Higher order Briot—Bouquet differential equations

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1. Introduction

Let $P(x, y)$ be a polynomial in x and y with constant coefficients, say

$$P_0(x)y^n + P_1(x)y^{n-1} + \ldots + P_n(x). \tag{1.1}$$

Let δ_j be the degree of the polynomial $P_j(x)$. A Briot—Bouquet DE of order k is

$$P[w(z), w^{(k)}(z)] = 0. \tag{1.2}$$

In this note k is supposed to be >2.

The main problem of integration is to find necessary and, if possible, sufficient conditions for the existence of what here will be called *canonical* solutions of (1.2), that is, solutions which are transcendental single-valued functions, holomorphic save for poles in the finite plane. A more general problem, not considered here, is to find the equations with fixed critical points (branch points and essential singularities). It does not seem to be significant for *BB* equations.

Various facts are known about this problem, a few valid for all k, more for $k=1$ or 2. Those which are needed or suggestive for the investigation are stated as lemmata.

Lemma 1. *Every elliptic function satisfies a BB equation of given order.*

For if $z \to f(z)$ is elliptic, so is its kth derivative and they have the same periods, so by a theorem proved by Charles Briot and Jean-Claude Bouquet in 1856 one is an algebraic function of the other, i.e. $f(z)$ satisfies an equation of type (1.2).

Lemma 2. *A BB equation can have a canonical solution only if the genus of the curve*

$$P(x, y) = 0 \tag{1.3}$$

is 0 *or* 1.

For Émile Picard proved in 1887 that an algebraic curve (1.3) can be represented parametrically by

$$x = S(z), \quad y = T(z) \tag{1.4}$$

where S and T are transcendental entire or meromorphic functions, only if the genus of the curve (1.3) is 0 or 1. In the Nevanlinna value distribution theory this is a consequence of the fact that a meromorphic function can have at most four completely ramified values.

Lemma 3. *If all solutions of* (1.2) *are canonical then* $P_0(x)$ *is a constant, say* $P_0(x) \equiv 1$ *and*

$$\delta_j \leqq (k+1)j, \quad j = 1, 2, \ldots, n. \tag{1.5}$$

For $k=1$ this was proved by Lazarus Fuchs [2] in 1884. For a general k it follows from results due to Jean Chazy [1, p. 378].

Lemma 4. *The canonical solutions are entire functions if*

$$\delta_j \leqq j, \ \forall j. \tag{1.6}$$

For this condition excludes poles.

Lemma 5. *If* $k=1$ *or* 2 *the Nevanlinna order of a transcendental meromorphic solution is* $\leqq 2$; *it is* $\leqq 1$ *if the solution is an entire function.*

For $k=1$ this follows from a result due to A. A. Gol'dberg [3] in 1956. For $k=2$ see Theorems 6 and 7 of [5]. The estimate is the best possible for a meromorphic solution could be an elliptic function in which case $T(r, w)$ exceeds a constant multiple of r^2.

Lemma 6. *If* $k=1$ *or* 2 *and if* $w(z)$ *is a solution which is an entire function with only a finite number of zeros then necessarily*

$$w(z) = Ce^{az} \tag{1.7}$$

where C *is a constant and* a *is a root of a characteristic equation.*

Lemma 7. *The determinateness theorem of Painlevé holds for* $k=1$ *and* 2, *i.e. analytic continuation of a solution along an arc of finite length always leads to a definite limit, finite or infinite.*

The original theorem holds for DE's of the form $w' = F(z, w)$ and is not valid for equations of second or higher order. For a proof of Lemma 7 see Theorems 3 and 4 of [6].

Lemma 8. *If* $k=1$ *or* 2 *and if all solutions are single-valued then the solutions are rational functions either of* z *or of* e^{az} *for some* a *or of a Weierstrass* $\wp$*-function and its derivative where the argument is linear in* z.

These are the functions for which Weierstrass proved algebraic addition theorems.

For $k=1$ the result is due to Fuchs [2]. For $k=2$ see [7] where ideas of Fuchs and Ludwig Schlesinger [9] are combined with the results of Paul Painlevé and René Gambier concerning second order *DE*'s with fixed critical points. See E. L. Ince [8, Chapter XIV]. Birational parametrization of the curve (1.3) leads to a second order *DE* of the Painlevé—Gambier type. Of the 53 types shown by them to have fixed critical points only 17 could possibly be transforms of a second order *BB*-equation. Of these 15 lead to solutions which are of the type specified in Lemma 8 and will preserve this form under the birational transdormation. The other two types have to be rejected because their solutions are $\wp$-functions of a non-linear argument. In one case the order would be 4 instead of 2 and in the other case there are infinitely many singularities which are limit points of poles and the solutions could not approach definite limits.

Our program in this note is to extend the results for $k=1$ and 2 to larger values of k.

2. The Nevanlinna order

We start with Lemma 5.

Suppose that $w(z)$ is a canonical solution of (1.2). Our first task is to study the nature of the *DE* in the neighborhood of a pole of order α of the solution. All poles may be of the same order but there are other possibilities and what they are is determined by the behaviour of the algebraic function $x \to A(x)$ defined by

$$P[x, A(x)] = 0 \tag{2.1}$$

for large values of $|x|$. The function $A(x)$ has a finite number of finite algebraic singularities where the various determinations of $A(x)$ are finite. Let R be so large that all the branch points lie in the disk $|x|<R$. Further restrictions will be imposed on R later when needed. There are n branches of $A(x)$ at infinity and they are of the form

$$A(x) = x^q \left\{a_0 + \sum_{j=1}^{\infty} a_j x^{-j/m}\right\} \tag{2.2}$$

where q is a rational number, $1<q\leqq k+1$. The values of q are determined from the diagram of Puiseux [6, section 2 for $k=1$ and 2]. The coefficients a_j are uniquely determined by $P(x, y)$. The series in (2.2) converges for $|x|\geqq R$. We may assume that R is so large that

$$\sum_{j=1}^{\infty} |a_j| R^{-j/m} < \frac{1}{2} |a_0| \tag{2.3}$$

so that for $|x| \geqq R$

(2.4) $$|x^{-q}A(x)-a_0| < \frac{1}{2}|a_0|.$$

At the poles of order α the corresponding branches of $A(x)$ have

(2.5) $$q = \frac{\alpha+k}{\alpha}$$

and m is an integer which divides α.

To each of the branches of $A(x)$ corresponds a reduced first degree DE

(2.6) $$w^{(k)}(z) = [w(z)]^q\{a_0+\sum_{j=1}^{\infty} a_j[w(z)]^{-j/m}\}.$$

Suppose now that $w(z)$ satisfies a reduced DE of type (2.6) and has a pole of order α at $z=z_0$. The solution is given by a convergent Laurent series

(2.7) $$w(z) = \sum_{p=0}^{\infty} c_p(z-z_0)^{-\alpha+p}.$$

The series is absolutely convergent in a punctured disk

(2.8) $$0 < |z-z_0| < r.$$

Here the coefficients c_p are independent of z_0 since the equation (2.6) is autonomous. The leading coefficient c_0 is a root of

(2.9) $$a_0(c_0)^{k/\alpha} = (-1)^k\alpha(\alpha+1) \ldots (\alpha+k-1).$$

Once c_0 has been chosen then all the other coefficients c_p are uniquely determined and this implies that r, the radius of convergence, is known. The radius can have a value taken from a set of at most kn elements. Let ϱ be the infimum of these kn positive numbers.

It is now seen that with each pole of $w(z)$ we can associate a polar neighborhood

(2.10) $$U_n = \{z;\ |z-z_n| < \varrho\}$$

which contains one and only one pole $z=z_n$. The multiplicity of the pole may differ from one pole to the next but α is an integer and at most equal to $A=k[\min q-1]^{-1}$. This means that we can estimate the enumerative function of the poles

(2.11) $$n(r, \infty; w) < 4A\varrho^{-2}r^2[1+o(1)].$$

The estimate is obtained by determining how many disks of radius $\frac{1}{2}\varrho$ can be placed in a large disk of radius r when no overlapping is allowed. Each small disk can contain a pole of multiplicity at most A. $N(r, \infty; w)$ is at most half of (2.11). Since the order is finite the proximity function of the poles

$$m(r, \infty; w) = O(\log r)$$

so that

$$T(r, w) < Cr^2. \tag{2.12}$$

Thus the Nevanlinna order is at most 2. On the other hand, this value 2 is reached whenever the solution is an elliptic function of z or of a linear function of z. Thus we have proved

Theorem 1. *The Nevanlinna order of a transcendental meromorphic solution of a* k*th order Briot—Bouquet differential equation is at most two and this value is reached if the solution is doubly-periodic.*

3. The case $q = k+1$

This case has many interesting features and merits a detailed study. It is present when $P_n(x)$ attains its maximum allowable degree $\delta_n = (k+1)n$ so that

$$P_{n,(k+1)n} \neq 0. \tag{3.1}$$

Then at infinity all branches of $A(x)$ take the form

$$A(x) = x^{k+1}\{a_0 + \textstyle\sum_{j=1}^{\infty} a_j x^{-j}\}. \tag{3.2}$$

Here $\alpha = 1$ so that all poles are simple, $m=1$ since m divides α. There are n distinct sets of coefficients $\{a_j\}$ and corresponding radii R_p, $p=1, 2, \dots, n$ so that for the pth branch the series (3.2) is absolutely convergent for $|x| > R_p$.

To each of these branches of $A(x)$ corresponds a reduced first degree *DE*

$$w^{(k)}(z) = [w(z)]^{k+1}\{a_0 + \textstyle\sum_{j=1}^{\infty} a_j [w(z)]^{-j}\}. \tag{3.3}$$

Besides these *DE*'s we are also interested in the *associated DE*'s

$$w'(z) = [w(z)]^2\{c_0 + \textstyle\sum_{j=1}^{\infty} c_j [w(z)]^{-j}\}. \tag{3.4}$$

Here right at the outset we are faced by the question: Given a canonical solution $w(z)$ of (1.2) which satisfies a particular branch of (3.3) does it also have to satisfy an equation of type (3.4) If so, how is the equation to be found? We shall sketch a method of construction for the second equation which amounts to an existence proof.

Theorem 2. *Let $A_i(x)$ be the ith branch of $A(x)$ at ∞. Let $w(z)$ be a solution of (1.2) which satisfies*

$$w^{(k)}(z) = A_i[w(z)] \tag{3.5}$$

in the polar neighborhoods U_{ij} of the poles z_{ij} of $w(z)$. Then there exist k convergent Laurent series

$$B_{i\nu}(x) = x^2\{c_{0\nu} + \sum_{j=1}^{\infty} c_{j\nu} x^{-j}\} \tag{3.6}$$

so that $w(z)$ satisfies one of the DE's

$$w' = B_{i\nu}(w) \tag{3.7}$$

in polar neighborhoods U_{ij}. Here c_0 is one of the kth roots of

$$a_0 = k!\, c_0^k \tag{3.8}$$

and the coefficients are determined uniquely in terms of c_0 and the a_j's.

Preliminary sketch. We shall discuss the case $k=3$ here. The general case will come out as a byproduct of other considerations. We are given

$$w^{(3)} = w^4\{a_0 + \sum_{j=1}^{\infty} a_j w^{-j}\}. \tag{3.9}$$

We postulate

$$w' = w^2\{c_0 + \sum_{j=1}^{\infty} c_j w^{-j}\} \equiv w^2 B(w). \tag{3.10}$$

Then

$$w'' = [2wB(w) + w^2 B'(w)]\, w' = 2w^3 B^2(w) + w^4 B(w) B'(w)$$

and

$$w^{(3)} = 6w^4 B^3(w) + w^5 B^2(w)[8B'(w) + wB''(w)] + w^6 B(w)[B'(w)]^2. \tag{3.11}$$

At $w=\infty$ these three terms are of order 4, 3 and 2, respectively. Here we substitute the series for B, B' and B'', multiply out, collect terms and equate the result to (3.9). The first two results are

$$6c_0^3 = a_0, \quad 12c_0^2 c_1 = a_1. \tag{3.12}$$

The coefficients c_j are uniquely determined once the value of c_0 has been chosen. For a particular c_j makes its first appearence among the coefficients of w^{4-j} in the first term of (3.11) where it is multiplied by a positive integer times c_0^{j-1}. In the second complex c_j goes with w^{3-j} and in the third complex with w^{2-j}. This shows that c_j is uniquely determined in terms of c_0 and $a_0, a_1, \ldots, a_j$. Questions of convergence are left open for the time being. ■

Our main interest in the associate DE (3.5) is due to our desire to study

$$w'(z)[w(z)]^{-2} \equiv Q(z) \tag{3.13}$$

for z in the open set

(3.14) $$U = \{z;\ |w(z)| > 2\}.$$

Such a study will lead to a number of important results. We shall need

Lemma 9. *Let $w(z)$ be a solution of* (1.2) *where* $p_{n\,(k+1)n} \neq 0$. *Then for* $z \in U$

(3.15) $$w^{(k)}(z) = B(z, w)[w(z)]^{k+1}$$

where

(3.16) $$|B(z, w)| \leqq 2M+1 \quad \text{and} \quad M = \max |p_{ij}|.$$

Proof. We have $|P_j(w)| \leqq 2M|w|^{(k+1)j}$ and well-known estimates of the roots of an algebraic equation in terms of the coefficients yield (3.16). ■

We aim to show that $Q(z)$ is holomorphic and bounded for z in U. We have an estimate of $w^{(k)}(z)$ in terms of $w(z)$ in U and we want to find an estimate of $w'(z)$ in terms of $w(z)$. This is a variant of the classical Hadamard—Kolmogorov—Landau problem: Given estimates of a function and its derivative of order k, find bounds for its jth derivative, $1 \leqq j \leqq k$. For the literature see [4] and [10]. In the H—K—L theory the estimates refer to normed vector spaces and take the form

(3.17) $$\|f^{(j)}\|^k \leqq C_{jk} \|f\|^{k-j} \|f^{(k)}\|^j.$$

If we have

(3.18) $$\|f^{(k)}\| \leqq (2M+1)\|f\|^{k+1}$$

then

(3.19) $$\|f'\| \leqq C\|f\|^2$$

for some fixed constant C. This is the desired estimate, but where is the space? We shall find one below when the situation is clearer.

Consider now a particular branch of $A(x)$ given by (3.2) in some subset of U, in particular the polar neighborhood U_{ij} of the pole z_{ij}. All the poles are simple and at $z = z_{ij}$ we have an expansion

(3.20) $$w(z) = \sum_{p=0}^{\infty} b_p (z - z_{ij})^{p-1}$$

and

(3.21) $$Q(z) = \sum_{p=0}^{\infty} q_p (z - z_{ij})^p.$$

Since the DE is autonomous the coefficients b_p and q_p are independent of z_{ij} and depends only upon what branch of $A(x)$ and what value of b_0 have been chosen. There are n different branches to which correspond kn sets of coefficients $\{b_p\}$ and $\{q_p\}$. There are also $2kn$ radii of convergence R_i^1 and R_i^2 for the two types of series (3.20) and (3.21).

Here R_i^1 is the distance from a pole in the i-series to the nearest pole of $w(z)$ and this pole may be either in the i-series or in a different j-series. In either case the direction from the pole to the nearest pole is the same for all poles of the i-series. If the nearest pole is also in the i-series, this means that the i-series contains a string of equidistant poles $z_{i0}+n\omega_i$. If the nearest pole is not in the i-series but say instead in the j-series, then at a distance R_j^1 there is another pole which may belong to the j-series or not. After a finite number of steps, at most n, we encounter a string of poles which have already figured in the process. If all n branches of $A(x)$ have been accounted for, we are through. If not, then pick a pole in one of the complementary sets and repeat the argument.

The second set of radii, the R_i^2, also gives important information. The only finite singularities of $Q(z)$ are the zeros of $w(z)$ and R_i^2 measures the distance from the pole z_{ij} to the nearest zero, say ζ_{ij}. Moreover, arg $(z_{ij}-\zeta_{ij})$ is the same for all j and depends only on i, i.e. on the branch of $A(x)$. Thus a string of equidistant poles brings with it a string of equidistant zeros. This suggests strongly that our solution $w(z)$ is periodic, simply or doubly periodic, with the periods ω_i where of course at most two of them can be linearly independent over the domain of the ntegers.

4. More on $Q(z)$

Suppose now that $w(z)$ is a single-valued solution, not a constant or a rational function or an entire function, and consider the boundary set $\partial U=\{z;\ |w(z)|=2\}$. It consists of infinitely many ovals V_{ij} and possibly one or more curves V which extend to infinity. Here $i=1, 2, \ldots, n$ and for a fixed i all ovals V_{ij} are congruent by the autonomous property of the *DE*. We may assume that each oval V_{ij} contains one and only one pole z_{ij}. If this is not true at the outset, it may be achieved by a suitable affine transformation $w=av$, $z=bs$. This changes the coefficients in the various expansions and affects the bound M of the coefficients in the resulting differential equation. This is immaterial for our purpuses but what counts is that if a is sufficienly small the ovals are forced to separate so that each of the new ovals contains one and only one pole. We assume that this has already been achieved.

Fix the value of i and consider an oval V_{ij}. Here there are two possibilities: either V_{ij} lies entirely within the circle $|z-z_{ij}|=R_i^2$ or partly outside the circle. For a fixed value of i the same alternative holds for all values of j. In the first case $|Q(z)|$ is bounded inside and on V_{ij} with the same bound for all values of j. If the second alternative holds we fall back on Lemma 9 and the Hadamard—Kolmogorov—Landau theory.

For a fixed i we can parametrize V_{ij} in terms of arclength, $0\leqq t\leqq L$. We consider the space of functions $t\to f(t)$ which are continuous and periodic with period L and use the metric defined by the sup norm. Then $w(z)$, $z\in V_{ij}$ belongs to this space and so do its derivatives of all orders. The H—K—L theory applies to this space and shows the existence of a number C such that

$$\|w'\|^k \leqq C\|w\|^{k-1}\|w^{(k)}\| = C2^{k-1}\|w^{(k)}\|. \tag{4.1}$$

But on V_{ij} Lemma 9 still holds so that

$$\|w'\| \leqq 4[C(2M+1)]^{1/k} \tag{4.2}$$

and hence

$$\|Q\| \leqq [C(2M+1)]^{1/k} \tag{4.3}$$

and this is now the bound of $|Q(z)|$ inside and on all ovals V_{ij} for a fixed i. Here C depends upon i which affects the parametrization and the arclength. But there are at most n different values for C that can occur so the conclusion is that $|Q(z)|$ is uniformly bounded inside and on all the ovals V_{ij}.

Next we have to prove that there can be no level curves V extending to infinity. To prove this we need Theorem 2.

Proof of Theorem 2. By (3.20) we have

$$w(z) = \sum_0^\infty b_p(z-z_{ij})^{p-1} \tag{4.4}$$

convergent for $0<|z-z_{ij}|<R_i^1$. The corresponding polar neighborhood is

$$U_{ij} = \{z, |z-z_{ij}| < \varrho\}, \quad \varrho = \min_i R_i^1. \tag{4.5}$$

These neighborhoods may be too large for our present purposes so we define reduced polar neighborhoods

$$U_{ij}^* = \{z, |z-z_{ij}| < \sigma\}, \quad \sigma \leqq \min_i R_i^2. \tag{4.6}$$

Then $Q(z)$ is holomorphic and uniformly bounded in all the U_{ij}^* 's, that is

$$\frac{w'(z)}{[w(z)]^2} = \sum_0^\infty q_p(z-z_{ij})^p \tag{4.7}$$

is holomorphic and bounded in U_{ij}^* uniformly in j. Then by the inverse function theorem

$$z-z_{ij} = \sum_1^\infty d_m[w(z)]^{-m} \tag{4.8}$$

and this is valid for large values of $|w(z)|$, that is in some neighborhood of z_{ij} independent of j. We may assume that the series (4.8) is absolutely convergent for $|w(z)|>3$, safely inside the oval V_{ij}, and that

$$\sum_1^\infty 3^{-m}|d_m| < \frac{1}{3}R_i^2. \tag{4.9}$$

There is evidently a B such that

$$|q_p| \leqq B2^p(R_i^2)^{-p}, \ \forall p. \tag{4.10}$$

It follows that

$$\frac{w'(z)}{[w(z)]^2} = \sum_0^\infty q_p\{\sum_1^\infty d_m[w(z)]^{-m}\}^p = \sum_{p=0}^\infty q_p \sum_{m=p}^\infty d_{pm} w^{-m} \tag{4.11}$$

$$= q_0 + q_1 d_{11} w^{-1} + (q_1 d_{12} + q_2 d_{22}) w^{-2} + \dots$$

$$= c_0 + \sum_1^\infty c_m [w(z)]^{-m}$$

where the double series is absolutely convergent and the rearrangement is permitted by the double series theorem of Weierstrass. This proves the validity of the expansion (2.11) in suitable polar neighborhoods for each fixed i independent of j.

We shall now prove that there can be no level curve $V: |w(z)| = 2$ which extends to infinity. To this end we take the reciprocal of (4.11) and obtain a result of the form

$$\frac{dz}{dw} = \sum_{j=0}^\infty e_j w^{-j-2} \tag{4.12}$$

convergent for large $|w|$, say for $|w| \geqq R$. Integration gives

$$z(w) = z_0 + \sum_0^\infty \frac{e_j}{j+1} \{w_0^{-j-1} - w^{-j-1}\} \tag{4.13}$$

also convergent for $|w| \geqq R$. This shows that the inverse image of the circle $|w| = R$ cannot possibly be an unbounded curve of type V. Without restricting the generality we may assume $R \leqq 2$. We have then finally

Theorem 3. *The set* $U = \{z, |w(z)| > 2\}$ *coincides with the union of the interiors of the ovals* V_{ij}, *one around each pole* z_{ij}, *and* $|Q(z)|$ *is uniformly bounded in* U.

This was proved under the assumption that $P_{n,(k+1)n} \neq 0$. We can dispense with this assumption as long as the parameter q of (2.2) exceeds 1. The poles are now of an order $\alpha > 1$ and $q = (\alpha + k)/\alpha$. In this case

$$Q(z) = w'(z)[w(z)]^{-1-1/\alpha}. \tag{4.14}$$

The formulas and the argument become rather messy so we shall not elaborate this point any further.

5. Determinateness

After these lengthy preparations we can now prove

Theorem 4. *Let $w(z)$ be a solution of*

$$[w^{(k)}]^n+\sum_{j=1}^n P_j(w)[w^{(k)}]^{n-j} = 0 \tag{5.1}$$

where the degrees of the coefficients are $\delta_j \leqq (k+1)j$. Analytic continuation of $w(z)$ along a path of finite length leads to a definite limit, finite or infinite.

Proof. Let the solution $w(z)$ be given by its initial values of $w, w', \ldots, w^{(k-1)}$ at $z=z_0$ and let the analytic continuation take place along a path C from $z=z_0$ to $z=\zeta$. Here C is supposed to be of finite length and the analytic continuation has encountered no singularities except possibly at the endpoint $z=\zeta$. Let the image of C in the w plane be $\Gamma, \Gamma=w(C)$. If $w(z)$ does not have a definite limit as $z\to\zeta$ then Γ cannot be of finite length.

Suppose to start with that Γ is confined to the disk $|w|<R$ where R is so large that all the branch points of the algebraic function $Y=A(x)$, defined by (2.1), belong to the disk. The only singularities of $A(x)$ for $|x|<R$ are algebraic branch-points where all the determinations of $A(x)$ are finite. This means that there is a constant $B=B(R)$ such that

$$|A(x)| \leqq B \quad \text{for} \quad |x| < R \tag{5.2}$$

and this holds for all the determinations of $A(x)$. Translated in terms of the *DE* (5.1), this means that if $|w(z_0)|<R$ and Γ stays in the disk $|w|\leqq R$, then everywhere on C we have

$$|w^{(k)}(z)| \leqq B. \tag{5.3}$$

Successive integrations show that $|w'(z)|$ is bounded by an expression of the form

$$\sum_{j=1}^k B_j \frac{[l(z)]^{j-1}}{(j-1)!} \quad \text{where} \quad B_k = B, \quad B_j \geqq |w^{(j)}(z_0)| \tag{5.4}$$

and where $l(z)$ is the length of the arc on C from z_0 to z and hence $\leqq l(C)$. Since the length of the arc Γ,

$$\int_{z_0}^{\zeta} |w'(s)|\,|ds|, \tag{5.5}$$

is evidently bounded, $w(z)$ must tend to a finite definite limit as $z\to\zeta$.

Hence for indetermination to be possible the image curve Γ must escape from the disk $|w|\leqq R$. It may cross the boundary $|w|=R$ a finite number of times or infinitely often. We start with the first case. There is then a last crossing after which the curve stays outside the disk. Let $z=z_1$ be the point on C where Γ has the last crossing and let Γ_1 be the image of C from $z=z_1$ to $z=\zeta$. We apply reciprocation to Γ_1

$$v(z)=\frac{1}{w(z)},\quad v'(z)=-\frac{w'(z)}{[w(z)]^2}. \tag{5.6}$$

Let $\varkappa$ be the image of Γ_1 under the reciprocation. The length of $\varkappa$ is

$$l(\varkappa)=\int_{z_1}^{\zeta}|v'(s)|\,|ds|. \tag{5.7}$$

Here we again recall the importance of the behavior of $A(x)$ for large values of $|x|$. If the exponent $q=k+1$, its maximal value, then $v'(z)=-Q(z)$ which is holomorphic and bounded in any domain bounded away from the zeros of the solution. This means that the integral in (3.7) has a finite value, $v(z)$ tends to a finite limit which may be zero. Hence $w(z)$ has a definite limit which may be infinity.

Now the same conclusion is valid for any q with $1<q<k+1$ for by (4.14) already $w'(z)[w(z)]^{-1-1/\alpha}$ is bounded for large values of $|w(z)|$ and *a fortiori* this is true if the exponent is lowered from $-1-1/\alpha$ to -2.

We don't have to consider the case $q=1$ for then the solution is an entire function and evidently tends to a finite limit for any path C of finite length.

Now if $0<q<1$ it is found that $w(z)$ becomes infinite with z,

$$w(z)=O(z^{k/(1-q)}) \tag{5.8}$$

so the existence of a definite infinite limit is obvious. Similar estimates hold for $q<0$ but now multi-valued solutions may appear.

It should be noted that our estimates are tied up with the assumption $\delta_j\leqq(k+1)j$. If this is violated the situation may change. Thus the *DE*

$$w^{(k)}=w^{2k+1}$$

is satisfied by a constant multiple of $z^{-1/2}$.

We have seen that under the assumptions made $w(z)$ must have a definite finite or infinite limit if the image curve Γ crosses $|z|=R$ a finite number of times. But if there are infinitely many crossings it is easy to show that the points of intersection as well as the intermediary arcs tend to a definite limit located on $|w|=R$. This takes care of all possibilities. ■

6. Entire solutions

Suppose that the equation is

(6.1) $$[w^{(k)}]^n+\sum_{j=1}^{n} P_j(w)[w^{(k)}]^{n-j} = 0$$

where $\delta_j \leqq j$. Then there can be no poles and if there exists a canonical solution it must be an entire function. Here we have

Theorem 5. *The Nevanlinna order of an entire solution of* (6.1) *is* $\leqq 1$.

Proof. Let

(6.2) $$U = \{z, |w(z)| > 2\}.$$

For $z \in U$ we have

(6.3) $$w^{(k)}(z) = B(z, w)w(z), \quad |B(z, w)| \leqq 2M+1$$

where $M = \max |p_{ij}|$ and p_{ij} is the coefficient of w^i in $P_j(w)$. To prove this just replace w' by $w^{(k)}$ in the proof of Lemma 3 of [5].

The set U is not necessarily connected and its components extend to infinity. There can be no finite "islands" since poles are excluded. The analogue of (2.6) reads

(6.4) $$w^{(k)}(z) = w(z)\left\{a_0+\sum_{j=1}^{\infty} a_j[w(z)]^{-j}\right\}$$

where a_0^k is a root of the characteristic equation

(6.5) $$C(t) \equiv t^n+p_{1,1}t^{n-1}+p_{2,2}t^{n-2}+\ldots+p_{n,n} = 0.$$

Let $w(z)$ be a solution of (6.1) with initial values

(6.6) $$w(0) = c_0,\ w'(0) = c_1,\ \ldots,\ w^{(k-1)}(0) = c_{k-1}$$

and set

(6.7) $$|c_j| = C_j,\ f(r) = \sum_0^{n-1} C_j r^j, \quad (2M+1)^{1/k} = B,$$

(6.8) $$\frac{1}{k}\sum_{j=1}^{k} \exp(\omega^j u) = K(u), \quad \omega = \exp\left(\frac{2\pi i}{k}\right),$$

(6.9) $$\sum_{n=1}^{\infty} \frac{u^{nk-1}}{(nk-1)!} = N_k(u).$$

Then

(6.10) $$N_k(u) = K'(u)$$

and

(6.11) $$F(r) = f(r)+B\int_0^r N_k[B(r-s)]f(s)\,ds$$

is a majorant of $w(re^{i\theta})$. It is clearly an entire function of r of order 1. This implies that the order of $w(z)$ is at most 1. ■

We shall see that this limit is reached in various special cases. We start by proving the extension of Lemma 6.

Lemma 10. *An entire solution of a BB equation of type* (6.1) *which has only a finite number of zeros is necessarily a constant multiple of* $\exp(az)$ *where* $C(a^k)=0$.

Proof. For the first and second order cases see Lemma 1 of [5]. The same type of argument applies for $k>2$. A solution of the desired property will be of the form

$$w(z) = Q(z)F(z) \quad \text{with} \quad F(z) = e^{az} \tag{6.12}$$

and where $Q(z)$ is a polynomial in z of degree $m \geqq 0$. The constant a is to be determined. Then

$$w^{(k)}(z) = \left\{\sum_{j=0}^{n} \binom{k}{j} a^j Q^{(k-j)}(z)\right\} F(z). \tag{6.13}$$

This is substituted in the *DE* giving a result of the form

$$\sum_{j=0}^{n} S_j(z)F(jz) = 0 \tag{6.14}$$

where the $S_j(z)$ are polynomials in z each of which must vanish identically if $w(z)$ is a solution. The terms of highest degree in S_n come from

$$C(a^k)[Q(z)]^n \tag{6.15}$$

and will vanish identically iff a^k is a root of the characteristic equation (6.5). This determines the admissible values of a. Terms of next highest order in S_n come from

$$KC'(a^k)[Q(z)]^{n-1}Q'(z). \tag{6.16}$$

Here there are two possibilities: either $Q'(z)\equiv 0$ or a^k is a multiple root of (6.5). In the first case $Q(z)$ is a constant and we have only to determine what conditions are imposed on the coefficients p_{ij} if Ce^{az} is a solution. By assumption $p_{ij}=0$ for $i>j$. The terms p_{ii} along the main diagonal determine the values of a^k. The coefficients above the main diagonal are not arbitrary. In fact we must have

$$\sum p_{i,i+m} a^{(n-i)k} = 0, \quad m = 1, 2, \ldots, n. \tag{6.17}$$

Any set of numbers $p_{i,j}$ that satisfies these conditions will do. The constant C is arbitrary.

If, however, it is the first factor $C'(a^k)$ which is zero, we notice that in $S_n(z)$ the terms of highest order now come from

$$Q^{n-1}Q'' \quad \text{and} \quad Q^{n-2}(Q')^2 \tag{6.18}$$

which are of the same degree $mn-2$ if $m>1$. Suppose that

$$Q(z) = b_0 z^m + b_1 z^{m-1} + \ldots.$$

Then

$$nQ^{n-1}Q'' = nm(m-1)b_0^n z^{mn-2} + n(m-1)(m-2)b_0^{n-1}b_1 z^{mn-3} + \dots$$

$$n(n-1)Q^{n-2}(Q')^2 = n(n-1)m^2 b_0^n z^{mn-2} + n(n-1)[(n-2)m^2+2m]b_0^{n-1}b_1 z^{mn-3} + \dots.$$

These two expressions have to be multiplied by polynomials in a and added. Actually the multipliers are simply $\frac{1}{2}k(k-1)a^{nk-2}$ and k^2a^{nk-2} so unless $a=0$ the two terms in (6.18) cannot be made to vanish. It follows that we must have $Q'(z)\equiv 0$ which brings us back to the first case and the same conclusion. ■

Lemma 11. *An exponential polynomial in e^{az} and/or e^{-az} will satisfy a BB-equation of the kth order and sufficiently high degree.*

Proof. For $k=1$ and 2 see Lemma 2 of [5]. The same argument applies for $k>2$. Let S_1 be a finite set of distinct integers and set $m=\max_{j\in S_1}|j|$. Let S_p be the pth sum-set, i.e. the set of all integers of the form

(6.19) $$j_1+j_2+\dots+j_p$$

where each $j\in S_1$ and are not necessarily distinct. Then the number of distinct element of S_p is at most $mp+1$. Suppose that

(6.12) $$w(z) = \sum_{j\in S_1} c_j e^{jaz}$$

is the given exponential sum. Then

$$[w^{(k)}(z)]^n = \sum_{j\in S_n} C_j(a)e^{jaz}$$

and the same exponentials figure in $p_v(w)[w^{(k)}]^{n-v}$. On the other hand, (6.1) involves $\frac{1}{2}(k+1)n(n+1)$ coefficients p_{ij}. We substitute (6.20) in (6.1), expand, collect terms and equate the coefficients of e^{jaz} to zero. This gives at most $mn+1$ linear equations for the determination of the p_{ij}'s. If $\frac{1}{2}(k+1)n(n+1)>m(n+1)$, i.e. $(k+1)n>2m$ there are certainly more unknowns then equations. The corresponding matrices may be assumed to have positive ranks. We can then determine a certain number of the p_{ij}'s in terms of the c_j's and the remaining arbitrarily chosen p_{ij}'s. This gives one or more *DE*'s of the desired type satisfied by the given exponential sum. ■

7. *Final remarks.* Does Lemma 8 hold for $k>2$? The available evidence favors such an hypothesis.

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Bibliography

1. CHAZY, J., Sur les équations differentielles du troisième ordre et d'ordre superieur dont l'intégrale générale a des points critiques fixes. *Acta Math.*, **24** (1911), 317—385.
2. FUCHS, L., Über Differentialgleichungen deren Integrale feste Verzweigungspunkte besitzen. *Sitzungsber. der Akademie, Berlin* (1884), 699—720.
3. GOL'DBERG, A. A., On single-valued solutions of first order differential equations. *Ukrain. Mat. Žurnal.* **8**:3 (1956), 254—261. (Russian, NRL Translation 1224, 1970).
4. HILLE, E., Generalizations of Landau's inequality to linear operators. *Linear Operators and Approximations, ISNM*, **20** (1972), 20—32.
5. HILLE, E., Remarks on Briot—Bouquet differential equations. I. *Comment. Mathematicae*, **21** (1978), 119—132.
6. HILLE, E., Remarks on Briot—Bouquet differential equations. II. *Math. Anal. Appl.*, **65** (1978).
7. HILLE, E., Second order Briot—Bouquet differential equations. *Acta Scientiarum Mathematicarum*, Szeged, **40** (1978), 63—72.
8. INCE, E. L., *Ordinary differential equations*. Dover, New York, (1944).
9. SCHLESINGER, L., *Einführung in die Theorie der Differentialgleichungen mit einer unabhängige Variabeln*. Sammlung Schubert XIII, Göschen Leipzig. (1904) [Chapter 8].
10. TREBELS, W., and WESTPHAL, U., A note on the Landau—Kallman—Rota—Hille inequality. *Linear Operators and Approximation. ISNM*, **20** (1972), 115—119.

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