oplus C''$. Hence $A_i'=A_i$ for some $1\leqq i\leqq n+1$ and therefore $\{B_1, \dots, B_n, A_i\}\in\mathscr{E}$.

b) The restriction of an independence structure to a subset is again an independence structure.

c) By definition $E\in\mathscr{E}\Rightarrow E\cap\mathscr{L}_A(M)\in\mathscr{E}_A$. Conversely, let $E\subset\mathscr{L}(M)$ and assume that $E\cap\mathscr{L}_A(M)\in\mathscr{E}_A$ for all $A\in\operatorname{supp} M$. To show that $E\in\mathscr{E}$ it can be assumed that E is a finite set. Let

$$E = \{A_{1,1}, \ldots, A_{1,r_1}, A_{2,1}, \ldots, A_{2,r_2}, \ldots, A_{s,1}, \ldots, A_{s,r_s}\}$$

where $A_{ij} \cong A_{kl}$ iff $i=k$. Hence

$$\begin{aligned} B_1 &= A_{1,1} \oplus \ldots \oplus A_{1,r_1} \lhd M \\ B_2 &= A_{2,1} \oplus \ldots \oplus A_{2,r_2} \lhd M \\ &\vdots \\ B_s &= A_{s,1} \oplus \ldots \oplus A_{s,r_s} \lhd M. \end{aligned}$$

Let $B_1 \oplus X_1 = M$. The module B_2 has the exchange property so $B_2 \oplus B_1' \oplus X_2 = M$, $B_1' \subset B_1$ and $X_2 \subset X_1$. Hence $B_2 \cong B_1'' \oplus X_1'$ where $B_1' \oplus B_1'' = B_1$ and $X_1' \oplus X_2 = X_1$. The Krull—Schmidt theorem implies that $B_1'' = (0)$, hence $B_1 \oplus B_2 \oplus X_2 = M$. We now use that B_3 has the exchange property. Since no direct summand of $B_1 \oplus B_2$ (different from (0)) is isomorphic to a direct summand of B_3 we get that $B_1 \oplus B_2 \oplus B_3 \oplus X_3 = M$ etc.

d) Follows from b) and c).

Since the independence structure on $\mathscr{L}(M)$ is the disjoint sum of the independence structures on $\mathscr{L}_A(M)$ we may concentrate on these structures.

Theorem 2.2. *Let A be an Le-module and let $\{f_i\}_{i \in I}$ be a family of split monomorphisms from A to M. Then the family $\{f_i(A)\}_I$ is independent in $\mathscr{L}_A(M)$ iff the family $\{\bar{f}_i\}_I$ is independent in $F_A(M)$. Furthermore, $\{f_i(A)\}_I$ is a basis for $\mathscr{L}_A(M)$ iff the family $\{\bar{f}_i\}_I$ is a basis for $F_A(M)$.*

Proof. We want to show $\{f_i(A)\}_I$ independent$\Leftrightarrow\{\bar{f}_i\}_I$ independent We can assume that $|I| < \infty$. Let $\bar{f}_1, \ldots, \bar{f}_n$ be an independent set in $F_A(M)$ and let $A_i = f_i(A)$ $(1 \leqq i \leqq n)$. Let $1 \leqq k \leqq n-1$ and assume that $A_1 \oplus \ldots \oplus A_k \lhd M$.

Let $A_1 \oplus \ldots \oplus A_k \oplus X = M$ and let $\pi_1, \ldots, \pi_k, \pi_x$ be the projections associated with this decomposition. Hence $\pi_1 + \ldots + \pi_k + \pi_x = 1_M$. Therefore,

$$f_{k+1} = \pi_1 f_{k+1} + \ldots + \pi_k f_{k+1} + \pi_x f_{k+1}.$$

Since im $\pi_i = \operatorname{im} f_i$ $(1 \leqq i \leqq k)$ and the f_i's are monomorphisms, we have maps $\varphi_i \in \operatorname{End} A$ $(1 \leqq i \leqq k)$ such that $\pi_i f_{k+1} = f_i \varphi_i$ $(1 \leqq i \leqq k)$. Hence

$$f_{k+1} = f_1 \varphi_1 + \ldots + f_k \varphi_k + \pi_x f_{k+1}.$$

Since the set $\{\bar{f}_1, \cdots, \bar{f}_k, \bar{f}_{k+1}\}$ is lin. independent it follows that $\overline{\pi_x f_{k+1}} \neq 0$. Hence $\pi_x f_{k+1}$ is a split mono. Therefore $\pi_x(A_{k+1})$ is a direct summand of X and π_x restricted to A_{k+1} is a monomorphism. Let $\pi_x(A_{k+1}) \oplus X' = X$. Then one easily proves that $A_1 \oplus \ldots A_k \oplus A_{k+1} \oplus X' = M$, hence $A_1 \oplus \ldots \oplus A_{k+1} \lhd M$. Since $A_1 \lhd M$, we can conclude that $A_1 \oplus \ldots \oplus A_n \lhd M$ which shows that $\{A_1, \ldots, A_n\}$ is independent.

Conversely assume that $A_1 \oplus \ldots \oplus A_n \oplus X = M$. We claim that $\{\bar{f}_i\}_{i=1,\ldots,n}$ are independent. Let $\pi_1, \ldots, \pi_n, \pi_x$ be the projections associated with the decomposition $A_1 \oplus \ldots \oplus A_n \oplus X = M$, and assume that $f = f_1\varphi_1 + \ldots + f_n\varphi_n \in J(A, M)$. Then $\pi_i f = f_i\varphi_i \in J(A, M)$. This implies that φ_i is not an isomorphism in End A, hence $\varphi_i \in J_A$. Hence $\bar{\varphi}_1 = \ldots = \bar{\varphi}_n = \bar{0}$ in Δ_A. Therefore $\{\bar{f}_1, \ldots, \bar{f}_n\}$ are independent over Δ_A.

It is now trivial that $\{f_i(A)\}_{i\in I}$ is a basis for $\mathcal{L}_A(M)$ iff $\{\bar{f}_i\}_{i\in I}$ is a basis for $F_A(M)$.

Corollary 2.3. $\dim \mathcal{L}_A(M) = [F_A(M) : \Delta_A]$.

Corollary 2.4. *Let $\{B_i\}_I$ and $\{C_j\}_J$ be two bases for $\mathcal{L}(M)$. Then $\bigoplus_I B_i \cong \bigoplus_J C_j$.*

Proof. Let $B = \{B_i\}_I$. Then $B \cap \mathcal{L}_A(M)$ is a basis for $\mathcal{L}_A(M)$. Hence $\# \{i \mid B_i \cong A\} = [F_A(M) : \Delta_A]$. This shows that $\bigoplus_I B_i \cong \bigoplus_J C_j$.

Let $A_1, \ldots, A_n$ be Le-modules and $M = A_1 \oplus \ldots \oplus A_n$. Then $\{A_1, \ldots, A_n\}$ is trivially a maximal independent set in $\mathcal{L}(M)$; in other words $\{A_1, \ldots, A_n\}$ is a basis for $\mathcal{L}(M)$. Let $\{B_1, \ldots, B_n\}$ be another basis. Then $B_1 \oplus \ldots \oplus B_n \lhd M$. If $B_1 \oplus \ldots \oplus B_n \neq M$, theorem 1.7i implies that $\{B_1, \ldots, B_n\}$ is not a maximal independent set.

We have shown

Theorem 2.5. *Let $A_1, \ldots, A_n$ be Le-modules and let $M = A_1 \oplus \ldots \oplus A_n$. Then* $\dim \mathcal{L}(M) = n$, *and $\{B_1, \ldots, B_n\}$ is a basis for $\mathcal{L}(M)$ iff $B_1 \oplus \ldots \oplus B_n = M$.*

Proposition 2.1, Theorem 2.2 and Theorem 2.5 reduce combinatorial problems on finite decompositions $M = A_1 \oplus \ldots \oplus A_n$ to similar problems on finite dimensional vector spaces over division rings.

As an example let us prove

Proposition 2.6. *Let $X = \{x_1, \ldots, x_n\}$ and $Y_n = \{y_1, \ldots, y_n\}$ be two bases for a vector space V over a division ring D. Then there exists a permutation $\sigma \in S_n$ such that $(Y - \{y_{\sigma(i)}\}) \cup \{x_i\}$ is a basis for all $1 \leqq i \leqq n$.*

Proof. Any element $x \in V$ is a linear combination $x = r_1y_1 + \ldots + r_ny_n$ $(r_i \in D)$. Let $\operatorname{supp} x = \{y_i \in Y \mid r_i \neq 0\}$. Then $(Y - \{y_i\}) \cup \{x\}$ is a basis iff $y_i \in \operatorname{supp} x$. Since $x \in \langle \operatorname{supp} x \rangle$, we note that

$$\# (\operatorname{supp} x_{i_1} \cup \operatorname{supp} x_{i_2} \cup \ldots \cup \operatorname{supp} x_{i_k}) \geqq k$$

for any set $1 \leqq i_1 < i_2 < \ldots < i_k \leqq n$. From Hall's theorem about distinct representatives it follows that the sets $\operatorname{supp} x_1, \ldots, \operatorname{supp} x_n$ have a distinct system of representatives, say, $y_{\sigma(1)}, \ldots, y_{\sigma(n)}$, and this proves Prop. 2.6.

Proposition 2.1, theorems 2.2 and 2.5 now imply

Theorem 2.7. *Let* $M=A_1\oplus\ldots\oplus A_n=B_1\oplus\ldots\oplus B_n$ *be two Le-decompositions. There exists a permutation* $\sigma\in S_n$ *such that* $B_{\sigma(1)}\oplus\ldots\oplus B_{\sigma(k-1)}\oplus A_k\oplus B_{\sigma(k+1)}\oplus\ldots\oplus B_{\sigma(n)}=M$ *for all* $k, 1\leqq k\leqq n$.

Proof. We leave it to the reader to verify the simple reduction to proposition 2.6.

Remark R. A. Brualdi [4] has shown that if B_1 and B_2 are bases of an independence structure then there is an injection $\sigma: B_1\to B_2$ such that $(B_2-\{\sigma(e)\}\cup\{e\}$ is a basis for all $e\in B_1$. The theorem of Brualdi together with Proposition 2.1 and Theorem 2.5 therefore imply Theorem 2.7

We shall extend theorem 2.5 to arbitrary Le-decompositions.

Definition. Two direct summands X and Y of a module M are *equivalent* provided they have identical sets of complements in M, and we write $X\sim Y$.

Lemma 2.8. *Let* $A_1, A_2\in\mathscr{L}(M)$. *Then* $A_1\sim A_2$ *iff* $A_1\oplus A_2\not\triangleleft M$.

Proof. Lemma 2.8 is proved in [3], but we repeat the simple proof.

Let $A_1\oplus X_1=A_2\oplus X_2=M$ and assume that $A_1\oplus X_2\neq M$. Since A_1 has the exchange property, we get that $A_1\oplus A_2\oplus X_2'=M$. Hence $A_1\oplus A_2\triangleleft M$.

Conversely, assume that $A_1\oplus A_2\triangleleft M$ and let $A_1\oplus A_2\oplus X=M$. Then $A_2\oplus X$ is a complement of A_1 but not a complement of A_2.

Lemma 2.9. *Let* $X\triangleleft M$ *and let* $A\in\mathscr{L}(M)$. *Then* $X\oplus A\not\triangleleft M$ *iff* $A\sim A'$ *for some* $A'\subset X$.

Proof. Assume first that $A\sim A'\subset X$. Let $A'\oplus X'=X$ and let $X\oplus A\oplus Y=M$. Then $A'\oplus X'\oplus A\oplus Y$, and lemma 2.8 implies that $A\nsim A'$. Hence $X\oplus A\not\triangleleft M$.

Conversely, assume that $X\oplus A\not\triangleleft M$. Let $X\oplus Y=M$ and let $\pi_X+\pi_Y=1_M$. Let $i\colon A\to M$ be the injection of A into M. Then $i=\pi_X i+\pi_Y i$. If $\pi_Y i$ is a split mono it is easily seen that $X\oplus A\triangleleft M$ (see proof of theorem 2.2). Hence we know that $\pi_Y i\in J(A, M)$. Since $i\notin J(A, M)$ it follows that $\pi_X i$ is a split monomorphism. Furthermore $\bar{i}=\overline{\pi_X i}$ in $F_A(M)$, and therefore $i(A)$ and $\pi_X(i(A))$ are dependent (theorem 2.2), i.e., $A\oplus\pi_X(A)\not\triangleleft M$. Therefore $A\sim\pi_X(A)\subset X$ (Lemma 2.8).

We are now prepared to state and prove the generalization of theorem 2.5.

Theorem 2.10. *The following properties for a module* M *are equivalent*

1) *For any basis* $\{A_i\}_I$ *in* $\mathscr{L}(M)$ *we have that* $M=\bigoplus_I A_i$.
2) M *has an Le-decomposition that complements direct summands.*

Proof 1)⇒2). Since any independence structure has a basis we may assume that M has an Le-decomposition $M=\bigoplus_I A_i$. Let $\{B_i\}_{i\in I}$ be any set of direct summands of M for which $B_i\sim A_i$ for all $i\in I$. We shall show that $\bigoplus_I B_i=M$ and the implication 1)⇒2) follows then from theorem 18, [3]. It therefore suffices to show that $\{B_i\}_I$ is a basis for $\mathscr{L}(M)$.

For any finite subset $\{i_1, \ldots, i_k\} \subset I$ we have $(A_{i_1} \oplus \ldots \oplus A_{i_k}) \oplus (\bigoplus_{j \in I'} A_j) = M$, and since $B_{i_1} \sim A_{i_1}, \ldots, B_{i_k} \sim A_{i_k}$ we get that $(B_{i_1} \oplus \ldots \oplus B_{i_1}) \oplus (\bigoplus_{I'} A_j) = M$. Hence the family $\{B_i\}_I$ is independent, and since $B_{i_1} \oplus \ldots \oplus B_{i_k} \sim A_{i_1} \oplus \ldots \oplus A_{i_k}$ one easily proves that $\{B_i\}_I$ is a maximal independent set.

2)⇒1). Assume that $M = \bigoplus_I A_i$ complements direct summands and let $\{B_j\}_J$ be a basis for $\mathscr{L}(M)$. For each $i \in I$, the set $\{A_i\} \cup \{B_j\}_J$ is dependent, so $(B_{j_1} \oplus \ldots \oplus B_{j_k}) \oplus A_i \prec\!\!\lhd M$ for some finite subset $\{j_1, \ldots, j_k\} \subset J$. Hence $A_i \sim C_i \subset B_{j_1} \oplus \ldots \oplus B_{j_k}$ (Lemma 2.9). Therefore $\bigoplus_J B_j \supset \bigoplus_I C_i$ where $C_i \sim A_i$ for all $i \in I$. Theorem 18 [3] implies that $\bigoplus_J B_j = M$.

Remarks. **I.** Let R be a local ring with maximal ideal m and let $P = R^{(I)}$ be a free module. We see that

$$F_R(P) = \mathrm{Hom}\,(R, P)/J(R, P) \cong P/mP.$$

Theorem 2.10 in this case says that $P = R^{(I)}$ complements direct summands iff mP is small in P. (Anderson and Fuller [1]).

II. Let $\{E_i\}_I$ be a family of injective indecomposable modules and let $E = \bigoplus_I E_i$. We observe from lemma 2.8 that two injective indecomposable submodules of a module M are equivalent iff they have a non-zero intersection. From lemma 2.9 we see that a family $\{F_j\}_J$ of injective indecomposable modules is a basis for E iff $\bigoplus_J F_j \cong E$ and the extension $\bigoplus_J F_j \subset E$ is essential. In particular, we get from theorem 2.10 that an Le-decomposition of an injective module complements direct summands (Warfield [6]).

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