Mathematical aspects of 't Hooft's eigenvalue problem in two-dimensional quantum chromodynamics

Part II. Behavior of the eigenfunctions of BEP and HEP at the singular boundary points

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The aim of this paper is to prove the following

Theorem 1. *Each eigenfunction* $\Phi(x)$, $0<x<1$, *of 't Hooft's eigenvalue problem (HEP) is Hölder continuous on the closed interval* $0 \leqq x \leqq 1$, *and*

$$\Phi(0) = 0, \quad \Phi(1) = 0.$$

Moreover, $\Phi(x)$ *disappears at the singular end points* $x=0$ *and* $x=1$ *at least like a positive power of* x *and* $1-x$, *respectively. That is, there are positive numbers* $\beta_0, \beta_1, c_0, c_1$ *such that*

$$|\Phi(x)| \leqq c_0 \cdot x^{\beta_0} \quad \text{for} \quad 0 \leqq x \leqq 1$$

and

$$|\Phi(x)| \leqq c_1 \cdot (1-x)^{\beta_1} \quad \text{for} \quad 0 \leqq x \leqq 1.$$

This result is an immediate consequence of Theorem 2 which is stated and proved at the end of our paper.

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The physical significance of HEP has been explained by 't Hooft in [4]. Further details and references have been stated in **part 1** of our investigations (cf. [2]), to which we in the following shall briefly refer as I.

For the convenience of the reader, we shall repeat the definition of HEP, and state once more some of the results of I.

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Definition of HEP. Determine real numbers λ, and real valued functions $\Phi(x)$ on $0<x<1$ with

$$\int_0^1 |\Phi(x)|^2\,dx = 1, \quad \int_0^1\int_0^1 \frac{|\Phi(x)-\Phi(y)|^2}{|x-y|^2}\,dx\,dy < \infty,$$

and with a Hölder continuous first derivative on $0<x<1$ such that

$$\lambda\Phi(x) = \left\{\frac{\alpha_1}{x}+\frac{\alpha_2}{1-x}\right\}\Phi(x) - \mathscr{P}\int_0^1 \frac{\Phi(\xi)}{(\xi-x)^2}\,d\xi, \quad 0<x<1, \tag{1}$$

and that

$$\Phi(0)=0, \quad \Phi(1)=0 \quad \text{(i. g. s.)}. \tag{2}$$

Here, α_1 and α_2 denote real parameters >-1, i.g.s. stands for "in the generalized sense", that is, $\lim_{\varepsilon\to+0}\frac{1}{\varepsilon}\int_0^\varepsilon |\Phi(x)|^2\,dx=0$, $\lim_{\varepsilon\to+0}\frac{1}{\varepsilon}\int_{1-\varepsilon}^1 |\Phi(x)|^2\,dx=0$, and $\mathscr{P}\int_0^1\ldots$ denotes the "regular cut-off" defined as

$$\lim_{\varepsilon\to+0}\int_0^1 \frac{1}{2}\left[(\xi-x-i\varepsilon)^{-2}+(\xi-x+i\varepsilon)^{-2}\right]\Phi(\xi)\,d\xi.$$

In the following, let $\mathscr{H}=\{(x,y)\in\mathbf{R}^2\colon y>0\}$ be the upper half plane, and denote by $\hat{\mathscr{H}}$ the closure $\overline{\mathscr{H}}=\{(x,y)\colon y\geqq 0\}$ of $\mathscr{H}$ minus the two end points $x=0$ and $x=1$ of the interval $0<x<1$ on the x-axis.

The main role in tackling HEP has been played by the eigenvalue problem BEP.

Definition of BEP. Determine real numbers λ and real valued functions $v(x,y)$ with Hölder continuous first derivatives on $\hat{\mathscr{H}}$ which are harmonic in $\mathscr{H}$ and satisfy

$$\int_0^1 |v(x,0)|^2\,dx = 1 \quad \iint_{\mathscr{H}} |\nabla v|^2\,dx\,dy < \infty,$$

as well as the boundary conditions

$$v(x,0)=0 \quad \text{for} \quad x\notin[0,1], \tag{3}$$

$$v(x,0)=0 \quad \text{(i. g. s.)} \quad \text{for} \quad x=0 \quad \text{and} \quad x=1, \tag{4}$$

$$-\pi v_y(x,0)+\left\{\frac{\alpha_1}{x}+\frac{\alpha_2}{1-x}\right\}v(x,0) = \lambda v(x,0) \quad \text{for} \quad 0<x<1. \tag{5}$$

We have proved in I, *2.3*, that HEP and BEP are equivalent problems in the following sense:

If $v(x,y)$ is an eigenfunction of BEP to the eigenvalue λ, then $\Phi(x)=v(x,0)$, $0<x<1$, is an eigenfunction of HEP corresponding to the eigenvalue λ. Conversely, if $\Phi(x)$ is an eigenfunction of HEP, then

$$v(x,y) = \operatorname{Im}\frac{1}{\pi}\int_0^1 \Phi(\xi)(\xi-x-iy)^{-1}\,d\xi, \quad (x,y)\in\mathscr{H},$$

is eigenfunction of BEP, where Φ and v belong to the same eigenvalue.

Moreover, we have verified in I, *4.1—5* and *5.2,* that there exists a sequence $\{\lambda_n\}$ of real numbers λ_n such that

$$0 < \lambda_1 < \lambda_2 \leqq \lambda_3 \leqq \ldots \leqq \lambda_n \leqq \lambda_{n+1} \leqq \ldots, \lim_{n\to\infty} \lambda_n = +\infty,$$

and a sequence of eigenfunctions $e_1(x, y)$, $e_2(x, y)$, $e_3(x, y), \ldots,$ which are real analytic and harmonic in $\hat{\mathscr{H}}$. Every $e_n(x, y)$ is an eigenfunction of BEP to the eigenvalue λ_n, while $\Phi_n(x)=e_n(x, 0)$, $0<x<1$, is an eigenfunction of HEP to λ_n, and $\{\Phi_n\}_{n=1,2,\ldots}$ forms a complete orthonormal system in $L_2([0, 1])$ whence the spectrum is purely discrete and consists only of denumerably many eigenvalues of finite multiplicity.

Each eigenfunction $e_n(x, y)$ is element of the Hilbert space H consisting of all functions $\psi(x, y)$ which are of the Sobolev class $W^1_{2,\mathrm{loc}}(\mathscr{H})$ and satisfy

$$\psi(x, 0) = 0 \quad \text{for} \quad x \notin [0, 1]$$

and

$$\iint_{\mathscr{H}} |\nabla\psi|^2 \, dx\, dy < \infty.$$

Moreover, each function $\psi \in H$ satisfies

$$(6)\quad \begin{aligned} &\int_0^1 \left\{\frac{1}{x}+\frac{1}{1-x}\right\} |\psi(x, 0)|^2 \, dx + \frac{1}{2}\int_0^1\int_0^1 \frac{|\psi(x, 0)-\psi(y, 0)|^2}{|x-y|^2}\, dx\, dy \\ &\leqq \pi \iint_{\mathscr{H}} \{|\psi_x|^2+|\psi_y|^2\}\, dx\, dy. \end{aligned}$$

For the sake of brevity, let us fix the following *notation*:

Let $\Phi(x)$ be an eigenfunction of HEP to the eigenvalue λ, and set

$$(7)\quad \begin{aligned} F(z) &= \frac{1}{\pi}\int_0^1 \frac{\Phi(\xi)}{\xi - z}\, d\xi = u(x, y)+iv(x, y), \quad z = x-iy, \\ u(x, y) &= \mathrm{Re}\, F(z), \quad v(x, y) = \mathrm{Im}\, F(z). \end{aligned}$$

Then,

$$(8)\quad \Phi(x) = v(x, 0), \quad 0 < x < 1,$$

and $v(x, y)$ is an eigenfunction of BEP of the class H and satisfies

$$(9)\quad \begin{aligned} &\pi \iint \nabla v \cdot \nabla\zeta \, dx\, dy + \int_0^1 \left\{\frac{\alpha_1}{x}+\frac{\alpha_2}{1-x}\right\} v(x, 0)\zeta(x, 0)\, dx \\ &= \lambda \int_0^1 v(x, 0)\zeta(x, 0)\, dx \quad \text{for all} \quad \zeta \in H, \end{aligned}$$

cf. I, (4.28).

Now we are going to prove Theorem 1.

Firstly, we note that there is a number $k>1$ such that the following can be achieved:

For each $R>0$, and for each $z_0=x_0+iy_0 \in \mathbf{C}$, there is a real-valued function $\eta(x, y)$ of the class $C_c^\infty(B_{2R}(z_0))$ such that

(10) $$\eta(x,y) \equiv 1 \quad \text{on} \quad B_R(z_0), \quad 0 \leqq \eta(x,y) \leqq 1 \quad \text{otherwise},$$

and

(11) $$|\nabla \eta(x,y)| \leqq k/R \quad \text{on } \mathbf{C}.$$

Here, $B_r(z_0)$ denotes the open disc in $\mathbf{R}^2 \cong \mathbf{C}$ of center z_0, and of radius r.

Furthermore, by a well known reasoning (cf. [6], pp. 81—86) we can prove the following "Poincaré inequalities":

There exist numbers K^* and $K^{**}>0$ such that, for all $x_0 \in \mathbf{R}$, and all $R>0$, the following holds:

(12) $$\iint_{T_{2R}(x_0)} \psi^2 \, dx\, dy \leqq K^* R^2 \iint_{T_{2R}(x_0)} |\nabla \psi|^2 \, dx\, dy$$

for all $\psi \in W_2^1(T_{2R}(x_0))$ satisfying $\psi(x, 0)=0$ on

$$x_0-2R < x < x_0-R, \quad \text{or on} \quad x_0+R < x < x_0+2R,$$

and,

(13) $$\int_{x_0}^{x_0+2R} |\psi(x,0)|^2 \, dx \quad \left(\text{or} \int_{x_0-2R}^{x_0} |\psi(x,0)|^2 \, dx\right) \leqq K^{**} R \iint_{S_{2R}(x_0)} |\nabla \psi|^2 \, dx\, dy$$

for all $\psi \in W_2^1(S_{2R}(x_0))$ satisfying $\psi(x, 0)=0$ on

$$x_0-2R < x < x_0 \quad (\text{or on } x_0 < x < x_0+2R, \text{ resp.}).$$

Here, we have set

$$S_R(x_0) = B_R(x_0) \cap \mathscr{H}, \quad T_{2R}(x_0) = S_{2R}(x_0) - S_R(x_0).$$

Now we define the following numbers:

(14) $$\alpha_0' = \min\{\alpha_1, 0\}, \quad \alpha_1' = \min\{\alpha_2, 0\}, \quad \alpha = \min\{\alpha_1, \alpha_2, 0\},$$

that is, $-1<\alpha_0', \alpha_1', \alpha \leqq 0$.

Furthermore, we set

(15) $$R_0 = \min\left\{\frac{1}{4}, \frac{(1+\alpha_0')\pi}{2(\lambda-2\alpha_1')K^{**}}\right\}, \quad R_1 = \min\left\{\frac{1}{4}, \frac{(1+\alpha_1')\pi}{2(\lambda-2\alpha_0')K^{**}}\right\}$$

$$\sigma_j = \frac{\log\left[1+\left(\frac{1+\alpha_j'}{2k^2K^*}\right)\right]}{2\cdot\log 2}, \quad M_j = R_j^{-\sigma_j}\sqrt{\frac{\lambda}{\pi(1+\alpha)}}, \quad j=0,1.$$

Note that $R_0, R_1, \sigma_0, \sigma_1$ are positive, and that

(16) $$\lim_{\alpha_1\to -1} \sigma_0(\alpha_1) = 0, \quad \lim_{\alpha_2 \to -1} \sigma_1(\alpha_2) = 0.$$

Lemma 1. *We obtain the following estimates:*

(17) $$\iint_{S_R(0)} |\nabla v|^2\,dx\,dy \leqq M_0 R^{2\sigma_0} \quad \text{for} \quad 0 < R \leqq R_0,$$

and

(18) $$\iint_{S_R(1)} |\nabla v|^2\,dx\,dy \leqq M_1 R^{2\sigma_1} \quad \text{for} \quad 0 < R \leqq R_1.$$

Proof. Fix some R with $0<R\leqq 1/4$, and let $\eta(x, y)$ be a cut-off function belonging to R and $z_0=0$, and satisfying (10) and (11). Set $S_R=S_R(0)$, $T_R=T_R(0)$. Clearly, the function $\zeta=\eta^2 v$ is in H, hence it is an admissible test function for (9). Thus,

$$\pi\iint_{\mathscr{H}} \nabla v\cdot\nabla(\eta^2 v)\,dx\,dy+\int_0^1\left\{\frac{\alpha_1}{x}+\frac{\alpha_2}{1-x}\right\}|\eta(x,0)|^2|v(x,0)|^2\,dx$$

$$=\lambda\int_0^1|\eta(x,0)|^2|v(x,0)|^2\,dx.$$

Obviously, $w=\eta v$ is also in H, and

$$\nabla v\cdot(\eta^2 v) = |\nabla w|^2 - v^2|\nabla\eta|^2.$$

Therefore,

$$\pi\iint_{\mathscr{H}}|\nabla w|^2\,dx\,dy+\alpha_1\int_0^1\frac{1}{x}\cdot|w(x,0)|^2\,dx$$

$$\leqq \pi k^2 R^{-2}\iint_{T_{2R}} v^2\,dx\,dy+(\lambda-2\alpha_1')\int_0^1|w(x,0)|^2\,dx.$$

In virtue of (6),

(19) $$\int_0^1\frac{1}{x}|w(x,0)|^2\,dx \leqq \pi\cdot\iint_{\mathscr{H}}|\nabla w|^2\,dx\,dy.$$

Thus we infer that

$$(1+\alpha_0')\pi\iint_{S_{2R}}|\nabla w|^2\,dx\,dy \leqq \pi k^2R^{-2}\iint_{T_{2R}} v^2\,dx\,dy+(\lambda-2\alpha_1')\int_R^{2R}|w(x,0)|^2\,dx.$$

Next, we apply (12) to $\psi=v$, and (13) to $\psi=w$, and obtain that

$$[(1+\alpha_0')\pi-(\lambda-2\alpha_1')K^{**}R]\iint_{S_{2R}}|\nabla w|^2\,dx\,dy \leqq \pi k^2K^*\iint_{T_{2R}}|\nabla v|^2\,dx\,dy.$$

Note that $v(x, y)=w(x, y)$ on S_R. Then, for $0<R\leqq R_0$, we obtain that

(20) $$\iint_{S_R}|\nabla v|^2\,dx\,dy \leqq K_0\cdot\iint_{T_{2R}}|\nabla v|^2\,dx\,dy$$

where we have set

$$K_0=\frac{2k^2K^*}{1+\alpha_0'}.$$

Now we are ready to apply Widman's hole filling technique (cf. [8]): Firstly, (20) yields that

$$\iint_{S_R} |\nabla v|^2\,dx\,dy \leqq \theta_0 \cdot \iint_{S_{2R}} |\nabla v|^2\,dx\,dy \tag{21}$$

where $\theta_0 = K_0/(1+K_0) < 1$.

By an iteration of (21), we derive that

$$\begin{gathered}\iint_{S_R} |\nabla v|^2\,dx\,dy \leqq \left(\frac{R}{R_0}\right)^{2\sigma_0} \cdot \iint_{S_{2R_0}} |\nabla v|^2\,dx\,dy \\ \text{for} \quad 0 < R \leqq R_0, \quad \text{and} \quad \sigma_0 = -\frac{\log \theta_0}{2 \cdot \log 2}.\end{gathered} \tag{22}$$

On account of (9), and of

$$\int_0^1 |v(x,0)|^2\,dx = 1,$$

we have that

$$\pi \iint_{\mathscr{H}} |\nabla v|^2\,dx\,dy + \int_0^1 \left\{\frac{\alpha_1}{x} + \frac{\alpha_2}{1-x}\right\} |v(x,0)|^2\,dx = \lambda$$

whence

$$\iint |\nabla v|^2\,dx\,dy \leqq \frac{\lambda}{\pi(1+\alpha)}. \tag{23}$$

Combining (22) and (23), we verify (17), and (18) is analogously proved.

Lemma 2. *Set* $N_j = 2^{1+2\sigma_j} \cdot M_j^2 \cdot (1+k^2K^*)$, $j=0, 1$. *Then*

$$\int_0^R \frac{1}{x} \cdot |v(x,0)|^2\,dx \leqq N_0^2 R^{2\sigma_0} \quad \textit{for} \quad 0 < R \leqq R_0/2, \tag{24}$$

and

$$\int_{1-R}^1 \frac{1}{1-x} \cdot |v(x,0)|^2\,dx \leqq N_1^2 R^{2\sigma_1} \quad \textit{for} \quad 0 < R \leqq R_1/2. \tag{25}$$

Proof. On account of (19),

$$\int_0^{2R} \frac{1}{x} |v(x,0)|^2 |\eta(x,0)|^2\,dx \leqq \iint_{S_{2R}} |\nabla(\eta v)|^2\,dx\,dy$$

whence, by (12) and (17),

$$\begin{aligned}\int_0^R \frac{1}{x} \cdot |v(x,0)|^2\,dx &\leqq 2\iint_{S_{2R}} |\nabla v|^2\,dx\,dy + 2k^2R^{-2} \iint_{T_{2R}} v^2\,dx\,dy \\ &\leqq 2(1+k^2K^*) \iint_{S_{2R}} |\nabla v|^2\,dx\,dy \\ &\leqq 2^{1+2\sigma_0} M_0^2 (1+k^2K^*) R^{2\sigma_0} \quad \text{for} \quad 0 < R \leqq R_0/2,\end{aligned}$$

and the lemma is proved.

Lemma 3. *There are numbers H_0 and H_1 depending only on $\alpha_1, \alpha_2, \lambda$, and μ such that*

$$|u(x,0)-u(x',0)| \leqq H_0|x-x'|^{\mu} \quad \text{for} \quad 0 \leqq x, x' \leqq \frac{R_0}{2}, \quad \text{and for each} \tag{26}$$
$$\mu < \min\left\{\sigma_0, \frac{1}{2}\right\},$$

and

$$|u(x,0)-u(x',0)| \leqq H_1|x-x'|^{\mu} \quad \text{for} \quad 1-\frac{R_1}{2} \leqq x, x' \leqq 1, \quad \text{and for each} \tag{27}$$
$$\mu < \min\left\{\sigma_1, \frac{1}{2}\right\}.$$

Proof. We infer from (24) that

$$\int_0^R |v(x,0)|^2\,dx \leqq N_0^2 R^{1+2\sigma_0} \quad \text{for} \quad 0<R\leqq R_0/2.$$

Then, for each τ with $0<\tau<\sigma_0$,

$$\int_0^R x^{-1-2\tau}|v(x,0)|^2\,dx = \sum_{j=0}^{\infty}\int_{2^{-j-1}R}^{2^{-j}R} x^{-1-2\tau}|v(x,0)|^2\,dx$$
$$\leqq \sum_{j=0}^{\infty} 2^{(j+1)(1+2\tau)}R^{-1-2\tau}\int_0^{2^{-j}R}|v(x,0)|^2\,dx$$
$$\leqq \sum_{j=0}^{\infty} 2^{(j+1)(1+2\tau)}R^{-1-2\tau}N_0^2 2^{-j(1+2\sigma_0)}R^{1+2\sigma_0}$$

whence

$$\int_0^R \frac{1}{x^{1+2\tau}}|v(x,0)|^2\,dx \leqq \frac{2^{1+2\sigma_0}\cdot N_0^2}{2^{2(\sigma_0-\tau)}-1}R^{2(\sigma_0-\tau)} \quad \text{for} \quad 0<R\leqq\frac{R_0}{2} \tag{28}$$

for each τ satisfying $0<\tau<\sigma_0$.

On $\{0<x<1,\ y=0\}$, we know that

$$u_x(x,0) = v_y(x,0) = \frac{1}{\pi}\cdot\left[\frac{\alpha_1}{x}+\frac{\alpha_2}{1-x}-\lambda\right]v(x,0).$$

Therefore,

$$|u_x(x,0)|^2 \leqq [k_1x^{-2}+k_2]|v(x,0)|^2 \quad \text{for} \quad 0<x\leqq\frac{1}{2}$$

where

$$k_1 = 2\alpha_1^2\pi^{-2}, \quad k_2 = (\alpha_2-2\lambda)^2 2^{-1}\pi^{-2}.$$

Then we get for each τ with $0<\tau<\sigma_0$ that

$$\int_0^R x^{1-2\tau}|u_x(x,0)|^2\,dx$$
$$\leqq k_1\int_0^R\frac{1}{x^{1+2\tau}}|v(x,0)|^2\,dx + k_2\int_0^R|v(x,0)|^2\,dx$$
$$\leqq \{k_1+R^{1+2\tau}k_2\}\cdot\int_0^R\frac{1}{x^{1+2\tau}}|v(x,0)|^2\,dx$$

whence, by (28),

$$\int_0^R x^{1-2\tau}|u_x(x,0)|^2\,dx \leqq \{k_1+k_2\}\frac{2^{1+2\sigma_0}}{2^{2\sigma_0-2\tau}-1}N_0^2R^{2\sigma_0-2\tau} \tag{29}$$

$$\text{for}\quad 0<R\leqq\frac{R_0}{2},\quad\text{and}\quad 0<\tau<\sigma_0.$$

Next, we choose τ such that $0<\tau<\min\{\sigma_0, 1\}$, and p with $1<p<2$ and $p<1/(1-\tau)$, i.e.,

$$\frac{p}{2-p}(2\tau-1)>-1.$$

Then, in virtue of Hölder's inequality, for $0<R'<R<R_0/2$

$$\int_{R'}^R|u_x(x,0)|^p\,dx \leqq \Big\{\int_{R'}^R x^{\frac{p(2\tau-1)}{2-p}}\,dx\Big\}^{\frac{2-p}{2}}\cdot\Big[\int_{R'}^R x^{1-2\tau}|u_x(x,0)|^2\,dx\Big]^{\frac{p}{2}}$$

$$\leqq c(p,\tau)R^{1-p(1-\tau)}\Big[\int_0^R x^{1-2\tau}|u_x(x,0)|^2\,dx\Big]^{\frac{p}{2}}$$

where $c(p,\tau)=[(2-p)/(2(1-p+p\tau))]^{\frac{2-p}{2}}$. By (29), the right hand side is bounded independently of R. Letting R' tend to zero, we arrive at

$$\int_0^R|u_x(x,0)|^p\,dx\leqq N_0^pC_0^pR^{1-p(1-\sigma_0)}\quad\text{for}\quad 0<R\leqq\frac{R_0}{2} \tag{30}$$

where

$$C_0(\alpha_1,\alpha_2,\tau,p)=\Big\{\frac{2\alpha_1^2}{\pi^2}+\frac{(\alpha_2-2\lambda)^2}{2\pi^2}\Big\}\cdot c(p,\tau)\cdot N_0\cdot\Big[\frac{2^{1+2\sigma_0}}{2^{2(\sigma_0-\tau)}-1}\Big]^{1/2}.$$

In particular,

$$\Big\{\int_0^{R_0/2}|u_x(x,0)|^p\,dx\Big\}^{1/p}\leqq C_0N_0. \tag{31}$$

Hence, for all x, x' satisfying $0<x, x'\leqq R_0/2$, and all $\tau<\sigma_0$, we obtain that

$$|u(x,0)-u(x',0)|\leqq\Big|\int_x^{x'}|u_\xi(\xi,0)|\,d\xi$$

$$\leqq|x-x'|^\mu\cdot\Big\{\int_0^{R_0/2}|u_\xi(\xi,0)|^p\,d\xi\Big\}^{1/p}\leqq N_0C_0|x-x'|^\mu$$

where $\mu=1-1/p<\min\{\tau, 1/2\}$. Then we can extend $u(x,0)$ continuously to $0\leqq x\leqq R_0/2$, and $u(x,0)$ will satisfy (26). Similarly, one proceeds at the singular point $x=1$. Thus, the lemma is proved.

Now we shall state the main result of the present paper which has Theorem 1 as an immediate consequence.

Theorem 2. *Every eigenfunction $v(x,y)$ of BEP to the eigenvalue λ is Hölder continuous on the closed upper half plane $\bar{\mathscr{H}}$. The Hölder exponent on $\bar{\mathscr{H}}\cap B_R(0)$, $0<R<1$, can be each positive β_0 less than* $\min\{\sigma_0/2, 1/4\}$, *and on $\bar{\mathscr{H}}\cap B_R(1)$, $0<R<1$, the Hölder exponent can be each $\beta_1>0$ less than* $\min\{\sigma_1/2, 1/4\}$.

Moreover,

(32) $$v(0,0) = 0, \quad \text{and} \quad v(1,0) = 0,$$

and there are numbers c_0 and c_1 depending on α_1, α_2, and λ such that

(33) $$|v(x, y)| \leqq c_0 r^{\beta_0} \quad \text{for} \quad 0 \leqq r \leqq 1, \; r = \sqrt{x^2+y^2},$$

and all $\beta_0 < \min\{\sigma_0/2, 1/4\}$, and

(34) $$|v(x, y)| \leqq c_1 r^{\beta_1} \quad \textit{for} \quad 0 \leqq r \leqq 1, \; r = \sqrt{(1-x)^2+y^2},$$

and all $\beta_1 < \min\{\sigma_1/2, 1/4\}$.

Remark. By (15), $\sigma_0 = \sigma_0(\alpha_1)$ depends on α_1 but not on α_2 and λ, while $\sigma_1 = \sigma_1(\alpha_2)$ depends on α_2 but not on α_1 and λ, and σ_1, σ_2 tend to zero as α_1, α_2 tend to -1, cf. (16). This is, probably, not due to some weakness of our technique but inherent to the problem as the numerical computations by Višnjić seem to confirm (cf. [3], Fig. 3—5).

Proof of Theorem 2. Let us consider the function

$$F(z) = u(x, y) + iv(x, y), \quad z = x+iy,$$

which is holomorphic on $\mathscr{H}$, and has a vanishing imaginary part on $\mathbf{R}-[0, 1]$. Thus, we can extend $F(z)$ to a holomorphic function on the slit domain $\mathscr{S} = \mathbf{C}-[0, 1]$, by setting

$$u(x, y) = u(x, -y), \quad v(x, y) = -v(x, -y) \quad \text{for} \quad y < 0.$$

Let us denote by E^+ and E^- the upper and the lower "edges" of the slit [0, 1]. The real part $u(x, y)$ of $F(z)$ passes continuously through the slit, that is,

$$\operatorname{Re} F(z) = \operatorname{Re} F(z^+)$$

on opposite points $z=r$ and $z^+ = re^{2\pi i}$, $0 \leqq r \leqq 1$, of the slit. Let $B = \{\zeta \in \mathbf{C}: |\zeta| < 1\}$, $C = \partial B$, and $C^+ = C \cap \{\operatorname{Im}\zeta \geqq 0\}$, $C^- = C \cap \{\operatorname{Im}\zeta \leqq 0\}$. Consider a conformal mapping $\tau = \tau(\zeta)$ of B onto $\mathscr{S}$ mapping C^+ onto E^+, and C^- onto E^-, and, in particular, $\tau(-1) = 0$, $\tau(1) = 1$. Such a map can explicitly written down (cf. [5], pp. 357—359). Set

$$G(\zeta) = F[\tau(\zeta)], \quad \zeta \in B.$$

Then $\operatorname{Re} G(\zeta)$ is continuous on $\bar{B}$, and there is a positive number ϱ such that $\operatorname{Re} G(\zeta)$ is Hölder continuous for every exponent μ less than $\min\{\sigma_0, 1/2\}$ or $\min\{\sigma_1, 1/2\}$ on the arc $C \cap B_\varrho(-1)$ or $C \cap B_\varrho(1)$, respectively.

(Note that $\tau(\zeta)$ behaves close to each of the branch points $\zeta = -1$ or 1 similar as the mapping $z = \zeta^2$ behaves at $\zeta = 0$, and $\zeta = \tau^{-1}(z)$ behaves similar at $z = 0$ or 1 as $\zeta = \sqrt{z}$.)

By an easily proved "local version" of the Korn—Priwalow-theorem (cf. [1], pp. 380, 401—403), also $G(\zeta)$ is Hölder continuous on $\bar{B}\cap\bar{B}_\varrho(-1)$, or on $\bar{B}\cap\bar{B}_\varrho(1)$, for every exponent $\mu<\min\{\sigma_0, 1/2\}$, respectively.

Reversing the transformation $z=\tau(\zeta)$, we see that $F(z)$ is Hölder continuous on $\{\mathscr{I}\cup E^+\cup E^-\}\cap\bar{B}_R(0)$ with each exponent less than $\min\{\sigma_0/2, 1/4\}$, and on $\{\mathscr{I}\cup E^+\cup E^-\}\cap\bar{B}_R(1)$ with each exponent less than $\min\{\sigma_1/2, 1/4\}$, for each $R<1$, if we take into account that $F(z)$ is analytic on the interior parts of the edges E^+ and E^-. Then we can infer (32) from (4), and the estimates (33) and (34) follow directly from the Hölder estimates. Thus, Theorem 2 is proved.

Added in proof. H. Lewy has proved that $F(z)=u(x,y)+iv(x,y)$ can be expanded in a convergent series of fractional powers of a certain holomorphic fuoction of $z=x+iy$ provided that v is a *bounded* solution of BEP. Combining his results with those of our paper, one obtains the complete description of the behavior of v at the two singular points. (Cf. Hans Lewy, *manuscripta mathematica* **26** (1979), 411—421.)

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