Some representation theorems for Banach lattices

Sten Kaijser

0. Introduction

We prove that the notions of a Banach lattice with "quasi-interior elements" and of a cyclic $C(X)$-module are essentially equivalent. This leads to various representation theorems for such Banach lattices. The most interesting result is perhaps that every separable non-atomic Banach lattice, such that the dual space has a weak order unit, can be represented by Lebesgue-measurable functions on the unit interval.

1. The notion of "ideal center" was first introduced in the setting of C^*-algebras by Effros [2] and Dixmier [1]. In [6] W. Wils redefined the notion in the setting of partially ordered spaces. He defines there the ideal center of a partially ordered vector space E to be the set of all endomorphisms of E that are bounded (for the order of operators) by a multiple of the identity operator. We shall denote the center of E by $Z(E)$ (Wils writes Z_E) and we shall use the ideal center in a context which was essentially avoided by Wils. According to Wils "Z_E turns out to be a very useful tool in digging up remnants of lattice structure", and since the main interest of Wils is to study more general partially ordered spaces, he almost entirely avoids the case of Banach lattices. In passing he does however prove that the ideal center of a Banach lattice is always a $C(X)$-space (both as a Banach lattice and as a Banach algebra), and it was observed by Hackenbroch [3] that if the given Banach lattice E has a weak order unit then $Z(E)$ is isomorphic (as a vector lattice) to the order ideal generated by u.

Using this fact Hackenbroch proves strong uniqueness theorems for representation spaces of Banach lattices (having a weak unit). Since thus the "endomorphism algebra" of a Banach lattice with weak order unit contains a "large $C(X)$-space" the lattice itself may be considered as a module over its ideal center.

Let A now be a Banach algebra and let M be a Banach space. We shall then say that M is a left Banach module over A, or a left A-module, if M is a representation space for A, that is if there exists a **contracting** homomorphism of A into $L(M)$. Likewise N is called a **right** A-module if there exists a contracting anti-homomorphism of A into $L(M)$. We shall usually write "$a \cdot m$" or only "am" for the "product" in a left A-module and "ma" for the product in a right A-module. It is well-known that if M is a left A-module then the dual space M' is a right A-module. Now it is a general rule that the simplest modules are the cyclic modules, and it is therefore a natural problem to decide when a Banach lattice B is a cyclic module over its center. It turns out that this holds if and only if the positive cone of B has a so called "quasi-interior point" u. We shall in this paper call such an element a "topological (order) unit". Now the main importance of a topological unit is that B may then be considered as the completion of $Z(B)$ for a smaller norm, and therefore we also get a representation of the dual space B' as measures on the maximal ideal space of $Z(B)$. Conversely, it is also natural to consider cyclic modules over a given (real) $C(X)$-space, and it is then easily proved that such a module is in fact a Banach lattice. The ideal center of this Banach lattice may be properly bigger than the given space and in particular we prove that B is separable if and only if it is a cyclic module over a $C(M)$-space, where M is a metrizable compact space. (In the following a **compact space** is always assumed to be a Hausdorff space.)

It even turns out that every reflexive, separable, non-atomic Banach lattice is a cyclic module over $C(I)$ (I the unit interval), while the ideal center of B is then isomorphic to $L^\infty(I, dx)$, and in fact B may be represented as an order ideal in $L^1(I, dx)$, i.e. as a space of Lebesgue-measurable functions on I.

The close relations between Banach lattices and $C(X)$-modules also hold for complex spaces and it is easily proved that a cyclic module over a complex $C(X)$-space is a complex Banach lattice (defined e.g. as the complexification of a real Banach lattice). In the following all Banach spaces are assumed to be real spaces but it would require only minor modifications to consider complex spaces.

We shall conclude this introduction with the remark that we follow the standard notations of writing $L(B)$ for the algebra of all endomorphisms of the given Banach space B, and we write $U(B)$ to denote the unit ball of B. Finally, $C(X)$ always denotes the algebra of all (real-valued) continuous functions on the **compact** space X.

1. Preliminaries

1. We shall begin by fixing terminology and notations. First of all a Banach lattice is a partially ordered Banach space where each finite subset has a least upper bound and such that if $|f| \leqq |g|$ then also $\|f\| \leqq \|g\|$ (we denote as usual $|f| = \sup(f, -f)$).

We also adopt the convention that a **subspace** of a Banach space is always closed, and if for some reason we consider a subset L of a Banach space, such that L is stable under the linear operations then we shall say that L is a **linear subspace** of the given Banach space. In accordance with this convention a sublattice and an ideal in a Banach lattice are always assumed to be closed, while e.g. a linear ideal need not be closed. Furthermore, we shall say that an ideal β in the Banach lattice B is a **band** if whenever a subset E of β has a least upper bound e in B, then $e\in\beta$. Since every band is topologically closed there is no need for 'linear band's'.

To fix notations we shall make the following

Definition 1.1. Let S be a subset of the Banach lattice B. We shall then write $m(S)$ to denote the linear ideal generated by S, i.e.

$$m(S)=\left\{x\in B\middle|\,|x|\leqq \textstyle\sum_{i=1}^{N} r_i\cdot|s_i|,\ r_i\in\mathbf{R},\ s_i\in S\right\}.$$

$I(S)$ to denote the ideal generated by $S(I(S)$ is the closure of $m(S))$
$\beta(S)$ to denote the band generated by S.

If the set S consists of a single element s, then we shall write $m(s)$, $I(s)$ and $\beta(s)$ instead of $m(\{s\})$, $I(\{s\})$ and $\beta(\{s\})$. The element a is called an *atom* if $I(a)$ is 1-dimensional. A Banach lattice is said to be non-atomic if it has no atoms.

We shall next introduce a perhaps not quite standard terminology by

Definition 1.2. Let B be a Banach lattice and let u be a positive element of norm 1. We shall then say that u is a

K-unit (for Kakutani-unit or Krein-unit) if $m(u)=B$
t-unit (for topological unit) or simply unit if $I(u)=B$
F-unit (for Freudenthal-unit) if $\beta(u)=B$.

Since the notion of a t-unit will be the most useful we shall say that a Banach lattice is *unital* if it has a t-unit. To simplify notations we shall often write (B,u) to denote that u is a given t-unit in the unital Banach lattice B.

We shall in the following need the following well-known

Proposition 1.3. a) *For every element s in the Banach lattice P, the linear ideal $m(s)$ can be renormed by*

$$\|f\|_m=\inf\{t\,|\,|f|\leqq t\cdot|s|\}.$$

With this norm $m(s)$ becomes an AM-space in the sense of Kakutani so by the Kakutani representation theorem $m(s)$ is isomorphic to $C(X)$ for some compact Hausdorff space X. Under this isomorphism the function 1_X corresponds to $|s|$.

b) *Every separable Banach lattice contains a t-unit.*

2. A concept which will turn out to be very useful when considering 'simultaneous' representations of a given Banach lattice B and its dual space B' is given by

Definition 1.4. Let B be a Banach lattice with dual space B'. A pair (u, u'), $u \in B$, $u' \in B'$ will be called a *dual pair* if u is a t-unit in B, if u' is an F-unit in B' and if $\langle u, u' \rangle = 1$. Even if both B and B' are separable so that both have units we do not know if they will always have a dual pair. However, it follows from the following proposition that it is no real loss of generality to assume that whenever both B and B' have units (or if B has a t-unit and B' an F-unit) then they also have a dual pair.

Proposition 1.5. *Let (B, u) be a unital Banach lattice, and suppose that the dual space B' has an F-unit v. Let further $\alpha > 1$ be a real number. There exists then a Banach lattice B_α, obtained from B by a slight change of norm, and an element u' in B'_α, such that (u, u') is a dual pair for B_α, B'_α and such that for all $b \in B$, $\|b\| \leqq \|b\|_\alpha \leqq \alpha \cdot \|b\|$ (where $\|b\|$ denotes the norm of b in B).*

Proof. Choose $u_1 \in B'$ such that $\langle u, u_1 \rangle = 1$ and put $u_2 = u_1 + (\alpha - 1) \cdot v$. Let further $k = \langle u, u_2 \rangle$, and put $u' = k^{-1} \cdot u_2$. Defining now $\|b\|_\alpha = \sup(\|b\|, \langle |b|, u' \rangle)$ we obviously have $\|u\|_\alpha = \langle u, u' \rangle = 1$. Furthermore, u' dominates the unit v so u' is a unit and since $1 \leqq \|u'\|_{B'} \leqq \alpha$, we also have $\|b\| \leqq \|b\|_\alpha \leqq \alpha \cdot \|b\|$ and this proves the proposition.

3. We shall conclude this introductory chapter by introducing a notion which was first used in the context of C^*-algebras by Dixmier [3], and was then used for studying partially ordered spaces by Wils [*W*].

Definition 1.6. Let B be a Banach lattice. The set of all operators z in $L(B)$ that are bounded for the ordering of operators by a multiple of the identity is called the *ideal center* of B and will be denoted $Z(B)$.

The basic properties of the ideal center are given by

Proposition 1.7. (Wils [W]). *Let B be a Banach lattice and let $Z(B)$ be the ideal center of B. $Z(B)$ is then a closed subalgebra of $L(B)$, and with the order structure inherited from $L(B)$ it is also a Banach lattice with 1_B as a K-unit. Furthermore, $Z(B)$ is with respect to both these structures isomorphic to $C(X)$ for some compact Hausdorff space X. If B has an F-unit u, then the map*

$$U\colon Z(B) \to B \quad \textit{defined by}$$

$$U(z) = z(u), \quad z \in Z(B)$$

is a lattice isomorphism onto $m(u)$, If u is a t-unit then B is a cyclic $Z(B)$-module with u a cyclic vector. If B $(=E')$ is a dual space (so that $L(B)$ is the dual space to $E\hat{\otimes}E'$) then $Z(B)$ is also a dual space (since it is weak closed in $L(B)$).*

Remark. It follows from Proposition 1.6 that if u_1 and u_2 are both (not necessarily normalized) F-units in B, then $m(u_1)$ and $m(u_2)$ are isomorphic vector lattices (since both are isomorphic to $Z(B)$). This fact was used by Hackenbroch [3] to obtain intrinsic characterizations of representation spaces for Banach lattices.

2. Unital Banach lattices and cyclic $C(X)$-modules

1. We ended the previous paragraph by stating Wils theorem which implies in particular that every unital Banach lattice is a cyclic $C(X)$-module for some $C(X)$-space. In the present paragraph we shall prove various converses and refinements of Wils' theorem, starting with

Proposition 2.1. *Let X be a compact space, let B be a Banach module over $C(X)$ and let $u\in B$, $\|u\|=1$ be a cyclic vector in B, i.e. the linear subspace $C(X)\cdot u$ is dense in B. Then B can be given an order structure and becomes with this order a unital Banach lattice with u as a t-unit.*

Proof. We define a new norm on $C(X)$ by

$$\|f\|' = \|f\cdot u\|_B$$

and we observe that by assumption the completion of $C(X)$ for this norm is isomorphic to B. (Strictly speaking $\|\ \|'$ need only be a seminorm but it is easy to see that the set of all f having $\|f\|'=0$ is an ideal I in $C(X)$ so $\|\ \|'$ is a norming of $C(X)/I\approx C(Y)$). The positive cone is now defined as the closure of the positive cone in $C(X)$ for the natural order of $C(X)$. We have then a positive injective map $h\colon C(X)\to B$ having dense renge. The adjoint of h is therefore an injective map $h'\colon B'\to M(X)$. Since h' is injective we identify B' with a subspace of $M(X)$. It is then clear that the dual order in P' coincides with the order obtained from $M(X)$. We observe next that B' is a linear ideal in $M(X)$, in fact B' is the set of all measures $\mu\in M(X)$ for which

$$\int_X f(x)\,d\mu(x) \leqq C(\mu)\cdot\|f\|'$$

and it is obvious that if $|\nu|\leqq|\mu|$ and $\mu\in B'$ then also $\nu\in B'$. In particular B' is a linear sublattice of $M(X)$ and is with its own norm a Banach lattice. Therefore B'' is a Banach lattice and then B being the closure of the linear sublattice $C(X)$ in B'' is a closed sublattice of B'' which means that B is itself a Banach lattice.

At this point it seems worth pointing out that while we do have an obvious inclusion $C(X) \subset Z(B)$ it is not in general true that $C(X) = Z(B)$. We have in fact

Proposition 2.2. *Let X, B and u be as in Proposition 2.1 and let (B, u) be the unital Banach lattice obtained from B when ordered as in proposition (2.1). The unit ball $U(Z(B))$ of $Z(B)$ is then lattice isomorphic to the order interval $[-u, u]$ in B and may be defined as the completion of the unit ball of $C(X)$ for the metric given by $\| \ \|$, (or equivalently as the closure of $U(C(X))$ in B).*

Proof. Obvious from the Wils theorem.

2. So far we have proved that every unital Banach lattice can be represented as the completion of a $C(X)$-space for some modulenorm on $C(X)$. We shall next prove that this completion can be represented by functions on X. Towards this we shall need

Definition 2.3. Let X be a compact space, let (B, u) be a cyclic $C(X)$-module with dual space $B' \subset M(X)$. We shall say that a subset E of X is a *B'-null set* if for every $\varepsilon > 0$ there exist an open set 0_ε, such that $E \subset 0_\varepsilon$, and such that for every positive measure μ in the unit ball of B'

$$\mu(0_\varepsilon) < \varepsilon.$$

It is easily seen that the family of all B'-null sets is a σ-ideal of subsets of X. This σ-ideal will be denoted $\mathcal{N}_{B'}$ and we shall say that a property holds B'-almost everywhere if it holds outside a B'-null set.

Besides the concept of a B'-null set we shall need the concept of a B'-measurable function as given by

Definition 2.4. Let $X, B\ u$ and $B' \subset M(X)$ be as in definition (2.3). We shall say that a function f defined B'-almost everywhere on X is *Lusin B'-measurable* if for every $\varepsilon > 0$, there exists an open set 0_ε which is of measure less than ε for all positive μ in $U(B')$, and a function $g \in C(X)$ such that $f = g$ outside 0_ε.

In terms of the preceding definitions we can now state and prove

Theorem 1. *Let X, B, u and $B' \subset M(X)$ be as in definition (2.3). The elements of the space B can then be represented by Lusin B'-measurable functions. Two such functions represent the same element of B if they are equal B'-almost everywhere. Furthermore, the element u is represented by the function 1 and the duality between B and B' is given by*

$$\langle b, b' \rangle = \int_X b(x)\, db'(x).$$

Proof. Let $b \in B$. By assumption there exists then a sequence $\{f_k\}_{k=1}^{\infty}$, $f_k \in C(X)$, such that $\|f_k - b\| \to 0$. Choosing if necessary a subsequence we may assume that

$$\|f_k - f_{k-1}\|_B < 4^{-k}.$$

Defining now $g_0 = f_0$ and $g_k = f_k - f_{k-1}$ for $k \geqq 1$ we have

$$b = \sum_{k=0}^{\infty} g_k$$

and

$$\|g_k\|_B < 4^{-k}.$$

We now define

$$S_n = \sum_{k=0}^{n} |g_k|$$

and we obtain an increasing sequence of continuous functions. For every μ in $U(B')^+$ we now have

$$\langle S_n, \mu \rangle < \|g_0\| + \sum_{k=1}^{\infty} 4^{-k}$$

and this means that the sequence S_n is B'-almost everywhere convergent to a finite limit.

It follows that the series $\sum g_k$ is B'-almost everywhere absolutely convergent to a limit function $b(x)$. We shall prove that the function $b(x)$ is Lusin B'-measurable. To do this we shall use the truncating function $a\colon \mathbf{R} \to \mathbf{R}$ defined by

$$a(t) = \begin{cases} t & \text{if } \ |t| \leqq 1 \\ \operatorname{sign}(t) & \text{if } \ |t| \geqq 1. \end{cases}$$

We put namely

$$h_k = 2^{-k} \cdot a(2^k \cdot g_k).$$

Let now $\varepsilon > 0$ be given and choose N such that $2^{-N} < \varepsilon$. We define

$$g_\varepsilon = \sum_{k=0}^{N} g_k + \sum_{k=N+1}^{\infty} h_k.$$

Since all g_k and all h_k are continuous and since $\|h_k\|_\infty < 2^{-k}$ it follows that the series is uniformly convergent so that g_ε is continuous. We put

$$0_k = \{x \mid g_k(x) - h_k(x) \neq 0\} = \{x \mid |g_k(x)| > 2^{-k}\}.$$

It is clear that $g_\varepsilon = b$ outside $0_\varepsilon = \bigcup_{N+1}^{\infty} 0_k$. Now let $\mu \in \bigcup (B')^+$.

We then have

$$\int_X |g_k(x)| \, d\mu(x) = \langle |g_k|, \mu \rangle \leqq \| |g_k| \| = \|g_k\| < 4^{-k}.$$

By Tchebycheff's inequality $(2^{-k} \cdot \mu(0_k) < \int |g_k| \, d\mu < 4^{-k})$ we then have $\mu(0_\varepsilon) \leqq \sum \mu(0_k) < \sum_{k=N+1}^{\infty} 2^{-k} < \varepsilon$, and this proves that $b(x)$ is Lusin B'-measurable. That u is represented by 1 is obvious and the integral relation follows from Lebesgue's theorem of dominated convergence.

3. So far we have proved that cyclic $C(X)$-modules are unital Banach lattices and can be represented by measurable functions but we have not considered any relations between properties of X and properties of B. We shall in the following prove that order completeness properties of B are related to connectedness properties (or rather disconnectedness properties) of X and that separability of B is related to metrizability of X, i.e. to separability of $C(X)$. We shall begin by considering the Wils representation of B and we then have

Proposition 2.5. *Let (B, u) be a unital Banach lattice with ideal center $Z(B)$ and Wils space $W(B)$.*
Then

(i) *If B is countably order complete, then so is $Z(B)$ and $W(B)$ is quasi-stonian.*
(ii) *If B is order complete, then so is $Z(B)$ and $W(B)$ is stonian.*
(iii) *If B is a dual space, then so is $Z(B)$ and $W(B)$ is hyperstonian.*
(iv) *If $Z(B)$ is separable then $Z(B)$ is lattice isomorphic to B.*

Proof. (i) and (ii) follow from the simple observation that if B is (countably) order complete then so is $m(u)$ and therefore also $Z(B)$ combined with well-known properties of the maximal ideal spaces of (countably) order complete AM-spaces (see e.g. [S, p. 107]). (iii) follows from Wils' theorem and the definition of a hyperstonian space as the maximal ideal space of a dual $C(X)$-space. To prove (iv) we assume that the norm of B is not equivalent to the uniform norm in $Z(B)$. This implies then that there exist functions $f_k \in Z(B)$ $k=1, 2, \dots,$ such that $f_k \geqq 0$, $\|f_k\|_B \leqq 2^{-k}$, while $\|f_k\|_\infty \geqq k$. We define next $w = u + \sum f_k$. We consider then the linear ideal $m(w)$ with its intrinsic AM-space norm as a Banach lattice in se. Since we have $\|w\|_B \leqq 2$, it is easy to see that for $f \in C(X)$,

$$\|f\|_B \leqq 2 \cdot \|f\|_{m(w)} \leqq 2 \cdot \|f\|_\infty. \tag{2.1.1}$$

Using Proposition 2.2 we can now determine $Z(m(w))$ as follows. Let 0 be the open set where $w < \infty$, then $Z(m(w)) \approx C_b(0)$ (the ring of all bounded continuous functions on 0. It is well-known that $C_b(0)$ is isomorphic to $C(\beta(0))$, where $\beta(0)$ is the Stone—Čech compactification of 0. It follows now from proposition (2.2) and the inequalities (2.1.1) that $Z(B) = C(W(B)) \subset Z(m(w)) = C(\beta(0)) \subset Z(B)$, and hence $W(B)$ is homeomorphic to $\beta(0)$ and since $\beta(0)$ is non-metrizable, $Z(B)$ is non-separable.

As we have just seen the Wils representation of a Banach lattice will usually represent it as a module over a non-separable $C(X)$-space. On the other hand it is clear that any cyclic $C(X)$-module over a separable $C(X)$-space is itself separable and we shall presently see that every separable unital Banach lattice can be represented as a cyclic $C(X)$-module over a separable $C(X)$. We state this as

Theorem 2. *A unital Banach lattice (B, u) is separable iff there exists a metrizable compact space M such that (B, u) is represented as a cyclic $C(M)$-module. If B is countably order complete then M can be chosen totally disconnected.*

Proof. Since the if part is trivial we let (B, u) be a given separable unital Banach lattice and we shall construct a metrizable space M satisfying the assertions of the proposition. To do this we first choose a dense (for the norm of B) countable subset $\{\varphi_k\}$ of the order interval $[0, u]$. Considering the φ_k as elements of $Z(B)$, that is as elements of $C(X)$, we may consider

$$\Phi = (\varphi_1, \varphi_2, \ldots)$$

as a continuous map of X into the Tychonov cube $[0, 1]^{+\infty}$. We put

$$M = \mathrm{Im}\,(\Phi)$$

and we see that M is compact and metrizable. Since we have a natural homomorphism: $\Phi: C(M) \to C(X)$ B is clearly a $C(M)$-module. Furthermore, M is constructed as a subset of $[0, 1]^{+\infty}$ so in $C(M)$ we have in particular the coordinate functions t_k, and since $\Phi^*(t_k) = t_k \circ \Phi = \varphi_k$ we see that

$$\Phi^*(C(M)) \cdot u$$

is dense in B, i.e. B is a cyclic $C(M)$-module. If B is countably order complete, then the finite linear combinations of idempotents are dense in $Z(B)$, and choosing a sequence j_k of idempotents we obtain a map of X into the Cantor space $\{0, 1\}^\infty$ with the same properties.

§ 3. Dual pairs

1. In the preceding chapter we obtained representations of unital Banach lattices as $C(X)$-modules and as 'measurable' functions on various compact spaces proving in particular that a unital Banach lattice is separable iff it can be represented as a $C(M)$-module for a separable $C(M)$-space. In this chapter we shall make an additional assumption on the lattices under consideration, namely that the pair (B, B') has a dual pair. Since every separable lattice is unital it follows from proposition (1.5) that a separable reflexive lattice may without essential loss of generality be assumed to have a dual pair. We shall see that this assumption leads to a very precise and well-known description of the results obtained in the previous chapter. We shall begin by proving

Theorem 3. *Let (B, u) be a unital Banach lattice represented as a cyclic $C(X)$-module and suppose (u, u') is a dual pair for (B, B'). Let further $\mu \in M(X)$ be the measure representing u'. Then μ is a probability measure and all measures in B'*

are absolutely continuous with respect to it, so taking Radon—Nikodym derivatives we may represent B' by (equivalence classes of) μ-measurable functions on X. Furthermore B'-measurability is equivalent to μ-measurability so also B is represented by (equivalence classes of) μ-measurable functions on X. We have thus natural inclusions

$$C(X)\subset B\subset L^1(X,\mu), \quad L^\infty(X,\mu)\subset B'\subset L^1(X,\mu)$$

and these inclusions are 'adjoints of each other' so that if $b(=b(x))\in B$ and $b'\in B'$ then $b'\in L^1(\mu)$ and

$$\langle b, b'\rangle = \int_X b(x)b'(x)\,d\mu(x).$$

Proof. The first assertion is that μ is a probability measure and this follows from the fact that u' is positive so that μ is also, and that

$$1 = \langle u, u'\rangle = \int_X 1\,d\mu.$$

To see that all measures in B' are absolutely continuous with respect to μ we take $b'\in(B')^+$, and we put $b'_n=\inf(b', n\cdot u')$. Since u' is an F-unit the sequence b'_n order converges to b'. Now the measures b'_n increase to the measure b', and since all the b'_n are absolutely continuous with respect to μ, so is (e.g. by the Vitali—Hahn—Saks theorem) b'. As asserted in the theorem we may therefore identify B' with a linear ideal in $L^1(X,\mu)$ and using Radon—Nikodym derivatives the elements of B' are represented by equivalence classes of μ-measurable functions on X. It is worth observing that u' is then represented by the function 1. We next observe that the unit ball of B' is a weakly compact subset of $L^1(\mu)$ which is therefore uniformly integrable and this implies that every μ-null set is a B'-null set and since the converse holds a priori this proves that the σ-ideal of B'-null sets coincides with the σ-ideal of μ-null sets. Therefore B'-measurability and μ-measurability are also equivalent properties.

The natural representation of B as μ-measurable functions on X gives us the inclusions $C(X)\subset B\subset L^1(X,\mu)$, and the adjoints of these inclusions give us maps

$$L^\infty(X,\mu)\xrightarrow{\subset'} B'\xrightarrow{\subset'} M(X)$$

but as we saw above the range of the second map is contained in $L^1(\mu)$. Furthermore, $C(X)$ is dense in B and B is dense in L^1 and therefore the adjoint maps are injective and may be written as inclusions. Finally, the assertion about the duality given by integration is 'built into the construction'.

2. We shall presently see that the assumption of the existence of a dual pair is really a very strong condition on a Banach lattice. A first result in this direction is

Theorem 4. *Let B, B', u, u', and X be as in Theorem 3 and let $P \in M(X)$ be the probability measure representing u'. Then B is separable iff $L^1(X, P)$ is separable.*

Proof. If B is separable, then we may by Theorem 2 assume X to be metrizable (otherwise we may replace X by a metrizable quotient space) and then L^1 as well as B is the completion of the separable space $C(X)$. Assume now that $L^1(X, P)$ is separable. Imitating the proof of theorem 2 we choose then a sequence $\{\varphi_k\}_{k=1} \subset [0, u]$ which is dense for the $L^1(X, P)$-norm in $[0, u]$ and we consider

$$\Phi = (\varphi_1, \varphi_2, \dots)$$

as a continuous map onto the compact metrizable space $M \subset [0, 1]^\infty$. Hence Φ^* maps $C(M)$ into B and the problem is to decide whether $\Phi^*(C(M))$ is dense in B. Now we may consider $(\Phi^*)'$ as a map from $M(X)$ onto $M(M)$, and we observe that by construction $(\Phi^*)' | L^1(P)$ is injective. But since $B' \subset L^1(P)$ this means that $(\Phi^*)'$ is injective on B' and this holds if and only if $\mathrm{Im}\,(\Phi^*)$ is dense in B and this proves the theorem.

An immediate consequence of Theorem 4 is

Corollary 3.1. *Let B, B', u, u', X and P be as in theorem 4, and suppose P is a discrete measure. Then B is separable.*

Proof. If P is discrete then P is concentrated on a countable set $D \subset X$, and then $L^1(X, P) \approx l^1(D)$ which is separable since D is countable.

In order to give a natural interpretation of the preceding corollary we shall say that a Banach lattice is discrete if the dual space consists of discrete measures. It follows from the corollary that if B is discrete, if B has a topological unit, and if B' has an F-unit, then B is separable. A consequence of the corollary is therefore that if the Banach lattice B is a sequence space then the dual space is also a sequence space, if and only if B is separable, i.e. if B is non-separable then B' contains 'singular functionals'. It is also worth remarking that a discrete Banach lattice need not be 'atomic' as defined in [S, p. 143]. The space c of all convergent sequences is probably the simplest counterexample.

Another consequence of Theorem 4 is

Corollary 3.2. *Let B, B', u, u', X and P be as in theorem 4, suppose P is a continuous measure, and suppose that B is separable. Then there exists a function $\Psi: X \to [0, 1]$ such that $\Psi(P)$ is Lebesgue measure, and hence B and B' can both be represented by equivalence classes of Lebesgue measurable functions on $(0, 1)$ and the duality is given by the Lebesgue integral.*

Furthermore, $Z(B) \approx Z(B') \approx L^\infty((0, 1), dx)$.

Conversely, if B is a Banach lattice of (equivalence classes of) Lebesgue measurable functions on $(0, 1)$ containing L^∞ then the dual space B'

can also be represented by Lebesgue measurable functions if and only if B is separable.

Proof. For the converse result it suffices to observe that if B' can be represented by measurable functions, then the function 1 is an F-unit for B' as well as for B. We let $T: C[0,1] \to B$ now be the natural imbedding, and we observe that the adjoint map $T': B' \to M[0,1]$ is by assumption injective, and this proves the assertion. (*Remark.* In a more general setting the same argument proves that the seemingly more general definition of a dual pair to be a pair of F-units for B and B' is in fact equivalent to our definition.)

Let now B, B', u etc be as above with P continuous and B separable. We shall begin by determining the ideal center $Z(B)$. Towards this we observe that by proposition (2.2) $U(Z(B))$ is the closure of $U(C(X))$ in B and since $U(C(X))$ is convex the norm closure and the weak closure coincide. Since furthermore $B' \subset L^1(X, P)$, the weak topology from B' is weaker than the weak topology from L^1, but in fact on the unit ball of $C(X)$ even the weak topology from $C(X)$ (as a subset of L^1) is just as strong. Hence all these weak topologies coincide, and since the weak closure of $U(C(X))$ in $L^1(X, P)$ is $U(L^\infty(X, P))$ we have $Z(B) \approx L^\infty(X, P)$. We let I now be the set of all idempotents in L, and we let $\{j_k\}_{k=1}$ be a sequence in I which is dense in I for the B-norm. It follows that the finite linear combinations of the j_k's are dense in B.

Defining now

$$\theta: X \to [0,1]$$

by

$$\theta(x) = 2 \cdot \sum_{k=1}^{\infty} 3^{-k} \cdot j_k(x)$$

we get a measurable function from X into the Cantor set $\mathbf{D}$. It also follows from the construction that $C(\mathbf{D})$ is dense in B. Furthermore $\theta(P)$ is a continuous measure on $\mathbf{D}$, and defining $F(t) = \int_0^t d(\theta(P))(u)$ we have a continuous map of $\mathbf{D}$ onto $[0,1]$ such that $F(\theta(P))$ is Lebesgue measure on $[0,1]$, so we define thus $\Psi = F \circ \theta$. The remaining properties asserted follow from the construction and theorem 3.

3. So far we have considered lattices with dual pairs under the assumption that the dual unit is either discrete or continuous. However, it follows from the following proposition that the general case can be reduced to a sum of the preceding cases.

Proposition 3.3. *Let B be a Banach lattice and suppose (u, u') is a dual pair for (B, B'). Then B can be decomposed as a sum of two projection bands B_c and B_d where B_c is the continuous part of B and B_d is the 'discrete' part.*

Proof. We represent (B, u) as a cyclic $C(X)$-module and we let P be the measure representing u'. Writing

$$P = \mu_d + \mu_c,$$

where μ_d is the discrete part and μ_c the continuous part of P we may write the elements of B' uniquely as a sum of a discrete measure (absolutely continuous with respect to μ_d) and a continuous measure absolutely continuous with respect to μ_c. This decomposition is in fact a decomposition of B' into the sum of two projection bands naturally denoted B'_c and B'_d. Writing B_c for the annihilator $(B'_d)^0$ and B_d for the annihilator $(B'_c)^0$ it is obvious that $B_c \cap B_d = 0$ and that B_c and B_d are bands in B. It is however not a priori clear that $B_c + B_d = B$. This is in fact true (as is easily proved by standard arguments of functional analysis) if and only if both B'_c and B'_d are weak* closed. It follows from the Krein—Šmulian theorem that this holds if the unit balls are both weak* compact. Now the unit ball of B' is a weakly compact subset of $L^1(P)$ and is therefore uniformly integrable. Consequently, the discrete parts as well as the continuous parts are uniformly integrable with respect to P, but then the discrete parts are also uniformly integrable with respect to μ_d, while the continuous parts are uniformly integrable with respect to μ_c. Therefore, $U(B'_c)$ is relatively weakly compact in $L^1(\mu)$. But then the weak closures are still contained in the proper L^1-spaces and this means that e.g. the weak* closure of $U(B'_d)$ is contained in B'_d and this implies the weak* closure of B'_d. By the same argument the same holds for B'_c and this proves the proposition.

4. Using the preceding results we obtain the following representation for separable reflexive Banach lattices.

Theorem 5. *Let B be a separable reflexive Banach lattice. There exist then an atomic Banach lattice B_a, a dense reflexive linear ideal B_c in $L^1((0, 1), dx)$, and a contracting linear lattice isomorphism $B\colon B \to B_a \prod B_c$ (the product of B_a and B_c) such that* $\|\theta^{-1}\| \leqq 2$.

Proof. Since both B and B' are separable they both have units. By proposition (1.5) we may after a slight change of norm assume the existence of a dual pair. By proposition (3.3) we can then decompose B as a sum of B_c and a discrete lattice B_d. Since both of these are separable we apply corollary (3.2) to the lattice B_c and corollary (3.1) to B_d. However if B_d is reflexive then it must also be atomic (in fact it is easily proved that otherwise B_d would contain the non-reflexive lattice c (of convergent sequences) as a sublattice. This proves the existence of the lattice isomorphism θ. That θ is contracting follows from the fact that band projections are contractions, and from properties of the product norm. That θ^{-1} has norm $\leqq 2$, follows most easily from the fact that θ^{-1} factors over $B_a \coprod B_c$, and is a contraction from that space, while the imbedding of $B_a \prod B_c$ into $B_a \coprod B_c$ has norm 2.

For arbitrary reflexive lattices we have a similar theorem though necessarily less precise as follows.

Theorem 6. *Let B be a reflexive Banach lattice. There exist then complementary projection bands B_a and B_c such that B_a is atomic and B_c is non-atomic. Furthermore we have the following descriptions of B_a and B_c.*

(i) *There exists a set A (the atoms of B) and lattice homomorphisms $l^1(A)\subset B_a\subset c_0(A)$ and by transposition $l^1(A)\subset B'_a\subset c_0(A)$.*

(ii) *There exists a set Γ, and Banach lattices B_γ, $\gamma\in\Gamma$, such that each B_γ is a projection band in B_c, and*

$$\coprod_{\gamma\in\Gamma} B_\gamma\subset B_c\subset \prod_{\gamma\in\Gamma} B_\gamma \quad \textit{(and likewise } \coprod_{\gamma\in\Gamma} B'_\gamma\subset B'_c\subset \prod_{\gamma\in\Gamma} B'_\gamma),$$

and each B_γ has a representation as P_γ-measurable functions on a compact space X_γ, such that

$$L^\infty(X_\gamma, P_\gamma)\subset B_\gamma\subset L^1(X_\gamma, P_\gamma) \quad\text{and}\quad L^\infty(X_\gamma, P_\gamma)\subset B'\subset L^1(X_\gamma, P_\gamma).$$

Finally the ideal center $Z(B)$ of B is the product $Z(B_a)\prod Z(B_c)$ of the ideal centers of each of the projection bands, and $Z(B_a)\approx l^\infty(A)$, while $Z(B_c)$ is the dual space to the AL-space $L^1(\coprod X_\gamma, \sum P_\gamma)$

Proof. We start by writing B_c for the band generated by all atoms in B. Since B is reflexive this band is a projection band. There is then trivially a lattice homomorphism

$$l^1(A)\subset B_a.$$

If now B_a was not contained in $c_0(A)$, then B_a would contain the characteristic function of some infinite subset $E\subset A$, and then "over E" the B-norm and the sup norm would be equivalent, and this contradicts the reflexivity of B. That the ideal center of B_a is $l^\infty(A)$ is obvious (given $z\in Z(B_a)$, define $\{f_\alpha\}\in l^\infty(A)$ by $f_\alpha=z(\alpha)$, $\alpha\in A$, (we identify A with the set of normalized atoms) and given $\{f_\alpha\}\in l^\infty(A)$ we can define $z\in Z(B_a)$ by $z(\alpha)=f_\alpha\cdot\alpha$).

For the asserted representation of B_c we start by choosing a maximal orthogonal system $\{u_\gamma\}_{\gamma\in\Gamma}$ in B_c. We write then B_γ to denote the band generated by u_γ, Since B is reflexive each B_γ is a projection band. Furthermore, each u_γ is by construction an F-unit in B_γ, but since B_γ is reflexive, every F-unit is in fact a t-unit (this follows e.g. from [S, p. 91, Th (5.11).]). It follows then that also B'_γ has a t-unit so that B_γ, B'_γ may be assumed to have a dual pair u_γ, u'_γ. Representing now B_γ as a cyclic $C(X_\gamma)$-module and denoting the measure in $M(X_\gamma)$ representing u'_γ by P_γ, we get the asserted representation of B. The inclusions $\coprod B_\gamma\subset B_c\subset\prod B_\gamma$ are trivial consequences of the universal properties of "sums" and "products". Finally, for every B_γ, we have $Z(B_\gamma)\approx L^\infty(X_\gamma, P_\gamma)$, and then $Z(B_c)\approx\prod Z(B_\gamma)\approx\prod L^\infty(X_\gamma, P_\gamma)\approx (L^1(\coprod X_\gamma, \sum P_\gamma))'$. This proves the theorem.

§ 4. Concluding remarks

1. This paper grew out of an attempt to rewrite in a more legible form some of the results proved in [4]. In that paper I define a "local operator" on a Banach lattice to be an operator T such that $x \perp y \Rightarrow Tx \perp y$. One of the results of that paper is that the set of all local operators is in fact the ideal center of the lattice. The main result obtained in [4] is however a very general version of the following

Theorem 0. *Let (B, u) be a cyclic $C(X)$-module. Then the "module tensor product"*

$$B \underset{C(X)}{\otimes} B'$$

is an AL-space in the sense of Kakutani, and its dual space is the ideal center of B'.

Using theorem 0 I then consider various aspects of the relations between Banach lattices and $C(X)$-modules, in particular the result that every separable Banach lattice is a cyclic $C(M)$-module is proved in [4]. The measurability results of this paper proved by using Tchebycheff's inequality are however not proved in [4]. Some consequences of the relations between lattices and modules that I have not studied either in [4], or in the present paper are connected to interpolation of Banach lattices and constructions of lattices out of say "partially ordered function spaces".

I intend to rewrite the more algebraic results proved in [4] in connection with a more general study of Banach modules.

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Sten Kaijser
Department of Mathematics
Uppsala Universitet
Sysslomansg. 8
S—752 23 Uppsala
Sweden