Hardy classes on plane domains

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1. Introduction

This is an attempt to classify plane domains by means of Hardy classes. Let φ be a convex function, by which we mean throughout this paper any nonconstant, nonnegative, nondecreasing, convex function defined on $[-\infty, +\infty)$. For a connected domain D in the Riemann sphere S we shall denote by $H_\varphi(D)$ the set of functions f analytic on D for which $\varphi(\log|f(z)|)$ has a harmonic majorant on D. We shall denote by $\mathcal{O}_\varphi$ the class of connected domains D in S for which $H_\varphi(D)$ contains only constant functions. We are going to show the following

Theorem 1.1. *Let φ and ψ be convex functions with the following properties:*

(A) *for any fixed* $s > o$

$$\psi(t)/\varphi(t-s) = o(1), \quad t \to +\infty;$$

(B) $t/\varphi(t) = o(1)$, $t \to +\infty$.

Then $\mathcal{O}_\varphi$ strictly includes $\mathcal{O}_\psi$.

When $\varphi(t)=e^{pt}$ with $0<p<+\infty$, we use $H^p(D)$ (resp., $\mathcal{O}_p$) in place of $H_\varphi(D)$ (resp., $\mathcal{O}_\varphi$). We denote by $\mathcal{O}_G$ (resp., $\mathcal{O}_{AB}$) the class of connected domains D in S on which there exist no Green functions (resp., nonconstant, bounded, analytic functions) and by $\mathcal{O}_{AB^*}$ the class $\mathcal{O}_\varphi$ with $\varphi(t)=\max\{t, 0\}$. The preceding theorem then implies the following, in which the inequality sign $<$ means strict inclusion.

Corollary 1.2. (a) $\mathcal{O}_p^- < \mathcal{O}_p < \mathcal{O}_p^+$ *for* $0<p<+\infty$, *where* $\mathcal{O}_p^- = \cup\{\mathcal{O}_q\colon 0<q<p\}$ *and* $\mathcal{O}_p^+ = \cap\{\mathcal{O}_q\colon p<q<+\infty\}$.

(b) $\mathcal{O}_{AB^*} < \cap\{\mathcal{O}_q\colon 0<q<+\infty\}$, $\cup\{\mathcal{O}_q\colon 0<q<+\infty\} < \mathcal{O}_{AB}$.

Together with the known fact $\mathcal{O}_G = \mathcal{O}_{AB^*}$ (cf., Sario and Nakai [7, p. 280 and p. 332]), this finishes a classification of plane domains in the sense of Heins given in his monograph [2, Chapter III].

We explain earlier results briefly. Heins [2] obtained a classification scheme for general Riemann surfaces in terms of Hardy classes. For the plane domains, however, he only showed $\mathcal{O}_{AB^*}<\mathcal{O}_1$ and proposed further investigation. Hejhal [3, 4] proved the following scheme:

$$\mathcal{O}_G = \mathcal{O}_{AB^*} \leqq \mathcal{O}_1^- < \mathcal{O}_1 \leqq \mathcal{O}_{3/2}^- < \mathcal{O}_{3/2} \leqq \mathcal{O}_2^- < \mathcal{O}_2 \leqq \mathcal{O}_{5/2}^-$$

$$< \mathcal{O}_{5/2} \leqq \mathcal{O}_3^- < \mathcal{O}_3 \leqq \ldots < \cup \{\mathcal{O}_q\colon 0 < q < +\infty\} < \mathcal{O}_{AB}.$$

Kobayashi [5] improved this by showing $\mathcal{O}_{n/2}<\mathcal{O}_p$ for any integer $n\geqq 2$ and any real number $p>n/2$.

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2. Null sets of class N_φ.

For a convex function φ we say that a set E in S is a null set of class N_φ if E is a bounded, closed, totally disconnected set and $H_\varphi(V-E)=H_\varphi(V)$ for every connected domain V in S which contains E. When $\varphi(t)=e^{pt}$, $0<p<+\infty$, we write N_p in place of N_φ. A similar definition holds for the null classes N_{AB} and N_{AB^*}. As observed by Hejhal [4], every member of N_{AB^*} has zero logarithmic capacity. Heins [2, pp. 50—51] showed that a closed linear set belongs to N_1 if and only if it has zero linear measure. This fact, in a slightly extended form, was used extensively by Hejhal in his classification theorems. Various null classes N_φ will play an important role in this paper, too. As far as we are aware, however, not much seems to be known about general null classes, e.g., $N_{1/2}$. So we should begin with showing that the null class N_φ contains sufficiently many members with nonzero logarithmic capacity as soon as the function φ satisfies the condition (B). Since $\psi(t)/\varphi(t)=O(1)$, $t\to+\infty$, implies $N_\psi\subseteqq N_\varphi$, we have only to investigate null classes N_φ for small φ.

Lemma 2.1. *Let φ be a convex function satisfying* (B). *Then there exists a convex function λ such that*

(i) $\lambda(t+\log 2)\leqq 2\lambda(t)$ *for all large t;*

(ii) $\lambda(t)$ *satisfies* (B)*;*

(iii) $\lambda(t)/\varphi(t)=o(1)$, $t\to+\infty$, *and*

$\lambda(t)/t^2=o(1)$, $t\to+\infty$.

This can be proved by a simple construction, which is omitted. The following proposition shows the importance of the hypothesis $\varphi(t+\log 2)\leqq 2\varphi(t)$ or, more generally,

$$\varphi(t+\log 2)/\varphi(t) = O(1), \quad t\to+\infty,$$

which is equivalent to the Δ_2-condition in the theory of Orlicz spaces (cf., Krasnosel'skii and Rutickii [6]).

Proposition 2.2. *Let φ be a convex function such that $t/\varphi(t)=o(1)$ and $\varphi(t+\log 2)/\varphi(t)=O(1)$, $t\to+\infty$, and let E be a bounded, closed, totally disconnected set in S. Then $E\in N_\varphi$ if and only if $S-E\in\mathcal{O}_\varphi$.*

Proof. The necessity is trivial and the sufficiency can be shown by essentially the same method as that of [4, Theorems 4 and 5]. The details are left to the reader.

Before proceeding to actual construction of null sets of the class N_φ having nonzero logarithmic capacity, we state two probably well-known facts concerning harmonic measures. When we write, for example, $\{|z|>a\}$, we include the point at infinity ∞ in the set. We denote by $\Gamma(z_0; r_0)$, $|z_0|<+\infty$ and $0<r_0<+\infty$, the circumference $\{|z-z_0|=r_0\}$. When $z_0=0$, we write simply $\Gamma(r_0)$.

Lemma 2.3. *Let $0<a<b<+\infty$ and let F be a bounded closed set in $\{|z|\geqq b\}$ such that $D_0=\{|z|>a\}-F$ is a connected domain. Denote by μ the harmonic measure at the point ∞ with respect to D_0. Then*

$$\max\left\{\frac{ds}{d\mu}(z)\colon\ z\in\Gamma(a)\right\}\leqq A(a/b)\frac{2\pi a}{\mu(\Gamma(a))},$$

where ds denotes the arc-length element on $\Gamma(a)$ and $A(t)$, $t>0$, is a finite, positive, nondecreasing function in t.

Proof. Let $D_1=\{a<|z|<b\}$, $D_2=\{|z|>a\}$ and $c=(ab)^{1/2}$. By Harnack's inequality there exists a constant $A'=A'(a/b)$, depending only on the ratio a/b in a nondecreasing way, such that $U(z_1)\leqq A'U(z_2)$ for any z_1, z_2 in $\Gamma(c)$ and any nonnegative harmonic function U on D_1. For any arc e on $\Gamma(a)$ we denote by $U_j(e; z)$, $0\leqq j\leqq 2$, the bounded harmonic function on D_j whose boundary values are equal to 1 on e and to 0 elsewhere. We then have

$$U_1(e;z)\leqq U_0(e;z)\leqq U_2(e;z),\quad a<|z|<b \tag{1}$$

and

$$\max\{U_2(e;w)\colon |w|=c\}\leqq A\min\{U_1(e;w)\colon |w|=c\}, \tag{2}$$

where

$$A = A(a/b) = 2A'(a/b)\frac{1+(a/b)^{1/2}}{1-(a/b)^{1/2}}.$$

From (1) and (2) it follows at once that

$$\max\{U_0(e_1; w): |w| = c\} \leqq A \min\{U_0(e_2; w): |w| = c\}$$

for any arcs e_1, e_2 on $\Gamma(c)$ with the same length. This shows that $\mu(e_1) \leqq A\mu(e_2)$ if $|e_1| = |e_2|$. The desired inequality is an easy consequence of this. Q.E.D.

Lemma 2.4. *Let* $0 < a^2/b < a_0 < a < b < +\infty$ *and* $F_1, \dots, F_k$ *a finite number of bounded closed subsets of* $\{|z| > b\}$ *such that each* $S - F_i$, $1 \leqq i \leqq k$, *is a connected domain containing all* F_j, $j \neq i$. *Let* $\{l(n): n \geqq 1\}$ *be an increasing sequence of positive numbers such that* $l(n)/n = o(1)$, $n \to +\infty$ *and* $l(n) \geqq n^{1/2}$ *for all large n. Let* $w_{n,j} = a_0 \exp(2\pi ji/n)$ *and* $K_{n,j} = \{|z - w_{n,j}| \leqq a_0 e^{-l(n)}\}$, $1 \leqq j \leqq n$. *We set* $K_n = \bigcup_{j=1}^{n} K_{n,j}$. *Suppose that each* F_j, $1 \leqq j \leqq k$, *has nonvanishing logarithmic capacity and that we consider only large n so that* K_n *is contained in the disk* $\{|z| < a\}$ *and* $K_{n,j}$*'s,* $1 \leqq j \leqq n$, *are mutually disjoint. Let* μ *and* μ_n *be the harmonic measures at the point* ∞ *with respect to the domains* $\{|z| > a_0\} - F$ *and* $S - K_n - F$, *respectively, with* $F = \bigcup_{j=1}^{k} F_j$. *Then, for any* $\varepsilon > 0$, *there exists an integer* $N = N(\varepsilon) > 0$ *such that for* $n \geqq N$

$$|\mu(\Gamma(a_0)) - \mu_n(\partial K_n)| < \varepsilon,$$

$$|\mu(F_j) - \mu_n(F_j)| < \varepsilon, \quad 1 \leqq j \leqq k,$$

and

$$\mu_n(\partial K_{n,j}) \geqq (Bn)^{-1}\mu(\Gamma(a_0)), \quad 1 \leqq j \leqq n,$$

where $B = B(a/b)$ *is a constant depending only on the ratio* a/b *in a nondecreasing fashion.*

Proof. By applying the conformal transformation $z \to z/b$, we may assume without loss of generality that $b = 1$. Given any $\varepsilon > 0$, we first choose $\varepsilon' > 0$ with

$$\max\{(1+\varepsilon') - (1+\varepsilon')^{-3}, \varepsilon'\} < 2^{-1}\mu(\Gamma(a_0))^{-1}\varepsilon \tag{3}$$

and then choose $0 < a' < a_0 < a'' < a$ so close to a_0 as to have

$$(1+\varepsilon')^{-1}\mu(\Gamma(a_0)) < \mu'(\Gamma(a')) < \mu''(\Gamma(a'')) < (1+\varepsilon')\mu(\Gamma(a_0)), \tag{4}$$

where μ' (resp., μ'') denotes the harmonic measure at the point ∞ with respect to the domain $\{|z| > a'\} - F$ (resp., $\{|z| > a''\} - F$). Let $G(z, w) = \log(|1 - z\bar{w}|/|z - w|)$ denote the Green function for the unit disk with pole at w. We denote by ν_n the positive measure of mass 1 with support on ∂K_n and uniform density on ∂K_n, and by $U_n(z)$ the Green potential of ν_n, i.e.,

$$U_n(z) = \int_{\partial K_n} G(z, w)\, d\nu_n(w), \quad |z| \leqq 1.$$

$U_n(z)$ is continuous on $\{|z| \leqq 1\}$, harmonic on $\{|z| < 1\} - K_n$, and vanishes on $\Gamma(1)$. An explicit computation, in view of the property of $\{l(n)\}$, shows that (i) $U_n(z)$

converges uniformly on the circle $\Gamma(a')$ to $\log(1/a_0)$ and (ii) there exists an integer $N'=N'(\varepsilon')>0$ such that

$$(1+\varepsilon')^{-1}\log\frac{1}{a_0} < U_n(z) < (1+\varepsilon')\log\frac{1}{a_0} \quad \text{on} \quad K_n,\ n \geqq N'.$$

Let u' (resp., u_n) be the solution of the Dirichlet problem for the domain $\{|z|>a'\}-F$ (resp., $S-K_n-F$) with the boundary data equal to 1 on $\Gamma(a')$ (resp., ∂K_n) and to 0 elsewhere. Take $N''=N''(\varepsilon')$ so large that for $n\geqq N''$, K_n lies in the annulus $\{a'<|z|<a''\}$ and, by (i), $U_n(z)\geqq(1+\varepsilon')^{-1}\log(1/a_0)$ on $\Gamma(a')$. Suppose $n\geqq N(\varepsilon)=\max\{N', N''\}$. Then, by (ii),

$$u_n(z) = 1 \geqq (1+\varepsilon')^{-1}\left(\log\frac{1}{a_0}\right)^{-1} U_n(z)$$

on ∂K_n and therefore everywhere on $|z|\leqq 1$. So on $\Gamma(a')$ we have $u_n(z)\geqq(1+\varepsilon')^{-2}=(1+\varepsilon')^{-2}u'(z)$. Since $u_n(z)$ is superharmonic on $\{|z|>a'\}-F$ and has the same boundary value as $u'(z)$ on ∂F, we see that $u_n(z)\geqq(1+\varepsilon')^{-2}u'(z)$ everywhere on $\{|z|>a'\}-F$. This, together with (4), means in particular that

$$\mu_n(\partial K_n) = u_n(\infty) \geqq (1+\varepsilon')^{-2}\mu'(\Gamma(a')) \geqq (1+\varepsilon')^{-3}\mu(\Gamma(a_0)). \tag{5}$$

On the other hand, since K_n lies inside $\Gamma(a'')$, we have

$$\mu_n(\partial K_n) \leqq \mu''(\Gamma(a'')) \leqq (1+\varepsilon')\mu(\Gamma(a_0)). \tag{6}$$

Combining the inequalities (5), (6) and using (3), we get

$$|\mu(\Gamma(a_0))-\mu_n(\partial K_n)| < \varepsilon$$

and

$$|\mu(F_j)-\mu_n(F_j)| < \varepsilon, \quad 1 \leqq j \leqq k.$$

In order to show the last statement, we fix n so large that $a_0 e^{-l(n)}<a-a_0$. Let $c=a^{1/2}$. Let u_j (resp., u_{1j}, u_{2j}), $1\leqq j\leqq n$, be the solution of the Dirichlet problem for the domain $S-K_n-F$ (resp., $S-K_n$, $\{|z|<1\}-K_n$) with the boundary data equal to 1 on $\partial K_{n,j}$ and to 0 elsewhere. Because of symmetry, $m_i=\min\{u_{ij}(z)\colon |z|=c\}$ and $M_i=\max\{u_{ij}(z)\colon |z|=c\}$, $i=1, 2$, are independent of j and by Harnack's inequality $M_i\leqq A'(a)m_i$, $i=1, 2$, where the constant $A'=A'(a)$ already appeared in the proof of the preceding lemma. Since $\sum_{j=1}^n u_{1j}=1$ on $S-K_n$, we have $1\leqq nM_1\leqq A'nm_1\leqq A'$ and therefore $M_1\leqq A'/n$. As our observation in the preceding paragraph shows, there exists a positive integer N_1 such that

$$\sum_{j=1}^n u_{2j}(z) \geqq \frac{1}{2}\frac{\log a}{\log a_0} \geqq \frac{1}{4}, \quad z\in\Gamma(a), \quad n \geqq N_1.$$

We thus have $A'nm_2\geqq nM_2\geqq\sum_{j=1}^n u_{2j}(z)\geqq 1/8$, $z\in\Gamma(c)$, and so $A'm_2\geqq 1/8n\geqq$

$M_1/8A'$. Since we have $u_{2j}(z) \leqq u_j(z) \leqq u_{1j}(z)$, $1 \leqq j \leqq n$, on the annulus $\{a<|z|<1\}$, the above inequality shows

$$u_i(z) \geqq 8A'^2 u_j(z), \quad z \in \Gamma(c), \quad 1 \leqq i, j \leqq n.$$

The same inequality holds in $\{|z|>c\}$ so that

$$\mu_n(\partial K_{n,i}) = u_i(\infty) \leqq 8A'^2 u_j(\infty) = 8A'^2 \mu_n(\partial K_{n,j}), \quad 1 \leqq i, j \leqq n.$$

This implies $\mu_n(\partial K_{n,j}) \geqq (8A'^2 n)^{-1} \mu_n(\partial K_n)$. Since $\mu(\Gamma(a_0)) \neq 0$, we have $\mu_n(\partial K_n) \geqq \mu(\Gamma(a_0))/2$ for all large n. Hence we have $\mu_n(\partial K_{n,j}) \geqq B^{-1} n^{-1} \mu(\Gamma(a_0))$ with $B=B(a)=16A'^2$. Q.E.D.

We are now in a position to prove the main result of this section.

Theorem 2.5. *Let φ be a convex function which satisfies* (B) *and $\varphi(t+\log 2)/\varphi(t)=O(1)$, $t \to +\infty$. Let $0<a<b<+\infty$ and let F_j, $1 \leqq j \leqq k$, be a finite number of bounded closed subsets of $\{|z|>b\}$ such that each $S-F_i$ is a connected domain containing all other F_j, $j \neq i$. Then for any positive numbers $\varepsilon>0$ and $0<\delta<1$ there exists a set $E \in N_\varphi$ of nonzero logarithmic capacity such that $E \subseteq \{\delta a \leqq |z| \leqq a\}$,*

(7) $$|\mu(F_j) - \mu_E(F_j)| < \varepsilon, \quad 1 \leqq j \leqq k,$$

and

(8) $$|\mu(\Gamma(a)) - \mu_E(E)| < \varepsilon,$$

where μ (resp., μ_E) denotes the harmonic measure at the point ∞ with respect to the domain $\{|z|>a\}-F$ (resp., $S-E-F$) with $F=\bigcup_{j=1}^n F_j$.

Proof. By applying Lemma 2.1 to the function $t \to \varphi(t/2)$, we get a convex function $\lambda(t)$ such that (i) $t/\lambda(t)=o(1)$, $t \to +\infty$; (ii) $\lambda(t)/\varphi(t/2)=o(1)$, $t \to +\infty$; (iii) $\lambda(t) \leqq t^2$ for all large t. There exists a positive number t_0 such that both $\varphi(t)$ and $\lambda(t)$ are strictly increasing for $t \geqq t_0$. Let $t_1=\max\{\varphi(t_0), \lambda(t_0)\}$. Then the inverse functions $h(t)$ and $l(t)$ of $\varphi(t)$ and $\lambda(t)$, respectively, are uniquely determined as strictly increasing functions in t for $t \geqq t_1$. The properties (i)—(iii) imply the following: (i') $l(t)/t=o(1)$, $t \to +\infty$; (ii') for any $\varepsilon>0$ there exists a number $t(\varepsilon)$ ($\geqq t_1$) such that $h(t/\varepsilon) \leqq l(t)/2$, $t \geqq t_1$; (iii') $l(t) \geqq t^{1/2}$ for all large t. So the sequence $\{l(n): n \geqq t_1\}$ satisfies the conditions in Lemma 2.4. In order to construct a set E with the desired property, we assume without loss of generality that F has nonvanishing logarithmic capacity. We may assume also that ε is small, i.e., $0<\varepsilon<(1-\mu(F))/\mu(F)$. We set $\varrho=(b/a)^{1/4}$ and $B=B(a/b)$, the constant appearing in Lemma 2.4.

By induction we construct families $\mathscr{K}_n$, $\mathscr{K}'_n$, $n=0, 1, \ldots,$ of closed disks contained in $\{|z| \leqq a\}$. Each $\mathscr{K}_n$ (resp., $\mathscr{K}'_n$) consists of a finite number of mutually disjoint, closed disks of the same radius r_n (resp., r'_n) in $\{|z| \leqq a\}$, whose union is denoted by K_n (resp., K_n'). By μ_n (resp., μ_n') we mean the harmonic measure at the point ∞ with respect to the domain $S-K_n-F$ (resp., $S-K_n'-F$). As the 0-th step of our induction, we define $\mathcal{K}_0$ to be empty and $\mathcal{K}_0'$ to consist of only one member $\{|z|\leqq a\}$, so that $\mu_0'=\mu$ and $r_0'=a$. Suppose that we have finished the n-th step with $n\geqq 0$. Namely, we have constructed $\mathcal{K}_n'$ in such a way that $\mathcal{K}_n'$ consists of closed disks D_α', $1\leqq\alpha\leqq N(n)$, of centers w_α and of common radius r_n' so that the disks $\{|z-w_\alpha|\leqq\varrho^4 r_n'\}$ are mutually disjoint and also disjoint from F. This condition is clearly fulfilled by $\mathcal{K}_0'$, for $b/a=\varrho^4$.

We define $\mathcal{K}_{n+1}$ and $\mathcal{K}_{n+1}'$ as follows. Let $\max\{\varrho^{-1},\delta\}r_n'<r_{n+1}<r_n'$ and set $D_\alpha=\{|z-w_\alpha|\leqq r_{n+1}\}$, $1\leqq\alpha\leqq N(n)$. The family $\mathcal{K}_{n+1}$ consists of these disks. The value r_{n+1} is fixed so close to r_n' as to have

(9) $$\mu_{n+1}(F_j)-\mu_n'(F_j)\leqq\varepsilon\mu(F_j)/2^{n+1},\quad 1\leqq j\leqq k,$$

and

(10) $$\mu_{n+1}(\partial D_\alpha)\geqq\mu_n'(\partial D_\alpha')/2,\quad 1\leqq\alpha\leqq N(n).$$

In order to define $\mathcal{K}_{n+1}'$, we take an integer $N'(n+1)\geqq\max\{n+1, t_1\}$, which will be fixed later. We set

$$w_{\alpha,j}=w_\alpha+r_{n+1}\exp[2\pi ji/N'(n+1)],\quad 1\leqq j\leqq N'(n+1),$$

$$N(n+1)=N(n)N'(n+1),\quad r_{n+1}'=r_{n+1}\exp\big[-l\big(N(n+1)\big)\big]$$

and

$$D_{\alpha,j}'=\{|z-w_{\alpha,j}|\leqq r_{n+1}'\},\quad 1\leqq j\leqq N'(n+1).$$

The disks $\{D_{\alpha,j}'\colon 1\leqq\alpha\leqq N(n),\ 1\leqq j\leqq N'(n+1)\}$ form the family $\mathcal{K}_{n+1}'$. We fix $N'(n+1)$ so large that (a) the disks $\{|z-w_{\alpha,j}|\leqq\varrho^4 r_{n+1}'\}$ are mutually disjoint and also disjoint from F;

(b) $$|\mu_{n+1}(F_j)-\mu_{n+1}'(F_j)|\leqq\varepsilon\mu(F_j)/2^{n+2},\quad 1\leqq j\leqq k;$$

(c) $\mu_{n+1}'(\partial D_{\alpha,j}')\geqq\big(BN'(n+1)\big)^{-1}\mu_{n+1}(\partial D_\alpha),\quad 1\leqq j\leqq N'(n+1),\quad 1\leqq\alpha\leqq N(n);$

(d) $$r_{n+1}'\leqq 3^{-1}\min\{r_n'-r_n,\ r_n-\delta r_n'\};$$

and

(e) $$h\big((n+1)(2B)^{n+1}N(n+1)\big)\leqq l\big(N(n+1)\big)/2.$$

This is trivial for (a) and (d). Statements (b) and (c) follow from Lemma 2.4 and (e) from the property (i'). Hence, by induction, the families $\mathcal{K}_n$ and $\mathcal{K}_n'$, $n\geqq 0$, are constructed.

Combining (9) and (b), we get for $n\geqq 1$

(11) $$|\mu_n(F_j)-\mu_{n+1}(F_j)|\leqq\varepsilon\mu(F_j)/2^n,\quad 1\leqq j\leqq k.$$

It follows from (10) and (c) that for any $D_\alpha'\in\mathcal{K}_n'$ there is a $D_\beta'\in\mathcal{K}_{n-1}'$ such that $\mu_n'(\partial D_\alpha')\geqq\big(2BN'(n)\big)^{-1}\mu_{n-1}'(\partial D_\beta')$. Repeating this, we see that for any $D_\alpha'\in\mathcal{K}_n'$

(12) $$\mu_n'(\partial D_\alpha')\geqq\big(2BN(n)\big)^{-1}\mu\big(\Gamma(a)\big).$$

We set

$$E=\bigcap_{n=1}^{\infty}\left(\mathrm{Cl}\left(\bigcup_{s=n}^{\infty}K_s\right)\right)$$

and denote by μ_E the harmonic measure at the point ∞ with respect to the domain $S-E-F$.

The property (d) shows that the set E lies in the annulus $\{\delta a<|z|<a\}$ and also in the interior of $K'_n, n\geqq 1$. By use of (11) we see for $n\geqq 1$

$$\begin{aligned}\mu_n(F) &= \mu(F)+\textstyle\sum_{s=1}^{n}\left(\mu_s(F)-\mu_{s-1}(F)\right)\\ &= \mu(F)+\textstyle\sum_{s=1}^{n}\sum_{j=1}^{k}\left(\mu_s(F_j)-\mu_{s-1}(F_j)\right)\\ &\leqq (1+\varepsilon)\mu(F)\end{aligned}$$

and therefore

$$\mu_n(\partial K_n)=1-\mu_n(F)\geqq 1-\mu(F)-\varepsilon\mu(F)=\mu(\Gamma(a))-\varepsilon\mu(F).$$

The last member is strictly positive, because of our assumption $\varepsilon<(1-\mu(F))/\mu(F)$. Let u_n (resp., v_n) be the solution of the Dirichlet problem for the domain $S-\mathrm{Cl}(\bigcup_{s=n}^{\infty}K_s)-F$ (resp., $S-K_n-F$) with the boundary data equal to 1 on $\partial[\mathrm{Cl}(\bigcup_{s=n}^{\infty}K_s)]$ (resp., ∂K_n) and to 0 on ∂F. Clearly, the sequence $\{u_n: n\geqq 1\}$ is monotonically decreasing and bounded below, so that it converges, by Harnack's theorem, to a nonnegative harmonic function, $u(z)$, on $S-E-F$. Obviously, $0\leqq u(z)\leqq 1$ and $u(z)=0$ on ∂F. We know that $u_n(z)\geqq v_n(z)$ on the domain of u_n and that $v_n(\infty)=\mu_n(\partial K_n)\geqq 1-(1+\varepsilon)\mu(F)>0$. Hence $u(\infty)>0$. The function u is seen to be the solution of the Dirichlet problem with the boundary data equal to 1 on E and to 0 on ∂F. Thus $\mu_E(E)=u(\infty)>0$ and consequently E has nonzero logarithmic capacity. We have

$$\begin{aligned}\mu_E(E) &= u(\infty)=\lim_{n\to\infty}u_n(\infty)\geqq \limsup_{n\to\infty}v_n(\infty)\\ &= \limsup_{n\to\infty}\mu_n(\partial K_n)\geqq \mu(\Gamma(a))-\varepsilon\mu(F).\end{aligned}$$

In order to get the reverse inequality, we note that E is contained in $\{|z|\leqq a\}$. So $\mu_E(F_j)\geqq\mu(F_j)$ and therefore $\mu_E(F)\geqq\mu(F)$. Thus $\mu(\Gamma(a))-\varepsilon\mu(F)\leqq\mu_E(E)\leqq\mu(\Gamma(a))$, which proves (8). On the other hand, we have $\mu_E(F_j)-\mu(F_j)\leqq\mu_E(F)-\mu(F)=\mu(\Gamma(a))-\mu_E(E)\leqq\varepsilon\mu(F)$, which shows (7).

We shall show finally that $E\in N_\varphi$. Since $\varphi(t+\log 2)/\varphi(t)=0(1)$, $t\to+\infty$, we have only to prove, in view of Proposition 2.2, that $S-E\in\mathcal{O}_\varphi$. Let f be any nonzero element in $H_\varphi(S-E)$ and u a harmonic majorant of $\varphi(\log|f|)$ on $S-E$. To show that f is a constant function, we first note that E is contained in the interior of $K'_n, n\geqq 0$. Let us fix $n\geqq 0$ and let $D'=\{z:|z-w'|\leqq r'\}$ be one of the members in $\mathscr{K}'_n$. Then the property (a) says that the annulus $\{r'\leqq|z-w'|\leqq\varrho^4 r'\}$

does not meet with E, F and $K'_n - D'$. Let ξ_w be the harmonic measure with respect to the annulus $\{r' < |z-w'| < \varrho^2 r'\}$ at a point w in it. If $|w-w'| = \varrho r'$, then Harnack's inequality implies that $d\xi_w/ds$ is bounded by $A'/(4\pi r')$ on $\Gamma(w'; r')$ and by $A'/(4\pi\varrho^2 r')$ on $\Gamma(w'; \varrho^2 r')$, where $A' = A'(\varrho^{-2})$ is the constant appearing in the proof of Lemma 2.3. Since f is bounded and analytic on the annulus $\{r' < |z-w'| < \varrho^2 r'\}$, we have for any w in this annulus

$$\log |f(w)| \leqq \int \log |f(\zeta)|\, d\xi_w(\zeta).$$

Applying the convex function $\varphi(t)$ to both sides and using Jensen's inequality, we get

$$\begin{aligned}\varphi(\log |f(w)|) &\leqq \int \varphi(\log |f(\zeta)|)\, d\xi_w(\zeta)\\ &= \int_{\Gamma(w'; r')} + \int_{\Gamma(w'; \varrho^2 r')}\\ &= I_1 + I_2.\end{aligned}$$

We now assume that $|w-w'| = \varrho r'$. We note that $\Gamma(w'; r') = \partial D'$. Let η, η' denote the harmonic measures at the point ∞ with respect to the domains $D_1 = \{|z-w'| > r'\} - E$ and $D_2 = \{|z-w'| > \varrho^2 r'\} - E$, respectively. Since $E \subseteqq K'_n$, we see that

$$\eta'(\Gamma(w'; \varrho^2 r')) \geqq \eta(\Gamma(w'; r')) \geqq \mu'_n(\partial D_1).$$

By use of (12) we have $\mu'_n(\partial D_1) \geqq (2B)^{-n} N(n)^{-1} \mu(\Gamma(a))$. By applying Lemma 2.3 to the annuli $\{r' < |z-w'| < \varrho^2 r'\}$ and $\{\varrho^2 r' < |z-w'| < \varrho^4 r'\}$, we see that $ds/d\eta$ is bounded by $2\pi A r'/\eta(\Gamma(w'; r'))$ on $\Gamma(w'; r')$ and $ds/d\eta'$ is bounded by $2\pi A \varrho^2 r'/\eta'(\Gamma(w'; \varrho^2 r'))$ on $\Gamma(w'; \varrho^2 r')$, where $A = A(\varrho^{-2})$ is the constant given in Lemma 2.3. So

$$\begin{aligned}I_1 &= \int_{\Gamma(w'; r')} \varphi(\log |f(\zeta)|) \frac{d\xi_w(\zeta)}{ds(\zeta)} \frac{ds(\zeta)}{d\eta(\zeta)}\, d\eta(\zeta)\\ &\leqq \frac{AA'}{2\eta(\Gamma(w'; r'))} \int_{\partial D_1} \varphi(\log |f(\zeta)|)\, d\eta(\zeta)\\ &\leqq \frac{1}{2} C(2B)^n N(n),\end{aligned}$$

where $C = AA'u(\infty)/\mu(\Gamma(a))$. Similarly, we have $I_2 \leqq 2^{-1} C(2B)^n N(n)$. Consequently, we have $\varphi(\log |f(w)|) \leqq C(2B)^n N(n)$ for $|w-w'| = \varrho r'$. Since $f \not\equiv 0$, we have $u(\infty) \neq 0$, so that the right-hand side exceeds t_1 for all sufficiently large $n \geqq n_0$, say. So we can apply the inverse function $h(t)$ and get the following inequality:

$$|f(w)| \leqq \exp\left[h\left(C(2B)^n N(n)\right)\right].$$

We integrate this along the circle $\Gamma(w'; \varrho r')$. Since $r'=r_n'=r_n \exp[-l(N(n))]$, we have, by use of the property (*e*),

$$(13) \qquad \int_{\Gamma(w';\varrho r')} |f(w)|\, ds(w) \leqq 2\pi\varrho r_n \exp[-l(N(n))+h(C(2B)^n N(n))]$$

$$\leqq 2\pi\varrho r_n \exp\left[-\frac{1}{2} l(N(n))\right],$$

if $n \geqq \max\{n_0, C\}$. We now look at the union Γ_n of paths $\Gamma(w'; \varrho r')$ corresponding to all $D' \in \mathscr{K}_n'$. Then, as we have seen, the set E is contained in the interior of Γ_n, $n \geqq 1$. Since the function f is analytic at ∞, it has an expansion $f(z)=\sum_{j=0}^{\infty} c_j z^{-j}$. Since we have $\varphi(t+\log 2)/\varphi(t)=O(1)$, $t \to +\infty$, it is easily seen that $f(z)-c_0$ also belongs to $H_\varphi(S-E)$, so that we may assume $f(\infty)=0$. If f does not vanish identically, let c_p, $p \geqq 1$, be the first nonzero coefficient of f. If n is sufficiently large, then we can use the inequality (13) and get

$$|c_p| \leqq \frac{1}{2\pi} \int_{\Gamma_n} |f(w)|\, |w|^{p-1} ds(w)$$

$$\leqq \varrho r_n b^{p-1} N(n) \exp\left[-\frac{1}{2} N(n)^{1/2}\right].$$

This is a contradiction, for the last member tends to zero as $n \to \infty$. Hence f must be a constant function, as was to be proved. Q.E.D.

In the preceding theorem we constructed a set in N_φ with nonzero logarithmic capacity. Trivially, the same result holds for an annulus with arbitrary center.

3. Classification of plane domains

Lemma 3.1. *Let* $0<a<b<c<+\infty$ *and let* F *be a bounded closed set contained in* $\{|z| \geqq c\}$ *such that* $\{|z|>b\}-F$ *is a connected domain. Let* μ *and* ν *be the harmonic measures at the point* ∞ *with respect to the domains* $\{|z|>b\}-F$ *and* $\{|z|>a\}-F$, *respectively. Then*

$$\frac{\log(c/b)}{\log(c/a)} \mu(\Gamma(b)) \leqq \nu(\Gamma(a)).$$

Lemma 3.2. *Let* F_j, $0 \leqq j \leqq k$, *be a finite number of bounded closed sets such that* $S-F_i$ *for each* $0 \leqq i \leqq k$ *is a connected domain containing all* F_j, $j \neq i$, *and let* $z_0 \in S-F$, $F=\bigcup_{j=0}^{k} F_j$, *be any finite point. Suppose that each* F_i *has positive logarithmic capacity. Let* μ *be the harmonic measure at the point* ∞ *with respect to the domain* $S-F$ *and, for each* $0<a<\inf\{|z-z_0|: z \in F\}$, *let* μ_a *be the harmonic* *measure at the point* ∞ *with respect to the domain* $\{|z-z_0|>a\}-F$. *Then there exists, for any* $\varepsilon>0$, *a number* a *with the above property such that* $\mu(F_j)-\mu_a(F_j)<\varepsilon$, $0\leqq j\leqq k$.

These lemmas, whose proof is omitted, are used to show the following

Lemma 3.3. *Let* φ *be a convex function satisfying* (B), $\{b_n: n\geqq 0\}$ *a sequence of positive numbers and* $0<\varrho<\delta<1$. *Then there exist a sequence* $\{a_n: n\geqq 0\}$ *of positive numbers and a sequence* $\{E_n: n\geqq 0\}$ *in* N_φ *such that* $a_{n+1}/a_n\leqq\varrho$, $a_n\leqq b_n$, $E_n\subseteqq\{\delta a_n\leqq|z|\leqq a_n\}$, $n\geqq 0$, and

$$d \leqq \varphi(-\log(na_n))m(E_n) \leqq 1, \quad n \geqq 1, \tag{14}$$

where m *is the harmonic measure at the point* ∞ *with respect to the domain* $S-E$ *with* $E=\bigcup_{n=0}^{\infty}E_n\cup\{0\}$ *and* d *is a constant,* $0<d<1$.

Proof. Let $\{d_n\}$, $0<d_n<1$, be a strictly decreasing sequence with limit $d>0$. We denote by μ_n (resp., ν_n) the harmonic measure at the point ∞ with respect to the domain $S-(\bigcup_{k=0}^{n}E_k)$ (resp., $\{|z|>a_{n+1}\}-(\bigcup_{k=0}^{n}E_k)$). We write also $\alpha_n=\varphi(-\log(na_n))$.

In order to construct E by induction, we first choose any a_0, satisfying $0<a_0\leqq b_0$ and $\varphi(-\log a_0)>0$, and any $E_0\in N_\varphi$ which has nonzero logarithmic capacity and is contained in the annulus $\{\delta a_0\leqq|z|\leqq a_0\}$. This is possible by Theorem 2.5. Suppose that we have chosen $E_k\in N_\varphi$, having nonzero logarithmic capacity, in $\{\delta a_k\leqq|z|\leqq a_k\}$, $0\leqq k\leqq n$, with the following property (P_n): $a_k\leqq b_k$ for $0\leqq k\leqq n$, $0<a_{k+1}/a_k\leqq\varrho$ for $0\leqq k\leqq n-1$ and

$$d_n\alpha_k \leqq \mu_n(E_k) \leqq \alpha_k, \quad 1 \leqq k \leqq n. \tag{15}$$

Then we choose a_{n+1} so small that

$$a_{n+1} \leqq b_{n+1}; \quad 0 < a_{n+1}/a_n \leqq \varrho; \tag{16}$$

$$d_{n+1}\alpha_k \leqq \nu_n(E_k) \leqq \alpha_k, \quad 1 \leqq k \leqq n; \tag{17}$$

$$\nu_n(\Gamma(a_{n+1})) > \varphi(-\log((n+1)a_{n+1}))^{-1} = \alpha_{n+1}. \tag{18}$$

There is no problem about (16). Statement (18) is clear from (B) and Lemma 3.1, for $\nu_n(\Gamma(a_{n+1}))$ decreases as $a_{n+1}\to 0$ no faster than $(-\log a_{n+1})^{-1}$ while $\varphi(-\log((n+1)a_{n+1}))^{-1}$ decreases much faster than this. For (17) we have only to set, in Lemma 3.2, $F_i=E_i$, $0\leqq i\leqq n$, $\mu=\mu_n$ and $\mu_a=\nu_n$ with $a=a_{n+1}$, and then use (15). By use of Theorem 2.5 we can find a set $E_{n+1}\in N_\varphi$ in $\{\delta a_{n+1}\leqq|z|\leqq a_{n+1}\}$ satisfying $d_{n+1}\alpha_{n+1}\leqq\mu_{n+1}(E_{n+1})\leqq\alpha_{n+1}$. Since $\nu_n(E_k)\leqq\mu_{n+1}(E_k)\leqq\mu_n(E_k)$, $1\leqq k\leqq n$, the property (P_{n+1}) is fulfilled. By induction we can construct $\{E_n: n\geqq 0\}$ which satisfies (P_n) for all $n\geqq 1$. Set $E=\bigcup_{n=0}^{\infty}E_n\cup\{0\}$. It is easy to see that, for each fixed k, the sequence $\{\mu_n(E_k): n \geqq k\}$ tends to $m(E_k)$. So, (17) implies (14), for $d_n \to d$. Q.E.D.

Let $\{a_n: n \geqq 0\}$ be a sequence of positive numbers and let $0 < \varrho < \delta < 1$ be constants. Suppose that $a_{n+1}/a_n \leqq \varrho$, $n \geqq 0$. For each $n \geqq 0$ let E_n be a closed totally disconnected set contained in $\{\delta a_n \leqq |z| \leqq a_n\}$ with nonzero logarithmic capacity. We set $E = \bigcup_{n=0}^{\infty} E_n \cup \{0\}$, so that E is bounded, closed and totally disconnected. We call such a set E a circular set with center at the origin. The definition of circular sets with center at an arbitrary point is obvious. By m and m_n, $n \geqq 1$, we denote the harmonic measures at the point ∞ with respect to the domains $S-E$ and $\{|z| > a_n\} - E$, respectively. We use various circular sets in order to obtain our classification result. Namely, we have the following

Theorem 3.4. *Let φ and ψ be convex functions satisfying* (A) *and* (B). *Then there exists a circular set E with center at the origin and with $E_n \in N_\varphi$, $n \geqq 0$, such that the function z^{-1} belongs to $H_\psi(S-E)$, while $H_\varphi(S-E)$ contains only constant functions.*

Proof. Let $0 < \varrho < \delta < 1$ be fixed. By use of (A) we can find a sequence $\{b_n: n \geqq 0\}$ of positive numbers such that $b_0 = 1$ and for $n \geqq 1$

$$\psi(\log(\delta^{-1} t))/\varphi(\log(n^{-1} t)) \leqq 2^{-n}, \quad t \geqq b_n^{-1}.$$

By Lemma 3.3 we get a sequence $\{a_n: n \geqq 0\}$ of positive numbers and a sequence $\{E_n: n \geqq 0\}$ in N_φ such that $a_{n+1}/a_n \leqq \varrho$, $a_n \leqq b_n$ and $E_n \subseteqq \{\delta a_n \leqq |z| \leqq a_n\}$ for $n \geqq 0$; and $d \leqq \varphi(-\log(na_n)) m(E_n) \leqq 1$, $n \geqq 1$, where m and d have the same meaning as in Lemma 3.3.

We see first that $z^{-1} \in H_\psi(S-E)$, for we have

$$\begin{aligned}\int_E \psi(\log|z^{-1}|)\, dm(z) &= \sum_{n=0}^{\infty} \int_{E_n} \psi(\log|z^{-1}|)\, dm(z) \\ &\leqq \sum_{n=0}^{\infty} \psi(-\log(\delta a_n)) m(E_n) \\ &\leqq \psi(-\log(\delta a_0)) m(E_0) + \sum_{n=1}^{\infty} \psi(-\log(\delta a_n))/\varphi(-\log(na_n)) \\ &\leqq \psi(-\log(\delta a_0)) m(E_0) + \sum_{n=1}^{\infty} 2^{-n} < +\infty.\end{aligned}$$

To show the latter half, we set $\sigma = (\delta/\varrho)^{1/4}$, so that $\{a_n < |z| < \sigma^4 a_n\} \subseteqq S-E$ for $n \geqq 1$. Let ξ_w be the harmonic measure with respect to the annulus $\{a_n < |z| < \sigma^2 a_n\}$ at the point w, $|w| = \sigma a_n$. Then we see that $d\xi_w/ds$ is bounded by $A'/(4\pi a_n)$ on $\Gamma(a_n)$ and by $A'/(4\pi\sigma^2 a_n)$ on $\Gamma(\sigma^2 a_n)$, where $A' = A'(\sigma^{-2})$ is the constant appearing in the proof of Lemma 2.3. Let $f \in H_\varphi(S-E)$ and u a harmonic majorant of $\varphi(\log|f|)$ on $S-E$. Since each E_n belongs to the class N_φ, we see that f can only be singular at the origin, i.e., $f(z) = \sum_{j=0}^{\infty} c_j z^{-j}$, $0 < |z| \leqq +\infty$. Let $n \geqq 1$. Since $f(z)$ is bounded analytic on the annulus $\{a_n < |z| < \sigma^2 a_n\}$, we have

$\log|f(w)|\leqq\int\log|f(\zeta)|\,d\xi_w(\zeta)$ for any w, $|w|=\sigma a_n$. Applying $\varphi(t)$ to both sides and using Jensen's inequality, we get

$$\varphi(\log|f(w)|)\leqq\left\{\int_{\Gamma(a_n)}+\int_{\Gamma(\sigma^2 a_n)}\right\}\varphi(\log|f(\zeta)|)\,d\xi_w(\zeta).$$

Using the fact $m_n(\Gamma(a_n))\geqq m(E_n)\geqq d\varphi(-\log(na_n))^{-1}$ and computing as in the proof of Theorem 2.5, we see

$$\varphi(\log|f(w)|)/\varphi(-\log a_n)\leqq C,\quad |w|=\sigma a_n, \tag{19}$$

where $C=A'(\sigma^{-2})A(\sigma^{-1})u(\infty)/d$. Take an integer $N\geqq 1$ so large that $\varphi(t)/t$ is positive and nondecreasing for $t\geqq a_N^{-1}$. Then (19) implies

$$|f(w)|\leqq a_n^{-C-1},\quad |w|=\sigma a_n\quad\text{and}\quad n\geqq N.$$

Letting $n\geqq N$ and writing $r=\sigma a_n$, we have

$$|c_k|=\left|\frac{1}{2\pi i}\int_{\Gamma(r)}f(z)z^{k-1}\,dz\right|\leqq r^k a_n^{-C-1}=\sigma^k a_n^{k-C-1}.$$

Letting $n\to+\infty$, we see that $c_k=0$, $k>C+1$, and therefore $f(z)$ is a polynomial in z^{-1}. Let c_p be the highest nonzero coefficient of $f(z)$. Suppose $p\geqq 1$. Then we would have $|f(z)|\geqq 2^{-1}|c_p|\,|z|^{-p}$ for all sufficiently small z, say $|z|\leqq a_{N'}$, with $N'\geqq N$. So,

$$\int_E\varphi(\log|f(z)|)\,dm(z)\geqq\sum_{n=N'}^{\infty}\varphi(\log(2^{-1}|c_p|a_n^{-1}))m(E_n)$$
$$\geqq d\sum_{n=N'}^{\infty}\varphi(\log(2^{-1}|c_p|a_n^{-1}))/\varphi(\log(na_n)^{-1})=+\infty.$$

Hence p should be zero. Thus $H_\varphi(S-E)$ contains only constant functions. This finishes the proof of the theorem and also of Theorem 1.1.

Proof of Corollary 1.2. To prove Corollary, we have only to apply Theorem 1.1 with suitable choices of φ and ψ. This is easy and is omitted.

4. Remarks

a) When two null classes of domains, e.g., $\mathcal{O}_p$ and $\mathcal{O}_q$, are distinct, their difference is very wide. Namely, we have

Theorem 4.1. *If convex functions φ and ψ satisfy* (A) *and* (B), *then there exists a connected domain $D\subseteqq S$ such that $H_\varphi(D)$ contains only constant functions, while $H_\psi(D)$ is infinite dimensional and contains functions having essential singularities.*

Sketch of Proof. For the sake of simplicity we construct a connected domain D for which $H_\varphi(D)$ is trivial but $H_\psi(D)$ contains functions with one essential singularity at the origin. Our domain will be obtained by omitting from S a countable number of circular sets. For any point $z_0 \in S$, $z_0 \neq \infty$, and any $r_0 > 0$ we denote by $N_\varphi(z_0; r_0)$ the totality of sets in N_φ contained in the annulus $\{15r_0/16 \leqq |z-z_0| \leqq r_0\}$. We define $b(n, j) > 0$ by the property: $\psi(\log(2t))/\varphi(\log(t/j)) \leqq 2^{-n-j}$ for $t \geqq b(n, j)^{-1}$, $n, j \geqq 1$. By induction we construct three sequences of positive numbers $\{A_n: n \geqq 0\}$, $\{a(n, j): n \geqq 1, j \geqq 0\}$, $\{z_n: n \geqq 1\}$; and two sequences of N_φ-sets $\{E'_n: n \geqq 0\}$, $\{E(n, j): n \geqq 1, j \geqq 0\}$. We first require the following:

$$A_{n+1}/A_n \leqq 1/8, \quad n \geqq 0; \quad A_n \leqq b(1, n), \quad n \geqq 1;$$

$$z_n = 3A_n/4, \quad n \geqq 1; \quad a(n, 0) \leqq A_n/8, \quad n \geqq 1;$$

$$a(n, j) \leqq b(n, j), \quad a(n, j+1)/a(n, j) \leqq 1/2, \quad n \geqq 1, \quad j \geqq 0;$$

$$E'_n \in N_\varphi(0; A_n), \quad n \geqq 0; \quad E(n, j) \in N_\varphi(z_n; a(n, j)), \quad n \geqq 1, \quad j \geqq 0.$$

We see that all E'_n and $E(n, j)$ are mutually disjoint. We set

$$E_n = \cup\{E(n, j): j \geqq 0\} \cup \{z_n\}, \quad n \geqq 1,$$

and finally E to be the union of all E_n with $n \geqq 1$, E'_n with $n \geqq 0$ and the origin. E is thus bounded, closed and totally disconnected. Let m be the harmonic measure at the point ∞ with respect to the domain $S-E$. Our final requirement is this:

$$d \leqq \varphi(-\log(nA_n))\, m(E'_n) \leqq 1, \quad n \geqq 1,$$

and

$$d \leqq \varphi(-\log(ja(n, j)))\, m(E(n, j)) \leqq 1, \quad n, j \geqq 1,$$

where d is a given constant, $0 < d < 1$. The induction arguments are similar to the ones used in the proof of Lemma 3.3 but are a little bit more involved. We omit the details.

Let $f \in H_\varphi(S-E)$ be any nonzero function. Since E'_n and $E(n, j)$ belong to the class N_φ, f can be extended analytically to points in these sets. So the possible singularities of f are z_n, $n \geqq 1$, and the origin. The point z_n for each $n \geqq 1$ is the center of the circular set E_n, for which we know

$$a(n, j+1)/a(n, j) \leqq 1/2, \quad j \geqq 0;$$

$$E(n, j) \subseteqq \{15a(n, j)/16 \leqq |z - z_n| \leqq a(n, j)\}, \quad j \geqq 0;$$

and

$$d \leqq \varphi(-\log(ja(n, j)))\, m(E(n, j)) \leqq 1, \quad j \geqq 1.$$

As we immediately see from our proof of Theorem 3.4, these conditions are enough to conclude that z_n is a removable singularity of f. For the origin we note that the domain $S-E$ contains annuli $\{A_n < |z| < 4A_n\}$, $n \geqq 1$, with constant modulus.

So our requirement on $m(E_n')$ again says that f has a removable singularity at the origin. Hence f is a constant function.

Finally it follows from the properties of $a(n, j)$ and $m(E(n, j))$ that $H_\psi(S-E)$ contains functions of the form

$$h(z) = \sum_{n=1}^{\infty} c_n(z-z_n)^{-1}, \quad c_n \neq 0,$$

when c_n are small enough. In this way we can show that $H_\psi(S-E)$ is infinite dimensional and contains functions with essential singularities.

b) Let E be a bounded, closed subset of an analytic curve. If $E \in N_{AB}$, then E has zero linear measure and thus belongs to N_1 (cf., [4, Theorem 9]). Namely, all N_p, $1 \leqq p < +\infty$, coincide on analytic curves. As shown by [4, Theorem 17], this is no longer true for $0<p<1$. There exists a bounded, closed set E, on the real axis, of zero linear measure such that $z^{-1} \in H^p(S-E)$ for all $0<p<1$. It looks like on open problem to find linear sets belonging to the class N_φ, when $t/\varphi(t)=o(1)$, $t \to +\infty$. For function-theoretic null sets, see also Ahlfors and Beurling [1].

c) The condition (A) cannot be replaced by $\psi(t)/\varphi(t)=o(1)$, $t \to +\infty$, as we see from the example $\varphi(t)=\exp(2e^t)$ and $\psi(t)=\exp(e^t)$.

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