Mean oscillation and commutators of singular integral operators

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0. Introduction

Let $T$ be a Caldéron–Zygmund transform

$$Tg(x) = \text{P.V.} \int_{\mathbb{R}^d} K(x-y) g(y) \, dy$$

where the kernel $K$ is homogeneous of degree $-d$, i.e. $K(x)=|x|^{-d}K(|x|)$, $\int_{\mathbb{R}^d} K = 0$ and $K$ satisfies some smoothness condition. $K \in C^\infty(S^{d-1})$ will always be sufficient. For the theory of these transforms, see e.g. Stein [7]. We need the result that $T$ is bounded on $L^p$, $1 < p < \infty$. $K$ and $T$ will be fixed throughout the paper and not identically zero.

Let $f$ be a function on $\mathbb{R}^d$, and let it also denote the operation of pointwise multiplication with $f$. We will study the commutator $[f, T]$ denoted by $C_f$.

Formally

$$C_f g(x) = fTg(x) - Tfg(x)$$

$$= f(x) \int K(x-y) g(y) \, dy - \int K(x-y) f(y) g(y) \, dy$$

$$= \int (f(x)-f(y)) K(x-y) g(y) \, dy.$$

For these formulas to make sense, $f$ has to be locally integrable. $C_f g$ is then defined a.e. as a principal value for $g$ bounded and with compact support. $C_f$ may be extended to all of $L^p$ when we have proved it to be continuous. $C_f g$ is clearly bilinear.

Let $Q$ be any cube in $\mathbb{R}^d$. We define $f_Q$, the mean value of $f$ on $Q$, as

$$|Q|^{-1} \int_Q f(x) \, dx$$

and $\Omega(f, Q)$, the mean oscillation of $f$ on $Q$, as

$$|Q|^{-1} \int_Q |f-f_Q| \, dx.$$
$|Q|$ is the Lebesgue measure. $BMO$ is the space of all functions of bounded mean oscillation, i.e. $f \in BMO$ if and only if $\Omega(f, Q) \leq C$ for every $Q$ ([4]). More generally, let $\varphi$ be a non-decreasing positive function and define $BMO_\varphi$ as the space of all functions $f$, with $\Omega(f, Q) \leq C \varphi(r)$ whenever $Q$ is a cube with edge-length $r$ ([6], [3]). The norms are defined as the least possible constants $C$ in the inequalities and the spaces are Banach spaces.

Coifman, Rochberg and Weiss [1] have proved that if $f \in BMO$, $C_f$ is a bounded operator from $L^p$ to itself, $1 < p < \infty$. They also proved a partial converse, viz. if $[f, R_j]$ is bounded on $L^p$ for every Riesz transform $R_j$, then $f$ belongs to $BMO$. The purpose of this paper is to show that it suffices to assume the boundedness of one of these commutators, or of any commutator $C_f$. More generally $f \in BMO_\varphi$ if and only if $C_f$ is a bounded operator from $L^p$ to a suitable Orlicz space.

1. Notation and basic lemmas

$C$ denotes different positive constants. $Q(x_0, r)$ denotes the cube with center $x_0$ and edge-length $r$. $nQ$ denotes the cube with the same center as $Q$, but enlarged $n$ times, i.e. $nQ(x_0, r) = Q(x_0, nr)$.

We state some lemmas without proofs. Cf. [3], [4], [6].

**Lemma 1.** $\Omega(f, Q) \leq 2 |Q|^{-1} \int_Q |f(x) - a| \, dx$ for every $a$.

**Lemma 2.** If $f \in BMO$, then $|f_Q - f_{nQ}| \leq C \|f\|_{BMO} \log n$.

**Lemma 3.** If $f \in BMO$ and $p < \infty$, then $|Q|^{-1} \int_Q |f(x) - f_Q|^p \, dx \leq C \|f\|_{BMO}^p$.

Let $A_\alpha$, $0 < \alpha \leq 1$, be the space of Lipschitz continuous functions, possibly unbounded, $A_\alpha = \{f; |f(x) - f(y)| \leq C |x - y|^\alpha\}$.

**Lemma 4.** $BMO_\alpha = A_\alpha$.

Let $\eta$ be an infinitely differentiable function with compact support such that $\int \eta = 1$. Define $f_\alpha(x)$ as $\int f(x - ry) \eta(y) \, dy$.

**Lemma 5.** If $\|f\|_{BMO_\alpha} = 1$, then $\|f - f_\alpha\|_{BMO} \leq C \varphi(r)$.

**Lemma 6.** If $\|f\|_{BMO_\alpha} = 1$, then $|f_\alpha(x) - f_\alpha(y)| \leq C \frac{\varphi(r)}{r} |x - y|$ and

$$|f_\alpha(x) - f_\alpha(y)| \leq C \int_r^{+|x-y|} \frac{\varphi(t)}{t} \, dt.$$

This gives the following estimate of the Lipschitz norm.
Lemma 7. If \( 0 < \alpha < 1 \) and \( t^{-\alpha} \varphi(t) \) is decreasing, or if \( \alpha = 1 \), then \( \|f_i\|_{A_{\alpha}} \leq Cr^{-\alpha} \varphi(r) \|f\|_{BMO_{\alpha}} \).

Let \( \psi \) be a non-decreasing convex function on \( \mathbb{R}^+ \) with \( \psi(0) = 0 \). \( \psi^{-1} \) denotes the inverse function. The Orlicz space \( L_\psi \) is defined as the set of functions \( f \) such that \( \int \psi(\lambda |f|) < \infty \) for some \( \lambda > 0 \). ([5], [8]). The norm is given by \( \|f\|_{L_\psi} = \inf \frac{1}{\lambda} \left( 1 + \int \psi(\lambda |f|) \right) \).

Lemma 8. If \( f \in L_\psi \) and \( \mathcal{E} \) is a set of finite measure, then \( \left| \int_E f(x) \, dx \right| \leq \|f\|_{L_\psi} \mathcal{E} \psi^{-1}(\mathcal{E}^{-1}) \).

We also need a result for maximal functions.

For \( q \geq 1 \) define
\[
M_q g(x) = \sup_{t \in Q} \left( \int_Q |g|^q \, dx \right)^{1/q}.
\]

\( M_q \) is a bounded operator on \( L_p, 1 < p < \infty \), see Stein [7]. Since \( M_1 = (M_1 |g|^q)^{1/q} \), this gives

Lemma 9. \( M_q \) is bounded on \( L_p, q < p < \infty \).

\( m_f \) denotes the distribution function. \( m_f(t) = \{|x; |f(x)| > t\}| \).

We have the following Marcinkiewicz-type interpolation theorem.

Lemma 10. Suppose \( 1 \leq p_2 < p < p_1 \leq \infty \), \( \varrho \) is a non-increasing function, \( A \) is a linear operator such that \( m_{A_\varrho}(t^{1/p_1} \varrho(t)) \leq \frac{C}{t} \), if \( \|g\|_{p_1} \leq 1 \), and \( m_{A_\varrho}(t^{1/p_2} \varrho(t)) \leq \frac{C}{t} \), if \( \|g\|_{p_2} \leq 1 \). Then \( \int_0^\infty m_{A_\varrho}(2t^{1/p} \varrho(t)) \, dt \leq C \), if \( \|g\|_p \leq (p/p_1)^{1/p} \).

Proof. Fix \( t \) for the moment. Set \( u = t^{1/p} \). Set \( g_1(x) = \min(|g(x)|, u) \cdot \text{sgn}(g(x)) \) and \( g_2 = g - g_1 \). Let \( m(s) \) denote \( m_g(s) \). Then
\[
m_{g_1}(s) = \begin{cases} m(s), & s < u \\ 0, & s \geq u \end{cases} \quad \text{and} \quad m_{g_2}(s) = m(s+u).
\]

Thus
\[
\|g_1\|_{p_1} = p_1 \int_0^u s^{p_1-1} m(s) \, ds
\]
and
\[
\|g_2\|_{p_2} = p_2 \int_0^\infty s^{p_2-1} m(s+u) \, ds \leq p_2 \int_u^\infty s^{p_2-1} m(s) \, ds.
\]

We have
\[
p_1 \int_0^u u^{p_1} s^{p_1-1} m(s) \, ds \leq p_1 \int_0^{u} s^{p_1-1} m(s) \, ds \leq \frac{p_1}{p} \|g\|_p^p \leq 1.
\]

Thus
\[
u^p \leq u^{p_1} \|g_1\|_{p_1}^{p_1} \quad \text{and} \quad \varrho(u^p) \geq \varrho(u^{p_1} \|g_1\|_{p_1}^{p_1}).
\]
We apply the assumptions to \( \frac{g_1}{\|g_1\|_{p_1}} \) and obtain
\[
m_{A_{g_1}}(uQ(u^p)) \equiv m_{A_{g_1}}(uQ(u^{p_1}\|g_1\|_{p_1}^{-p_1}))
\]
\[
= m_{A_{g_1}}(u\|g_1\|_{p_1}^{-1}Q(u^{p_1}\|g_1\|_{p_1}^{-1})) \equiv C u^{-p_1}\|g_1\|_{p_1}^{-p_1} = C u^{-p_1} \int_0^u s^{p_1-1} m(s)\,ds.
\]
Similarly
\[
m_{A_{g_2}}(uQ(u^p)) \equiv C u^{-p_1} \int_u^\infty s^{p_2-1} m(s)\,ds.
\]
Thus we have
\[
\int_0^\infty m_{A_g}(2t^{1/p}g(t))\,dt = p\int_0^\infty u^{p-1}m_{A_g}(2uQ(u^p))\,du
\]
\[
\equiv C \int_0^\infty \int_0^u u^{p-1-p_1}s^{p_1-1} m(s)\,ds\,du + C \int_0^\infty \int_u^\infty u^{p-1-p_2}s^{p_2-1} m(s)\,ds\,du
\]
\[
= C \int_0^\infty \int_0^u u^{p-1-p_1} du s^{p_1-1} m(s)\,ds + C \int_0^\infty \int_0^s u^{p-1-p_2} du s^{p_2-1} m(s)\,ds
\]
\[
= C \int_0^\infty s^{p-1} m(s)\,ds \equiv C.
\]

2. The main result

**Theorem.** Let \( 1 < p < \infty \), and let \( \varphi \) and \( \psi \) be two non-decreasing positive functions on \( \mathbb{R}^+ \) connected by the relation \( \varphi(r) = r^{d/q} \psi^{-1}(r^{-d}) \), or equivalently \( \psi^{-1}(t) = t^{1/p} \varphi(t^{-1/d}) \). We assume that \( \psi \) is convex, \( \psi(0) = 0 \) and \( \psi(2t) \equiv C \psi(t) \). Then \( f \) belongs to \( BMO_\varphi \) if and only if \( C_f \) maps \( L^p \) boundedly into \( L^{p_1} \).

**Remark.** By duality, \( f \) belongs to \( BMO_\varphi \) if and only if \( C_f \) maps \( L_{\psi*} \) into \( L^{p'} \).

Also, the proof may be generalized to show that \( f \) belongs to \( BMO_\varphi \) if and only if \( C_f \) maps \( L_{\psi_1} \) into \( L_{\psi_2} \) with
\[
\varphi(r) = \frac{\psi_2^{-1}(r^{-d})}{\psi_1^{-1}(r^{-d})},
\]
under suitable conditions on \( \psi_1 \) and \( \psi_2 \).

**Proof.** We first prove that the condition is sufficient. Assume that \( C_f \) maps \( L^p \) into \( L_{\psi_1} \).

\[
\frac{1}{K(z)}
\]
is many times infinitely differentiable in an open set. Consequently, we may choose \( z_0 \neq 0 \) and \( \delta > 0 \) such that \( \frac{1}{K(z)} \) can be expressed in the neighborhood \( |z - z_0| = \sqrt{d} \delta \) as an absolutely convergent Fourier series, \( \frac{1}{K(z)} = \Sigma a_n e^{i\alpha_n z} \). (The exact form of the vectors \( v_n \) is irrelevant.)
Set $z_1 = \delta^{-1}z_0$. If $|z-z_1|<\sqrt{d}$, we have the expansion

$$\frac{1}{K(z)} = \delta^{-d} = \delta^{-d} \sum a_n e^{i\nu_n \cdot \delta z}.$$ 

Choose now any cube $Q=Q(x_0, r)$. Set $y_0 = x_0 - rz_1$ and $Q' = Q(y_0, r)$ Thus, if $x \in Q$ and $y \in Q'$,

$$\frac{|x-y|}{r} - z_1 \leq \left|\frac{x-x_0}{r}\right| + \left|\frac{y-y_0}{r}\right| \leq \sqrt{d}.$$ 

Denote $\text{sgn} (f(x) - f_{Q'})$ by $s(x)$. This gives us

$$\int_Q |f(x) - f_{Q'}| \, dx = \int_Q (f(x) - f_{Q'}) s(x) \, dx = |Q|^{-1} \int_Q \int_{Q'} (f(x) - f(y)) s(x) \, dy \, dx$$

$$= r^{-d} \int_{R^d} \int_{R^d} (f(x) - f(y)) \frac{r^d K(x-y)}{K \left( \frac{x-y}{r} \right)} s(x) \chi_Q(x) \chi_{Q'}(y) \, dy \, dx$$

$$= C \int \int (f(x) - f(y)) K(x-y) \sum a_n e^{i\nu_n \cdot \frac{x-y}{r}} s(x) \chi_Q(x) \chi_{Q'}(y) \, dy \, dx$$

$$= C \sum a_n \int \int (f(x) - f(y)) e^{i\nu_n \cdot \frac{x-y}{r}} s(x) \chi_Q(x) e^{-i\frac{\delta}{r} \nu_n \cdot y} \chi_{Q'}(y) \, dy \, dx.$$ 

If we introduce

$$g_n(y) = e^{-i\frac{\delta}{r} \nu_n \cdot y} \chi_{Q'}(y)$$

and

$$h_n(x) = e^{i\frac{\delta}{r} \nu_n \cdot x} s(x) \chi_Q(x)$$

we have obtained

$$\int_Q |f(x) - f_{Q'}| \, dx = C \sum a_n \int \int (f(x) - f(y)) K(x-y) g_n(y) h_n(x) \, dy \, dx$$

$$= C \sum a_n \int C_f g_n(x) h_n(x) \, dx$$

$$= C \sum |a_n| \int |C_f g_n| \, dx.$$ 

However, $g_n$ belongs to $L^p$, and its norm is $|Q|^{1/p} = r^{d/p}$. Consequently, $\|C_f g_n\|_{L^p} \leq C r^{d/p}$ and, by Lemma 8,

$$\int_Q |C_f g_n| \leq C r^{d/p} |Q| \psi^{-1}(|Q|^{-1}).$$ 

Thus we have obtained

$$\int_Q |f(x) - f_{Q'}| \, dx \leq C \sum |a_n| r^{d/p} |Q| \psi^{-1}(|Q|^{-1}) = C |Q| r^{d/p} \psi^{-1}(r^{-d}) = C |Q| \varphi(r),$$

and $\Omega(f, Q) \leq C \varphi(r)$ by Lemma 1.
We prove the converse in several steps and begin with two special cases.

**Lemma 11.** If $\|f\|_{BMO} \leq 1$ and $\|g\|_{p} \leq 1$, $1 < p < \infty$, then $\|C_{f}g\|_{p} \leq C$.

This is proved in [1]. The following simpler proof was suggested to the author by Jan-Olov Strömberg.

**Proof.** We will estimate $(C_{f}g)^{(x)} = \sup_{x \in Q} \Omega(C_{f}g, Q)$. Choose $q$ and $r$ greater than 1 such that $p = qr$. Let $x$ and $Q = Q(x_{0}, s)$ be fixed with $x \in Q$. Set $g_{1} = g \cdot \chi_{2Q}$ and $g_{2} = g - g_{1}$. This gives

$$C_{f}g = C_{f - f_{Q}}g = (f - f_{Q})Tg - T(f - f_{Q})g_{1} - T(f - f_{Q})g_{2}.$$ 

We estimate the mean oscillation on $Q$ of each of these functions separately. Hölder’s inequality and Lemma 3 give

$$\left| Q^{-1} \int_{Q} |f - f_{Q}|Tg_{1} \right| \leq \left[ \left| Q^{-1} \int_{Q} |f - f_{Q}|^{q} \right|^{1/q} \left( \left| Q^{-1} \int_{Q} |Tg|^{r} \right|^{1/r} \right) \right] \leq CM_{q}Tg(x).$$

We also have

$$\left| Q^{-1} \int_{Q} |f - f_{Q}|^{r} \right| g_{1}^{r} = \left| Q^{-1} \int_{2Q} |f - f_{Q}|^{r} \right| g_{1}^{r} \leq \left( \left| Q^{-1} \int_{2Q} |f - f_{Q}|^{rs} \right|^{1/s} \left( \left| Q^{-1} \int_{2Q} |g|^{rs} \right|^{1/r} \right) \right)^{\frac{1}{s}} \leq C(M_{rQ}g(x))^{r}.$$ 

Thus

$$\left\| (f - f_{Q})g_{1} \right\|_{r} \leq C |Q|^{1/r} M_{rQ}(g)(x)$$

and consequently

$$\left| Q^{-1} \int_{Q} |T(f - f_{Q})g_{1}| \right| \leq |Q|^{-1/r} \left\| T(f - f_{Q})g_{1} \right\|_{r} \leq C |Q|^{-1/r} \left\| (f - f_{Q})g_{1} \right\|_{r} \leq CM_{rQ}g(x).$$

For the last term we have for any $y \in Q$

$$\left| T(f - f_{Q})g_{2}(y) - T(f - f_{Q})g_{2}(x_{0}) \right| = \left| \int (K(y - z) - K(x_{0} - z))(f(z) - f_{Q})g_{2}(z) \, dz \right|$$

$$\leq \int_{Q} \left| K(y - z) - K(x_{0} - z) \right| |f(z) - f_{Q}| |g(z)| \, dz$$

$$\leq C \int_{Q} \frac{|y - x_{0}|}{|x_{0} - z|^{d+1}} |f(z) - f_{Q}| |g(z)| \, dz$$

$$\leq C \sum_{n=2}^{\infty} \int_{\gamma_{n+1}Q \setminus \gamma_{n}Q} 2^{-n} |2^n \Omega|^{-1} \left( |f(z) - f_{\gamma_{n}Q} + f_{\gamma_{n}Q} - f_{Q}| \right) |g(z)| \, dz$$

$$\leq C \sum_{n=2}^{\infty} |2^n \Omega|^{-1} \int_{\gamma_{n}Q} |f(z) - f_{\gamma_{n}Q}| |g(z)| \, dz + C \sum_{n=2}^{\infty} |2^n \Omega|^{-1} \int_{\gamma_{n}Q} |g(z)| \, dz$$

$$\leq C \sum_{n=2}^{\infty} \left( |2^n \Omega|^{-1} \int_{\gamma_{n}Q} |f(z) - f_{\gamma_{n}Q}|^{q} \, dz \right)^{1/q} \left( |2^n \Omega|^{-1} \int_{\gamma_{n}Q} |g(z)|^{q} \, dz \right)^{1/q} + CM_{q}g(x)$$

$$\leq CM_{q}g(x) + CM_{q}g(x).$$
These estimates give
\[ \Omega(Cfg, Q) \leq 2 |Q|^{-1} \int_Q |C_f g(z) - T(f - f_Q) g_2(x_0)| \, dz \]
\[ \equiv CM_q Tg(x) + CM_{rq} g(x) + CM_q g(x) + CM_1 g(x) \equiv C(M_q Tg(x) + M_{rq} g(x)). \]
This holds for every \( Q \) containing \( x \), and thus
\[ (C_f g)^* \equiv C(M_q Tg + M_{rq} g) \in L^p. \]
This, however, implies
\[ \|C_f g\|_p \leq C \|C_f g\|_p^* \leq C \|M_q Tg\|_p + C \|M_{rq} g\|_p \leq C \|g\|_p, \]
see [2].

**Lemma 12.** If \( f \in A_u \) and \( g \in L^p \), \( 1 < p < \frac{d}{a} \), then \( \|C_f g\|_q \equiv C\|f\|_{A_u} \|g\|_p \), where \( \frac{1}{q} = \frac{1}{p} - \frac{a}{d} \).

**Proof.**
\[ |C_f g(x)| \leq \int |f(x) - f(y)| |K(x - y)| |g(y)| \, dy \]
\[ \equiv C \|f\|_{A_u} \int |x - y|^a |x - y|^{-a} |g(y)| \, dy = C \|f\|_{A_u} I_a(\|g\|)(x). \]
The theorem of fractional integration [7, p. 119] shows that this Riesz potential exists a.e. and belongs to \( L^q \) with the right norm.

To complete the proof of the theorem, let us assume that \( \|f\|_{BMO^q} \equiv 1 \). We note that there exists a \( q < \infty \) such that \( (2t)^{-q} \psi(2t) < t^{-q} \psi(t) \). Thus, replacing \( \psi \) by an equivalent Orlicz function if necessary, \( t^{-q} \psi(t) \) is decreasing. Consequently \( t^{-1/q} \psi^{-1}(t) \) is increasing and \( r^{q(1/q - 1/p)} \phi(r) \) is decreasing.

Let \( \alpha \) be the minimum of \( d(\frac{1}{p} - \frac{1}{q}) \) and 1. Assume that \( 1 < p_1 < \frac{d}{a} \), and that \( \|g\|_{p_1} \equiv 1 \). Lemma 7 shows that \( \|f\|_{A_u} \equiv C r^{-z} \phi(r) \), and Lemma 12 gives
\[ \|C_f, g\|_{q_1} \equiv C r^{-z} \phi(r), \quad \text{where} \quad \frac{1}{q_1} = \frac{1}{p_1} - \frac{a}{d}. \]
Lemmas 5 and 11 give
\[ \|C_{f, g}\|_{p_1} \equiv C \phi(r). \]
We set in these formulas \( r = t^{-1/d} \) and obtain a weak estimate.
\[ mc_{fg}(t^{1/p_1} \phi(t^{-1/d})) \equiv \left( \frac{2C \phi(r)}{t^{1/p_1} \phi(r)} \right)^{p_1} + \left( \frac{2Cr^{-z} \phi(r)}{t^{1/p_1} \phi(r)} \right)^{q_1} = C \frac{c}{t} + \frac{C}{t} \left( \frac{1}{p_1} - \frac{a}{d} \right) q_1 = C \frac{c}{t}. \]
Choosing \( 1 < p_2 < p < p_1 < \frac{d}{a} \). Let \( \phi(t) \) be \( \psi(t^{-1/d}) \) and let \( A \) be \( C_f \). We have just proved that the conditions in Lemma 10 are fulfilled. Thus, if \( \|g\|_p \equiv (p/p_1)^{1/p} \),
\[ \int \psi \left( \frac{1}{2} \|C_f g\|_p \right) = \int_0^\infty m_{C_f} (2\psi^{-1}(t)) \, dt \equiv C. \]
That is, \( \|C_f g\|_{L^\psi} \equiv C. \)
3. Examples

1. \( \varphi \equiv 1 \). We may take any \( 1 < p < \infty \) and \( \psi(t) = t^p \). Thus \( C_f \) maps \( L^p \) into \( L^p \) if and only if \( f \in BMO \), as asserted in the introduction.

2. \( \psi(t) = t^q \), \( 1 < p < q < \infty \). \( \varphi(r) = r^{d/p} r^{-d/q} \). Thus, by Lemma 4, \( C_f \) maps \( L^p \) into \( L^q \) if and only if \( f \in A_{\frac{1}{d} \left( \frac{1}{p} - \frac{1}{q} \right)} \). This holds even if \( d \left( \frac{1}{p} - \frac{1}{q} \right) > 1 \), then \( f \) has to be a constant.

3. \( \psi(t) = t^p (1 + \log^+ t)^a \), \( 1 < p < \infty \), \( a > 0 \). \( \psi^{-1}(t) \sim t^{1/p} (1 + \log^+ t)^{-a/p} \), i.e. \( \varphi(r) \sim (1 + \log^+ r)^{-a/p} \). Thus \( f \in BMO_{(1 + \log^+ r)^{-a/p}} \) if and only if \( C_f \) maps \( L^p \) into \( L^p (1 + \log^+ L)^a \).

Added in proof. There is an overlap between the results of this paper and those of A. Uchiyama, Compactness of operators of Hankel Type. Tōhoku Math. J. 30 (1978), 163-171.

References


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