A generalisation of Widman's theorem on comparison domains for harmonic measures

P. J. Rippon

1. Introduction

Let $P=(X, y)$ denote a point of R^m, $m \geqq 3$, with X in R^{m-1}, y real and suppose that $\varphi(t)$, $t>0$, is a positive, convex function for which

$$\int_0^1 \frac{\varphi(t)}{t^2}\, dt < \infty. \tag{1.1}$$

If

$$E_0 = \{P = (X, y)\colon\ |X| < 1,\ \varphi(|X|) < y < 2\}$$

and

$$E_1 = \{P = (X, y)\colon\ |X| < 1,\ -\varphi(|X|) < y < 2\}$$

then Widman has proved the following [9], Theorem 2.2 and Theorem 2.5.

Theorem A. (i) *If $\partial' E_1$ is that part of ∂E_1 lying in $\{y > -\varphi(|X|)\}$ while $u(P)$ is the harmonic measure of $\partial' E_1$ with respect to E_1 then*

$$u(P) \leqq \text{Const} \cdot y, \quad P = (0, y), \quad 0 < y < 2,$$

where the constant depends only on $\varphi(t)$ and m.

(ii) *Let $G(P, M)$ be the Green function of E_0. If P_0 is fixed in E_0 then*

$$G(P, P_0) \geqq \text{Const} \cdot y, \quad P = (0, y), \quad 0 < y \leqq 1,$$

where the strictly positive constant depends on $\varphi(t), m$ and P_0.

The domains E_0 and E_1 are used by Widman as interior and exterior comparison domains for estimating the Green function of a Liapunov—Dini domain. In fact, Widman also assumes that $\varphi'(t)/t$ is monotonic in the proof of (i), but a slight modification of his method (see [6], Theorem 4.2) shows that this assumption may be omitted.

It is our purpose here to prove that the above estimates hold in more general comparison domains of this type. Our methods of proof are inspired by ideas in Warschawski [8], Lemma 1 and Theorem 3, where a similar situation in R^2 is treated. For technical reasons our domains will be taken to have spherical rather than rotational symmetry and for this purpose we put $\cos\theta = y/|P|$ whenever $P=(X, y)$.

If $\varepsilon(r)$, $0<r<1$, is continuous and satisfies

$$0 \leqq \varepsilon(r) < \frac{\pi}{2}, \quad 0 < r < 1, \tag{1.3}$$

then we define

$$D_0 = \left\{P = (X, y)\colon |P| < 1,\ 0 \leqq \theta < \frac{\pi}{2} - \varepsilon(|P|)\right\},$$

and

$$D_1 = \left\{P = (X, y)\colon |P| < 1,\ 0 \leqq \theta < \frac{\pi}{2} + \varepsilon(|P|)\right\}.$$

For $i=0, 1$ let $\partial' D_i$ be that part of ∂D_i lying on $\{|P|=1\}$ and let $\omega_i(P)$ denote the harmonic measure of $\partial' D_i$ at P with respect to D_i.

Theorem. *If*

$$\int_0^1 \frac{\varepsilon(r)}{r}\, dr < \infty \tag{1.4}$$

then

$$\text{(i)} \quad \omega_1(0, y) \leqq \mathrm{Const} \cdot y, \quad 0 < y < 1,$$

where the constant depends only on $\varepsilon(r)$ and m.

If in addition there exists $c>0$ such that

$$|r\varepsilon(r) - s\varepsilon(s)| \leqq c\,|r-s|, \quad r, s \in (0, 1), \tag{1.5}$$

then

$$\text{(ii)} \quad \omega_0(0, y) \geqq \mathrm{Const} \cdot y, \quad 0 < y < 1,$$

where the strictly positive constant depends on $\varepsilon(r)$, m and c.

Some improvement to part (i) of the theorem is possible, as will become apparent in the course of the proof. However we should like to mention a result due to Eke [1], Theorem 1, which suggests that some additional hypothesis, like (1.5), in part (ii) of our theorem is essential.

Finally, for the sake of comparison, we shall deduce a

Corollary. *If $\varphi(t)$, $t>0$, is a positive, increasing, continuous function for which*

$$\int_0^1 \frac{\varphi(t)}{t^2}\, dt < \infty, \tag{1.6}$$

and E_0, E_1 are given by (1.2), *then Theorem A holds.*

2. Proof of Part (i)

Our proof uses a convexity property of the 'Carleman mean' of $\omega_1(P)$ which is given by

$$m(r) = \left\{\frac{1}{\sigma_m r^{m-1}} \int_{\theta_r} \omega_1(P)^2 d\sigma(P)\right\}^{1/2}, \quad 0 < r < 1. \tag{2.1}$$

Here $d\sigma(\cdot)$ denotes $(m-1)$-dimensional measure (or surface area) on

$$\theta_r = \{P\colon\ |P| = r\} \cap D_1,$$

and σ_m is the surface area of a unit ball in R^m. To describe this convexity property, it is necessary to define the so-called characteristic constant, α_E, of an open set E lying on $\{|P|=1\}$.

Let $\mathscr{F}(E)$ be the class of functions which are Lipschitzian, nonnegative and not identically zero on $\{|P|=1\}$ and which vanish outside E. Put

$$\lambda_E = \inf_{\mathscr{F}(E)} \frac{\int |\operatorname{grad} f|^2 d\sigma}{\int |f|^2 d\sigma},$$

where $d\sigma$ denotes surface area on $\{|P|=1\}$ and grad f is the gradient of f on $\{|P|=1\}$, and then define α_E to be the positive root of

$$\alpha(\alpha+m-2) = \lambda.$$

More generally, if E is an open subset of $\{|P|=r\}$ we put

$$\alpha_E = \alpha_{E'},$$

where

$$E' = \left\{\frac{P}{|P|}\colon P \in E\right\},$$

and then define $\alpha(r)$, $0<r<1$, to be the characteristic constant of

$$\theta_r = \left\{P\colon\ |P| = r,\ 0 \leqq \theta < \frac{\pi}{2} + \varepsilon(r)\right\}.$$

The following fundamental inequality was proved by Huber [5], p. 111 under the assumption that $\partial D_1 \diagdown \partial' D_1$ is smooth:

$$r\frac{d}{dr}\{\log A(r)\} \geqq 2\alpha(r)+m-2, \quad 0 < r < 1, \tag{2.2}$$

where

$$A(r) = r\frac{d}{dr}\left(m(r)^2 r^{m-2}\right). \tag{2.3}$$

In view of the possibility of exhausting D_1 by a monotone sequence of domains with the required smoothness we shall assume that (2.2) holds. To exploit this inequality we need a good lower estimate for the characteristic constant, α_ε say, of the spherical cap

$$E_\varepsilon = \left\{P = (X, y)\colon |P| = 1,\ 0 \leqq \theta < \frac{\pi}{2} + \varepsilon\right\}, \quad 0 \leqq \varepsilon < \frac{\pi}{2}.$$

Such an estimate has recently been obtained in the papers of Hayman and Ortiz [4] and Friedland and Hayman [2], Theorem 3. Amongst other results they prove that if

$$S(\varepsilon) = \frac{\sigma_{m-1}}{\sigma_m} \int_0^{\pi/2+\varepsilon} (\sin\theta)^{m-2}\, d\theta, \quad 0 \leqq \varepsilon < \frac{\pi}{2},$$

is the surface area of E_ε expressed as a proportion of σ_m, then

$$\alpha_\varepsilon \geqq 2(1 - S(\varepsilon)), \quad 0 \leqq \varepsilon < \frac{\pi}{2}. \tag{2.4}$$

We deduce that if

$$\sigma(r) = \frac{\sigma_{m-1}}{\sigma_m} \int_{\pi/2}^{\pi/2+\varepsilon(r)} (\sin\theta)^{m-2}\, d\theta, \quad 0 < r < 1,$$

then

$$\alpha(r) \geqq 1 - 2\sigma(r), \quad 0 < r < 1.$$

which, with (2.2), implies that

$$\frac{d}{dr}\{\log A(r)\} \geqq \frac{m}{r} - 4\frac{\sigma(r)}{r}, \quad 0 < r < 1. \tag{2.5}$$

Noting that $\sigma(r) \leqq \sigma_{m-1}\, \varepsilon(r)/\sigma_m$, and hence, by (1.4), that

$$\int_0^1 \frac{\sigma(r)}{r}\, dr < \infty,$$

we obtain on integrating (2.5) that

$$\frac{A(r_1)}{r_1^m} \exp\left[4\int_0^{r_1} \frac{\sigma(r)}{r}\, dr\right] \leqq \frac{A(r_2)}{r_2^m} \exp\left[4\int_0^{r_2} \frac{\sigma(r)}{r}\, dr\right], \quad 0 < r_1 < r_2 < 1. \tag{2.6}$$

Now

$$\frac{A(r)}{r^m} = (m-2)\left(\frac{m(r)}{r}\right)^2 + \frac{2m(r)m'(r)}{r}, \tag{2.7}$$

and

$$m(r)m'(r) = \frac{1}{\sigma_m r^{m-1}} \int_{\theta_r} \omega_1 \frac{\partial \omega_1}{\partial r}\, d\sigma = \frac{1}{\sigma_m r^{m-1}} \int_{D_1 \cap B(0,r)} |\nabla \omega_1|^2\, dP,$$

where dP denotes m-dimensional measure in R^m and by $B(P_0, r)$ we mean the open ball

$$B(P_0, r) = \{P\colon |P - P_0| < r\}.$$

Since $m'(r) \geqq 0$, while $0 \leqq m(r) \leqq 1$, the set of r in $(0, 1)$ where $m'(r) > 3$ can have length at most $\frac{1}{3}$. Thus there exists $r_0 > \frac{1}{2}$ such that $m'(r_0) \leqq 3$ and we deduce from (2.7) that

$$\frac{A(r_0)}{r_0^m} \leqq 4(m+1). \tag{2.8}$$

It follows, by (2.6), that

$$\frac{A(r)}{r^m} \leqq 4(m+1)\exp\left[4\int_0^1 \frac{\sigma(r)}{r}\,dr\right], \quad 0 < r \leqq \frac{1}{2},$$

and hence, by (2.7), that

$$m(r) \leqq \text{Const} \cdot r, \quad 0 < r \leqq \frac{1}{2}, \tag{2.9}$$

the constant depending only on m and $\varepsilon(r)$. Strictly speaking (2.9) has only been verified for a smooth domain approximating to D_1 from within. However the dependence of the above constant upon $\varepsilon(\cdot)$ show that (2.9) holds in D_1 by a limiting process.

Finally, for $0 < r \leqq \frac{1}{2}$, we put

$$h_r(P) = \frac{1}{\sigma_m r}\int_{\theta_r} \omega_1(Q)\frac{(r^2 - |P|^2)}{|P-Q|^m}\,d\sigma(Q), \quad |P| < r,$$

so that

$$\omega_1(P) \leqq h_r(P), \quad P \in D_1 \cap B(0, r).$$

In particular, for $0 < r \leqq \frac{1}{2}$,

$$\omega_1\left(0, \frac{r}{2}\right) \leqq h\left(0, \frac{r}{2}\right) \leqq \frac{3 \cdot 2^{m-2}}{\sigma_m r^{m-1}}\int_{\theta_r} \omega_1(Q)\,d\sigma(Q) \leqq 3 \cdot 2^{m-2} m(r),$$

by the Schwarz inequality.

Thus, by (2.9),

$$\omega_1(0, y) \leqq \text{Const} \cdot y, \quad 0 < y \leqq \frac{1}{4},$$

the constant depending only on m and $\varepsilon(r)$, which proves part (i).

Remark 1. For simplicity we have proved part (i) when D_1 has rotational symmetry but the proof can be adapted to more general situations. To do this we must be able to estimate, in (2.2), the characteristic constant of a set which is not necessarily a spherical cap. Here, however, we can use the result of Sperner [7] that if E is a set on $\{|P|=1\}$ and E^* is a spherical cap on $\{|P|=1\}$ with $\sigma(E) = \sigma(E^*)$ then

$$\lambda_E \geqq \lambda_{E^*}$$

and hence

$$\alpha_E \geqq \alpha_{E^*}.$$

The proof then goes through exactly as above with (1.4) replaced by an integral restriction on the surface areas of the sets θ_r.

Remark 2. Inequalities (2.6) and (2.8) imply that

$$\mu = \lim_{r \to 0} \frac{A(r)}{r^m}$$

exists finitely, and it is then easy to deduce from (2.3) and (2.7) that

$$\lim_{r \to 0} \frac{m(r)}{r} = \lim_{r \to 0} m'(r) = \sqrt{\frac{\mu}{m}}.$$

This suggests that, perhaps under additional hypotheses on $\varepsilon(r)$,

$$\lim_{r \to 0} \frac{\omega_1(0, r)}{r}$$

may exist. If this were so the correct value for the limit would be $\sqrt{2\mu}$ as may be seen by considering $\varepsilon(r) \equiv 0$. When $m=2$ a result like this is proved by Warschawski [8], Theorem 1 using conformal mapping.

Remark 3. Our method depends upon the inequality (2.4) due to Friedland, Hayman and Ortiz ([2] and [4]). Since the proof of this estimate is highly technical and relies to some extent on the use of computer techniques, we point out that when ε is sufficiently small elementary methods give

$$1 - \alpha_\varepsilon \leqq \text{Const} \cdot \varepsilon,$$

which can be used in place of (2.4). In particular, if $\varepsilon(r) \to 0$ as $r \to 0$ then part (i) of our theorem can be proved without the difficult estimate of Friedland, Hayman and Ortiz.

3. Proof of Part (ii)

It will be convenient to have available an elementary

Lemma. *Let*

$$H_\varrho = \{P = (X, y)\colon\ |P| < \varrho,\ y \geqq 0\}$$

and let $h_\varrho(P)$ denote the harmonic measure at P of that part of ∂H_ϱ lying on $\{|Q| = \varrho\}$ with respect to H_ϱ. Then

$$h_\varrho(P) \leqq 3 \cdot 2^m \frac{my}{\varrho}, \quad P = (X, y), \quad |P| < \frac{\varrho}{2}.$$

Proof. By a reflection in $\{y=0\}$ we see that

$$h_\varrho(P) = \frac{\varrho^2-|P|^2}{\sigma_m \varrho} \int_{\partial H_\varrho \cap \{|Q|=\varrho\}} \left[\frac{1}{|P-Q|^m} - \frac{1}{|P-Q^*|^m}\right] d\sigma(Q), \quad P = (X, y) \in H_\varrho,$$

where Q^* is the reflection of Q in $\{y=0\}$. If $d(Q)$ denotes the distance from Q to $\{y=0\}$ then, for Q in the range of integration,

$$|P-Q^*|^2 - |P-Q|^2 = 4yd(Q), \quad P = (A, y) \in H_\varrho,$$

and we deduce that

$$\frac{1}{|P-Q|^m} - \frac{1}{|P-Q^*|^m} \leqq \frac{2myd(Q)}{|P-Q|^{m+2}}, \quad P = (X, y) \in H_\varrho.$$

Thus

$$h_\varrho(P) \leqq \frac{3 \cdot 2^{m+1} my}{\sigma_m \varrho^{m+1}} \int_{\partial H_\varrho \cap \{|Q|=\varrho\}} d(Q)\, d\sigma(Q) \leqq 3 \cdot 2^m \frac{my}{\varrho},$$

as required, which proves the lemma.

To prove part (ii) we first remark that, by (1.4) and (1.5),

$$\varepsilon(r) = o(1), \quad r \to 0.$$

In particular, we can choose $r_0, 0<r_0<1/8$, so that

$$\varepsilon(8r) < 1, \quad 0 < r \leqq r_0. \tag{3.1}$$

Now put

$$D_r = D_0 \cup B(P_r, r), \quad P_r = (0, r), \quad 0 < r < r_0,$$

and define $\omega_r(P)$ in the same way as $\omega_0(P)$ but with respect to D_r.

If $K(r)=\omega_r(P_r)/r$ then

$$\omega_r(0, y) \geqq K(r) 2^{1-m} y, \quad 0 < y \leqq r. \tag{3.2}$$

In fact, an application of Harnack's inequality (Hayman and Kennedy [3], p. 35) to $B(P_r, r)$ yields

$$\begin{aligned} \omega_r(0, y) &\geqq \omega_r(P_r) y r^{m-2}/(2r-y)^{m-1}, \quad 0 < y \leqq r, \\ &\geqq K(r) 2^{1-m} y, \end{aligned}$$

as required.

In D_0 we consider

$$v_r(P) = \omega_r(P) - \omega_0(P),$$

so that $v_r(P)$ is harmonic and bounded in D_0, takes the same values as $\omega_r(P)$ on $\partial D_0 \setminus \partial D_r$ and vanishes at interior points of $\partial D_0 \cap \partial D_r$. We shall show that

$$v_r(0, y) \leqq N(m, c) K(r) y \int_0^{4r} \frac{\varepsilon(t)}{t}\, dt, \quad 0 < r \leqq r_0, \quad 0 < y < 1, \tag{3.3}$$

where $K(r)$ is as above. Here, and subsequently, $N(m, c)$ denotes a constant, $0<N(m, c)<\infty$, depending only on m and c and possibly varying from one occurrence to the next.

Part (ii) follows from (3.3) by choosing $\varrho>0$ so small that

$$v_\varrho(0, y) \leqq K(\varrho)2^{-m}y,$$

and observing that

$$\omega_0(0, y) = \omega_\varrho(0, y) - v_\varrho(0, y)$$
$$\geqq K(\varrho)2^{-m}y, \quad 0 < y \leqq \varrho.$$

To prove (3.3) it is first necessary to see that

$$\omega_r(P) \leqq \omega_r(0, |P|), \quad P \in D_r, \quad 0 < r \leqq r_0. \tag{3.4}$$

This is a consequence of the following geometric property of D_r. Let T_P denote the hyperplane in R^m which passes through the midpoint of the line segment joining P to $(0, |P|)$ and which is orthogonal to that segment. Note that T_P also passes through the origin $(0, 0)$. If D_P denotes that component of $D_r \setminus T_P$ which contains P then the reflection, D_P^*, of D_P in T_P lies entirely in D_r. Moreover any part of ∂D_P lying on $\{|Q|=1\}$ is reflected in this way to a part of ∂D_P^* which also lies on $\{|Q|=1\}$. Thus the reflection, $\omega_r^*(\cdot)$, of $\omega_r(\cdot)$ across T_P is majorised by $\omega_r(\cdot)$ at all points of ∂D_P^* apart from a set of harmonic measure zero. By the generalised maximum principle therefore $\omega_r^*(\cdot)$ is majorised by $\omega_r(\cdot)$ in D_P^* and in particular

$$\omega_r^*(0, |P|) \leqq \omega_r(0, |P|)$$

which is precisely (3.4).

Now, by (3.4) and the maximum principle,

$$\omega_r(P) \leqq \omega_r(0, \varrho) h_\varrho(P), \quad P \in D_r \cap B(0, \varrho), \quad 0 < r \leqq r_0, \quad 0 < \varrho < 1,$$

where $h_\varrho(P)$ is defined in the lemma. Putting $\varrho=4r$ we deduce from the lemma that

$$\omega_r(P) \leqq N(m)\, \omega_r(0, 4r)\, y/r, \quad P \in D_r \cap B(0, 2r), \quad 0 < r \leqq r_0.$$

In view of (3.1) Harnack's principle yields

$$\omega_r(0, 4r) \leqq N(m)\, \omega_r(0, r), \quad 0 < r \leqq r_0,$$

and so, since $\partial D_0 \setminus \partial D_r$ lies in $B(0, 2r)$, we obtain

$$v_r(P) \leqq N(m) K(r) y, \quad P \in \partial D_0 \setminus \partial D_r. \tag{3.5}$$

Now let $\varphi(r)=r\varepsilon(r)$, $0<r<1$, and put

$$k_r(P) = \frac{2y}{\sigma_m} \int_{|Q|<4r} \frac{\varphi(|Q|)}{|P-Q|^m}\, d\sigma(Q), \quad P = (X, y), \; y > 0,$$

where Q lies in R^{m-1} and $d\sigma$ denotes $(m-1)$-dimensional measure. Since $k_r(P)$ is positive and harmonic in $\{y>0\}$ the estimate

$$v_r(P) \leqq N(m,c)K(r)k_r(P), \quad P \in D_0, \tag{3.6}$$

can be proved by verifying it on $\partial D_0 \setminus \partial D_r$ (where we know that (3.5) holds).

If $P=(X,y)$ lies in $\partial D_0 \setminus \partial D_r$ we define

$$\Delta_r(P) = \{y=0\} \cap B\left(\frac{|P|}{|X|}X, \frac{\varphi(|P|)}{2c}\right), \quad 0<r\leqq r_0,$$

and since we may suppose, without loss of generality, that $c \geqq \frac{1}{2}$ it is clear, because $|P|<2r$ and $\varepsilon(|P|)<1$, that

$$\Delta_r(P) \subset \{y=0\} \cap B(0, 4r).$$

Thus

$$k_r(P) \geqq \frac{2y}{\sigma_m}\int_{\Delta_r(P)} \frac{\varphi(|Q|)}{|P-Q|^m}\, d\sigma(Q), \quad P=(X,y) \in \partial D_0 \setminus \partial D_r.$$

Now, by (1.5),

$$\varphi(|Q|) \geqq \varphi(|P|) - c\big||P|-|Q|\big|, \quad Q \in R^{m-1},$$

and so

$$\varphi(|Q|) \geqq \frac{1}{2}\varphi(|P|), \quad Q \in \Delta_r(P), \quad P \in \partial D_0 \setminus \partial D_r.$$

Also

$$|P-Q| \leqq \varphi(|P|) + \frac{\varphi(|P|)}{2c}, \quad Q \in \Delta_r(P), \quad P \in \partial D_0 \setminus \partial D_r,$$

and so

$$k_r(P) \geqq \frac{y}{\sigma_m}\int_{\Delta_r(P)} \frac{\varphi(|P|)}{\left\{\varphi(|P|)\left(1+\frac{1}{2c}\right)\right\}^m}\, d\sigma(Q), \quad P=(X,y) \in \partial D_0 \setminus \partial D_r,$$

$$= N(m,c)y.$$

Combining this inequality with (3.5) we obtain (3.6).

Finally we estimate $k_r(P)$ from above when $P=(0,y)$, $0<y<1$. In that case

$$k_r(0,y) \leqq \frac{2y}{\sigma_m}\int_{|Q|<4r} \frac{\varphi(|Q|)}{|Q|^m}\, d\sigma(Q) = N(m)y\int_0^{4r} \frac{\varepsilon(t)}{t}\, dt,$$

after an appropriate change of variable.

Combining this with (3.6) we obtain (3.3) and our proof of part (ii) is complete.

4. Proof of the Corollary

Since $\varphi(t)$ is increasing the graph $y=\varphi(t)$ may be represented in the polar form $\theta=\frac{1}{2}\pi-\varepsilon_1(r)$, $r>0$, $0\leqq\varepsilon_1(r)<\frac{1}{2}\pi$, where

$$r^2 = t^2+\varphi(t)^2$$

and

$$\tan\varepsilon_1(r) = \frac{\varphi(t)}{t}.$$

Using (1.6) and the monotonicity of $\varphi(t)$ we obtain

$$\varphi(t) = o(t), \quad t \to 0,$$

so that

$$r = t\big(1+o(1)\big), \quad t \to 0,$$

and

$$\varepsilon_1(r) = \frac{\varphi(t)}{t}\big(1+o(1)\big), \quad t \to 0.$$

We deduce from (1.6) that

$$\int_0^1 \frac{\varepsilon_1(r)}{r}\,dr < \infty,$$

and so we can apply part (i) of our theorem to the domain D_1 defined by $\varepsilon_1(r)$. When E_1 is defined by $\varphi(t)$ we evidently have

$$D_1 \subset E_1$$

while the part of ∂D_1 determined by $\varepsilon_1(r)$ is a subset of the part of ∂E_1 determined by $\varphi(t)$. Thus part (i) of Theorem A follows by an application of the maximum principle.

To prove part (ii) of Theorem A we shall construct a function $\varepsilon_0(r)$ satisfying the hypotheses of part (ii) of our theorem (with $c=\frac{1}{2}\pi$) in the interval $0<r\leqq r_0$, such that

$$D = \Big\{P = (X, y)\colon\ |P| < r_0,\ 0 \leqq \theta < \frac{\pi}{2}-\varepsilon_0(|P|)\Big\} \subset E_0 \tag{4.1}$$

and

$$\partial' D = \Big\{P = (X, y)\colon\ |P| = r_0,\ 0 \leqq \theta \leqq \frac{\pi}{2}-\varepsilon_0(|P|)\Big\} \subset E_0. \tag{4.2}$$

The function $\varepsilon_0(r)$ and r_0 will depend only on $\varphi(t)$.

We can then deduce part (ii) of Theorem A by noting that

$$D_0 = \{P\colon\ r_0 P \in D\}$$

satisfies the hypotheses of part (ii) of our theorem. We deduce that

$$\omega(P) \geqq \mathrm{Const}\cdot y, \quad P = (0, y), \quad 0 < y < r_0, \tag{4.3}$$

where $\omega(P)$ is the harmonic measure of $\partial' D$ with respect to D and the strictly positive constant depends only on $\varphi(t)$ and m. Now, by (4.2),

$$\min\{G(P, P_0)\colon\ P\in\partial' D\}$$

is strictly positive and depends on $\varphi(t)$, m and P_0 so that, by the maximum principle and (4.3),

$$G(P, P_0) \geqq \mathrm{Const}\cdot y, \quad P = (0, y), \quad 0 < y < r_0.$$

The constant here depends on $\varphi(t)$, m and P_0, and the proof of part (ii) is easily completed.

To construct $\varepsilon_0(r)$ we first need a Lipschitzian majorant for $\varphi(t)$ in $(0, 1)$. We put

$$\varphi_0(2^{-n}) = \varphi(2^{-n+1}), \quad n = 0, 1, 2, \ldots,$$

and then take $\varphi_0(t)$ to be linear in each of the intervals $[2^{-n-1}, 2^{-n}]$, $n=0, 1, 2, \ldots$. Since $\varphi(t)$ is increasing we have

$$\varphi_0(t) \geqq \varphi(t), \quad 0 < t \leqq 1,$$

and it is easy to check that

$$\int_0^1 \frac{\varphi_0(t)}{t^2}\, dt < \infty. \tag{4.4}$$

Because $\varphi_0(t)$ is itself increasing on $(0, 1)$, we deduce that

$$\varphi_0(t) = o(t), \quad t \to 0,$$

and for $n=0, 1, 2, \ldots$ we have

$$\begin{aligned}\varphi_0'(t) &= 2^{n+1}(\varphi_0(2^{-n}) - \varphi_0(2^{-n-1})), \quad 2^{-n-1} < t < 2^{-n},\\ &\leqq 2^{n+1}\varphi_0(2^{-n}).\end{aligned}$$

Thus we can choose r_0, $0<r_0<1$, such that

$$|\varphi_0(r) - \varphi_0(s)| < |r-s|, \quad r, s\in(0, r_0). \tag{4.5}$$

Now we put

$$\varepsilon_0(r) = \frac{\pi\varphi_0(r)}{2r}, \quad 0 < r \leqq r_0$$

so that, by (4.5),

$$0 \leqq \varepsilon_0(r) < \frac{\pi}{2}, \quad 0 < r \leqq r_0.$$

Also, by (4.4) and (4.5), we have

$$\int_0^{r_0} \frac{\varepsilon_0(r)}{r}\, dr < \infty,$$

and

$$|r\varepsilon_0(r)-s\varepsilon_0(s)| < \frac{1}{2}\pi|r-s|, \quad r, s\in(0, r_0),$$

so that $\varepsilon_0(r)$ satisfies the hypotheses of part (ii) of our theorem in $(0, r_0)$.

To complete the proof of the corollary it is only necessary to show that (4.1) and (4.2) are true. To do this we need to check that

$$r\sin\varepsilon_0(r) > \varphi\big(r\cos\varepsilon_0(r)\big), \quad 0 < r \leqq r_0,$$

holds. In fact we have

$$r\sin\varepsilon_0(r) > \frac{2}{\pi}r\varepsilon_0(r) = \varphi_0(r) \geqq \varphi(r) \geqq \varphi\big(r\cos\varepsilon_0(r)\big),$$

as required.

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P. J. Rippon
University of Sheffield
Department of Pure Mathematics
The Hicks Building
SHEFFIELD S3 7RH
England