Multidimensional extensions of the Grothendieck inequality and applications

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1. Introduction — Statement of the main theorem

In this paper we develop and study a multidimensional formulation of the Grothendieck inequality. The present work is a continuation and expansion of a line begun in [3], where the Grothendeick theorem was proved by exploiting the notion of $\Lambda(2)$-uniformizability. The presentation here, however, is entirely self-contained and familiarity with [3] is not required.

We employ basic notation and facts of commutative harmonic analysis as they are presented and followed in [10]. Γ, as usual, will be a discrete abelian group and $G=\Gamma^\wedge$ will denote its compact dual group. For the sake of simplicity, most of our work will be performed in the framework of $\otimes\mathbf{Z}_2=\Omega$, the (compact) direct product of $\mathbf{Z}_2$, and $\oplus\mathbf{Z}_2=\hat{\Omega}$, its (discrete) dual group, the direct sum of $\mathbf{Z}_2$. Throughout, $E=\{r_n\}_{n=1}^{\infty}\subset\hat{\Omega}$ will denote the system of Rademacher functions realized as characters in $\hat{\Omega}$: The n^{th} Rademacher function r_n is defined by

$$r_n(\omega) = e^{i\pi\omega(n)},$$

for all $\omega\in\otimes\mathbf{Z}_2$ $\left(=\{(\omega(j))_{j=1}^{\infty}=\omega\colon \omega(j)=0 \text{ or } 1\}\right)$. Ω^N and $\hat{\Omega}^N$ will denote the N-fold cartesian products of Ω and $\hat{\Omega}$, respectively. The characters in E^N, the N-fold cartesian product of $E=\{r_n\}$, will be designated as $(r_{i_1},\dots,r_{i_N})$ for all $i_1,\dots,i_N\in\mathbf{Z}^+$. For $F\subset\Gamma$, $B(F)$ will denote the Banach algebra of restrictions of Fourier—Stieltjes transforms to F, and $A(F)$ will be the Banach algebra of restrictions of Fourier transforms to F. That is,

$$B(F) = M(G)^\wedge/\{\mu\in M(G)\colon \hat{\mu}=0 \text{ on } F\},$$

and

$$A(F) = L^1(G)^\wedge/\{f\in L^1(G)\colon \hat{f}=0 \text{ on } F\}.$$

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$C_F(G)$ and $L_F^p(G)$, $1 \leqq p \leqq \infty$, will be the spaces of functions in $C(G)$ and $L^p(G)$, respectively, whose spectra lie in F. That is,

$$C_F(G) = \{f \in C(G) \colon \hat{f} = 0 \text{ on } \sim F\},$$

and

$$L_F^p(G) = \{f \in L^p(G) \colon \hat{f} = 0 \text{ on } \sim F\}.$$

We now state the classical Grothendieck inequality that was formulated and proved first by A. Grothendieck in [8].

Grothendieck's inequality.

There is a constant $K_G > 0$ with the following property. Let A be any bounded bilinear form on a Hilbert space H, and let $(x_i)_i$ and $(y_j)_j$ be arbitrary sequences in the unit ball of H. Let $f \in C_{E^2}(\Omega^2)$ be any trigonometric polynomial given by

$$f(\omega_1, \omega_2) = \sum_{i,j} a_{ij} r_i(\omega_1) r_j(\omega_2).$$

Then,

$$\left|\sum a_{ij} A(x_i, y_j)\right| \leqq K_G \|f\|_\infty \|A\|$$

$$\Big(\|A\| = \sup_{0 \neq x, y \in H} |A(x, y)| / \|x\| \, \|y\|\Big).$$

Equivalently, let $\varphi \in l^\infty((\mathbf{Z}^+)^2)$ be defined by $\varphi(i, j) = A(x_i, y_j)$. Then

$$\|\varphi\|_{B(E^2)} \leqq K_G \|A\|.$$

We are led to the following:

Definition 1.1. Let $N \geqq 2$, and A be an N-linear form on a Hilbert space H. A is said to be *projectively bounded* if there is a constant η_A with the following property. Let $(x_i^1)_i, \ldots, (x_i^N)_i$ be arbitrary sequences in the unit ball of H. Let $f \in C_{E^N}(\Omega^N)$ be any trigonometric polynomial given by

$$f(\omega_1, \ldots, \omega_N) = \sum_{i_1, \ldots, i_N} a_{i_1 \ldots i_N} r_{i_1}(\omega_1) \ldots r_{i_N}(\omega_N).$$

Then,

$$\left|\sum_{i_1, \ldots, i_N} a_{i_1 \ldots i_N} A(x_{i_1}^1, \ldots, x_{i_N}^N)\right| \leqq \eta_A \|f\|_\infty \|A\|.$$

Equivalently, define $\varphi \in l^\infty((\mathbf{Z}^+)^N)$ by

$$\varphi(i_1, \ldots, i_N) = A(x_{i_1}^1, \ldots, x_{i_N}^N),$$

for all $i_1, \ldots, i_N \in \mathbf{Z}^+$. Then

$$\|\varphi\|_{B(E^N)} \leqq \eta_A \|A\|.$$

Our main purpose here is to characterize within a natural class of bounded multilinear forms on a Hilbert space those which are projectively bounded.

Let $J \geqq K > 0$ and $N > 1$ be arbitrary integers. Let

$$\mathcal{F} = \{1, \dots, J\},$$

and $(S_\alpha)_{\alpha=1}^N$ be a sequence of K-subsets (sets containing K elements) of $\mathcal{F}$ with the following properties:

(1.1) $$\bigcup_{\alpha=1}^N S_\alpha = \mathcal{F};$$

(1.2) for each $1 \leqq \alpha \leqq N$,

$$S_\alpha \cap \left(\bigcup_{k \neq \alpha} S_k\right) = S_\alpha.$$

Also, each S_α is enumerated as

$$S_\alpha = (\alpha_1, \dots, \alpha_K).$$

Each S_α gives rise to P_α, a projection from $(\mathbf{Z}^+)^J$ onto $(\mathbf{Z}^+)^K$: For $i \in (\mathbf{Z}^+)^J$,

$$P_\alpha(i) = (i_{\alpha_1}, \dots, i_{\alpha_K}).$$

Next, let $\varphi \in l^\infty((\mathbf{Z}^+)^J)$ and define an N-linear form on $l^2((\mathbf{Z}^+)^K)$ by

(1.3) $$A_{(S_\alpha)_{\alpha=1}^N, \varphi}(x_1, \dots, x_N) = \sum_{i \in (\mathbf{Z}^+)^J} \varphi(i) x_1(P_1(i)) \dots x_N(P_N(i)),$$

where $x_1, \dots, x_N \in l^2((\mathbf{Z}^+)^K)$. In the sequel that follows, whenever $(S_\alpha)_{\alpha=1}^N$ is understood, we shall write $A_{N,\varphi}$ for $A_{(S_\alpha)_{\alpha=1}^N, \varphi}$; when $\varphi \equiv 1$, we shall write A_N. Observe that the boundedness of $A_{N,\varphi}$ follows from (1.2) and the Schwartz inequality. We state it formally and leave the details of the straightforward proof (by induction on J) to the reader.

Lemma 1.2. *For any* $x_1, \dots, x_N \in l^2((\mathbf{Z}^+)^K)$,

$$|A_{N,\varphi}(x_1, \dots, x_N)| \leqq \|\varphi\|_\infty \|x_1\|_2 \dots \|x_N\|_2.$$

Next, we define a new breed of spectral sets that will play a prominent role in the characterization of projectively bounded multilinear forms. Let

$$E = \{r_{i_1 \dots i_K}\}_{i_1, \dots i_K = 1}^\infty \subset \hat{\Omega}$$

be a K-fold enumeration of the Rademacher system, and define

(1.4) $$E_{(S_\alpha)_{\alpha=1}^N} = \{(r_{P_1(i)}, \dots, r_{P_N(i)})\}_{i \in (\mathbf{Z}^+)^J} \subset E^N.$$

Again, whenever $(S_\alpha)_{\alpha=1}^N$ is given and understood from the context, we shall write E_N for $E_{(S_\alpha)_{\alpha=1}^N}$. $\varphi \in l^\infty((\mathbf{Z}^+)^J$ is said to be in $B(E_N)$ if there is $\mu \in M(\Omega^N)$ so that for all $i \in (\mathbf{Z}^+)^J$

(1.5) $$\varphi(i) = \hat{\mu}((r_{P_1(i)}, \dots, r_{P_N(i)})).$$

As usual, the norm of φ in $B(F_N)$ is given by

$$\|\varphi\|_{B(F_N)} = \inf \{\|\mu\| : \mu \text{ satisfies } (1.5)\}.$$

We now state the main result.

Theorem 1.3. *$A_{N,\varphi}$ is projectively bounded if and only if $\varphi \in B(F_N)$. Moreover, for all $\varphi \in l^\infty((\mathbf{Z}^+)^J)$,*

$$\eta_{A_{N,\varphi}} \leqq \|\varphi\|_{B(E_N)} \eta_{A_N}.$$

The key to the 'only if' direction of Theorem 1.3 is the projective boundedness of A_N. This is proved in Section 2 where a crucial use is made of the $\Lambda(2)$-uniformizability of $E^K \subset \hat{\Omega}^K$. We also obtain estimates for η_{A_N} in terms of N, J and $\Lambda(2)$ uniformizing constants of E^K. The full proof of 1.3 is given in Section 3. An important consequence of the characterization of projectively bounded forms is the abundance of projectively unbounded multilinear forms which is displayed via the proposition that $E_{\{S_\alpha\}_{\alpha=1}^N}$ is a Sidon set if and only if $J=K$.

Some applications are given in the remaining parts of the paper. In Section 5 we consider a special instance of Th. 1.3 and place in a broader perspective Varopoulos' results regarding the failure of multidimensional polynomial inequalities for operators on a Hilbert space (see [12] and [13]): The failure of a general multidimensional Von Neumann inequality follows from the ($\Rightarrow$) direction of Th. 1.3. whereas the other direction of 1.3 yields polynomial inequalities for operators in the Hilbert—Schmidt class. In Section 6, Th. 1.3 is used to produce absolutely summing and non absolutely summing operators from $C_{E^N}(\Omega^N)$ into a Hilbert space.

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2. A_N is projectively bounded

The key ingredient in the proof of the projective boundedness of A_N is the $\Lambda(2)$ uniformizability of $E^K \subset \hat{\Omega}^K$. First, recall that $F \subset \Gamma$ is a $\Lambda(p)$ set for $1<p<\infty$ if

$$L^1_F(G) = L^p_F(G).$$

Definition 2.1. $F \subset \Gamma$ is said to be a uniformizable $\Lambda(2)$ set if for every $0<\delta<1$ there is $\beta_F(\delta)=\beta$ so that whenever $\varphi \in l^2(F)$ there is $f \in L^\infty(G)$ with the following properties:

(i) $\hat{f} = \varphi$ on F;

(ii) $\|f\|_\infty \leqq \beta \|\varphi\|_2$;

(iii) $\|\hat{f}|_{\sim F}\|_2 \leqq \delta \|\varphi\|_2$

($\hat{f}|_{\sim F}$ denotes the restriction of $\hat{f}$ to $\sim F$). $0<\delta<1$ and $\beta_F(\delta)$ are said to be $\Lambda(2)$ uniformizing constants of F.

Recall now that E^K is a $\Lambda(p)$ set for all $1<p<\infty$ and, in particular, it is a $\Lambda(2)$ set (see Appendix D in [11], for example). To establish the $\Lambda(2)$ uniformizability of E^K, we observe that we can find a measure $\mu\in M(\Omega^K)$ so that $\hat{\mu}=1$ on E^K and $\|\hat{\mu}|_{\sim(E^K)}\|_\infty\leqq\delta$, for any given $0<\delta<1$ (see Lemma 3.1 in [3] or p. 311 of [2]). The best result in this direction, however, was communicated to us by G. Pisier, with whose kind permission it is reproduced here.

Lemma 2.2. (Pisier). *Let $F=\{\gamma_j\}_j\subset\Gamma$ be a $\Lambda(p)$ set for some $p>2$. Then F is a uniformizable $\Lambda(2)$ set.*

Proof. Let $0<\delta<1$ be arbitrary and $\varphi\in l^2$. Since F is $\Lambda(p)$, there is $C_p>0$ so that

$$\left\|\sum_j\varphi(j)\gamma_j\right\|_p\leqq C_p\|\varphi\|_2. \tag{2.2.1}$$

Let

$$h_1=\begin{cases}\sum_j\varphi(j)\gamma_j & \text{if } \left|\sum\varphi(j)\gamma_j\right|\leqq\delta^{\frac{2}{(2-p)}}\left\|\sum\varphi(j)\gamma_j\right\|_p\\ 0 & \text{otherwise,}\end{cases}$$

and $h_2=\sum_j\varphi(j)\gamma_j-h_1$. A routine estimate yields

$$\|h_2\|_2\leqq\delta\left\|\sum\varphi(j)\gamma_j\right\|_p. \tag{2.2.2.}$$

Next, write

$$h_2=\sum\psi(j)\gamma_j+P,$$

where $\operatorname{spectrum}(P)\subset\sim F$. Clearly, $\|\psi\|_2\leqq\delta\cdot C_p\|\varphi\|_2$ ((2.2.1) and (2.2.2)), and since F is (a fortiori) a $\Lambda(2)$ set we can find $g\in L^2_F(G)$ so that

$$\left\|\sum\psi(j)\gamma_j+g\right\|_\infty\leqq\delta\cdot C_2\cdot C_p\|\varphi\|_2$$

(C_2 is the $\Lambda(2)$ constant of F). Finally, let

$$f=h_1+\sum\psi(j)\gamma_j+g.$$

Observe that

$$\hat{f}(\gamma_j)=\varphi(j),$$

$$\|f\|_\infty\leqq\|\varphi\|_2\left(C_p\delta^{\frac{2}{(2-p)}}+C_p\cdot C_2\delta\right),$$

and

$$\|\hat{f}|_{\sim F}\|_2\leqq\delta\|\varphi\|(C_p+C_p\cdot C_2). \qquad \square$$

Corollary 2.3.

$$\beta_{E^K}(\delta)\quad\text{is}\quad\mathcal{O}\left(\frac{1}{\delta}\right).$$

Theorem 2.4. *$A_N=A_{(S_\alpha)_{\alpha=1}^N}$ is projectively bounded. Moreover,*

$$\eta_{A_N}\leqq[2\beta_{E^K}(\delta)]^N/(2-(1+\delta)^J),$$

where $0<\delta$ satisfying $(1+\delta)^J<2$ and $\beta_{E^K}(\delta)$ are uniformizing constants of E^K.

Let H be a non-empty subset of $\{1, \dots, J\}$. Let $h_1, h_2 \in L^\infty(\Omega^J)$, and define $h_1 \underset{H}{*} h_2$ to be the convolution of h_1 and h_2 with respect to the coordinates indexed by members of H. If $H=\emptyset$, we let $h_1 \underset{H}{*} h_2 = h_1 \cdot h_2$ (ordinary pointwise multiplication). We illustrate: Let $J=3$, and $H=\{1, 3\}$; $h_1 \underset{H}{*} h_2 \in L^\infty(\Omega^3)$ is defined by

$$h_1 \underset{H}{*} h_2(\omega_1, \omega_2, \omega_3) = \int_{\Omega^2} h_1(\omega_1 - t_1, \omega_2, \omega_3 - t_3) h_2(t_1, \omega_2, t_3)\, dt_1\, dt_3.$$

Let $1 < \alpha \leqq N$, and

$$H_\alpha = S_\alpha \cap \left(\bigcup_{k=1}^{\alpha-1} S_k \right).$$

We define an N-linear form on $L^\infty(\Omega^K)$, denoted as $\hat{A}_{(S_\alpha)_{\alpha=1}^N} = \hat{A}_N$, in the following way: Let $f_1, \dots, f_N \in L^\infty(\Omega^K)$ be arbitrary, and consider f_α, $1 \leqq \alpha \leqq N$, as a function in $L^\infty(\Omega^J)$ that depends only on the $\alpha_1^{\text{th}}, \dots, \alpha_K^{\text{th}}$ coordinates (recall that $S_\alpha = (\alpha_1, \dots, \alpha_K)$). For example, f_1 is thought to be the function in $L^\infty(\Omega^J)$ whose value at $(\omega_1, \dots, \omega_J) \in \Omega^J$ is $f_1(\omega_{1_1}, \dots, \omega_{1_K})$. Define

$$\hat{A}_N(f_1, \dots, f_N) = f_1 \underset{H_2}{*} \cdots \underset{H_N}{*} f_N(0, \dots, 0).$$

The key observation (whose easy verification is left to the reader) is that

$$\hat{A}_N(f_1, \dots, f_N) = A_N(\hat{f}_1, \dots, \hat{f}_N) \tag{2.1}$$

($\hat{f} \in L^\infty(\Omega^K)\hat{}$ is thought of as an element in $l^2((\mathbf{Z}^+)^K)$ in the canonical way: Let $(w_j)_{j=1}^\infty$ be an enumeration of $\hat{\Omega}$. $\hat{f}$ is the element in $l^2((\mathbf{Z}^+)^K)$ whose value at $(i_1, \dots, i_K) \in (\mathbf{Z}^+)^K$ is $\hat{f}((w_{i_1}, \dots, w_{i_K}))$.).

We now state and prove an integral representation of A_N which will subsequently be used for the proof of Theorem 2.4. In what follows $0 < \delta$ satisfying $(1+\delta)^J < 2$ and $\beta(\delta) = \beta$ will be $\Lambda(2)$ uniformizing constants for E^K.

Lemma 2.5. *For each* $1 \leqq \alpha \leqq N$ *there is a map from* $l^2((\mathbf{Z}^+)^K)$ *into*

$$\prod L^\infty(\Omega^K) = \{(f_n)_{n=1}^\infty : f_n \in L^\infty(\Omega^K)\}$$

given by $l^2((\mathbf{Z}^+)^K) \ni x \to ((f_{k,j}^{x,\alpha})_{j=1}^{(2^J-1)^{k-1}})_{k=1}^\infty$ so that the following holds:
(1) For any $x_1, \dots, x_N \in l^2((\mathbf{Z}^+)^K)$,

$$A_N(x_1, \dots, x_N) = \sum_{k=1}^\infty (-1)^{k-1} \sum_{j=1}^{(2^J-1)^{k-1}} \hat{A}_N(f_{k,j}^{x_1,1}, \dots, f_{k,j}^{x_N,N}).$$

(2) For each $k \geqq 1$

$$\sum_{j=1}^{(2^J-1)^{k-1}} \sup \left\{ \prod_{\alpha=1}^N \|f_{k,j}^{x_\alpha,\alpha}\|_\infty : x_1, \dots, x_N \textit{ in unit ball of } l^2((\mathbf{Z}^+)^K) \right\}$$
$$\leqq \beta^N [(1+\delta)^J - 1]^{k-1}.$$

Proof. We agree on a one-one correspondence between E and $\sim E$,

$$E \ni r_n \leftrightarrow \chi_n \in \sim E.$$

Let

$$\Omega_J = \{\omega = \omega(j) \colon \omega(j) = 1 \text{ or } 0,\ 1 \leqq j \leqq J\} = \{\omega_i\}_{i=1}^{2^J},$$

and adopt the convention that $\omega_{2^J}(j)=0$ for all $1 \leqq j \leqq J$. Also, for each $\omega \in \Omega_J$, define

$$|\omega| = \sum_{j=1}^{J} \omega(j).$$

We construct the promised maps by induction. Let $x \in l^2((\mathbf{Z}^+)^K)$ be arbitrary. For each $1 \leqq \alpha \leqq N$, choose $f_{1,1}^{x,\alpha} \in L^\infty(\Omega^K)$ so that

$$\hat{f}_{1,1}^{x,\alpha}((r_{i_1}, \dots, r_{i_K})) = x(i_1, \dots, i_K)$$

for all $i_1, \dots, i_K = 1, 2, \dots,$

$$\|f_{1,1}^{x,\alpha}\|_\infty \leqq \beta \|x\|_2,$$

and

$$\|\hat{f}_{1,1}^{x,\alpha}|_{\sim(E^K)}\|_2 \leqq \delta \|x\|_2.$$

Next, for each $1 \leqq j \leqq 2^J - 1$ define $x_{1,j}^\alpha \in l^2((\mathbf{Z}^+)^K)$ by

$$x_{1,j}^\alpha(i_1, \dots, i_K) = \hat{f}_{1,1}^{x,\alpha}(\gamma_{i_1}^{\omega_j(\alpha_1)}, \dots, \gamma_{i_K}^{\omega_j(\alpha_K)}),$$

where

$$\gamma_n^{\omega_j(i)} = \begin{cases} r_n & \text{if } \omega_j(i) = 0 \\ \chi_n & \text{if } \omega_j(i) = 1. \end{cases}$$

We make the following observations. Let B_K be the unit ball of $l^2((\mathbf{Z}^+)^K)$ and $x_1, \dots, x_N$ be arbitrary elements in B_K. Observe that

$$\hat{A}_N(f_{1,1}^{x_1,1}, \dots, f_{1,1}^{x_N,N}) = A_N(x_1, \dots, x_N) + \sum_{j=1}^{2^J-1} A_N(x_{11,j}^1, \dots, x_{N1,j}^N). \tag{2.5.1}$$

(As above, $x_{\alpha 1,j}^\alpha \in l^2((\mathbf{Z}^+)^K)$, $1 \leqq \alpha \leqq N$, is defined by

$$x_{\alpha 1,j}^\alpha(i_1, \dots, i_K) = \hat{f}_{1,1}^{x_\alpha,\alpha}((\gamma_{i_1}^{\omega_j(\alpha_1)}, \dots, \gamma_{i_K}^{\omega_j(\alpha_K)})).).$$

That is, by applying $\hat{A}_N$ to $f_{1,1}^{x_1,1}, \dots, f_{1,1}^{x_N,N}$ we obtain $A_N(x_1, \dots, x_N)$ with an error that can be estimated as follows:
For each $1 \leqq j \leqq 2^J - 1$,

$$\sup\left\{\prod_{\alpha=1}^N \|x_{\alpha 1,j}^\alpha\|_2 \colon x_1, \dots, x_N \in B_K\right\} \leqq \delta^{|\omega_j|}.$$

Therefore,

$$\sum_{j=1}^{2^J-1} \sup\left\{\prod_{\alpha=1}^N \|x_{\alpha 1,j}^\alpha\|_2 \colon x_1, \dots, x_N \in B_K\right\} \leqq (1+\delta)^J - 1. \tag{2.5.2}$$

From (2.5.2) and Lemma 1.2 it follows that

$$\sum_{j=1}^{2^J-1} |A_N(x_{11,j}^1, \dots, x_{N1,j}^N)| \leqq (1+\delta)^J - 1. \tag{2.5.3}$$

Motivated by (2.5.1) and (2.5.3), the strategy of the induction is to 'feed' the error back to E^K and correct its effect.

Let $k>1$, $1\leqq j\leqq (2^J-1)^{k-1}$, and write

$$j=(2^J-1)n+v,$$

where $1\leqq v\leqq 2^J-1$. Define $x^{\alpha}_{k-1,j}\in l^2((\mathbf{Z}^+)^K)$ by

$$x^{\alpha}_{k-1,j}(i_1,\dots,i_K)=\hat{f}^{x,\alpha}_{k-1,n}(\gamma^{\omega_v(\alpha_1)}_{i_1},\dots,\gamma^{\omega_v(\alpha_K)}_{i_K}).$$

Next, let $f^{x,\alpha}_{k,j}\in L^\infty(\Omega^K)$ be so that

(i) $\hat{f}^{x,\alpha}_{k,j}((r_{i_1},\dots,r_{i_K}))=x^{\alpha}_{k-1,j}(i_1,\dots,i_K)$,

(ii) $\|f^{x,\alpha}_{k,j}\|_\infty\leqq\beta\|x^{\alpha}_{k-1,j}\|_2$,

and

(iii) $\|\hat{f}^{x,\alpha}_{k,j}|_{\sim(E^K)}\|_2\leqq\delta\|x^{\alpha}_{k-1,j}\|_2$.

The induction is complete.

We now observe by induction on k (the inductive step is similar to the step leading to (2.5.2)) that for all $k\geqq 1$

$$\text{(2.5.4)}\qquad \sum_{j=1}^{(2^J-1)^k}\sup\left\{\prod_{\alpha=1}^N\|x^{\alpha}_{\alpha k,j}\|_2\colon x_1,\dots,x_N\in B_K\right\}\leqq[(1+\delta)^J-1]^k.$$

Therefore it follows from (ii) that for all $k\geqq 1$

$$\sum_{j=1}^{(2^J-1)^{k-1}}\sup\left\{\prod_{\alpha=1}^N\|f^{x_\alpha,\alpha}_{k,j}\|_\infty\colon x_1,\dots,x_N\in B_K\right\}\leqq\beta^N[(1+\delta)^J-1]^{k-1}.$$

Part (2) in the statement of the lemma is now proved. To verify (1), note that for each $M\geqq 1$ (see (2.5.1), for example),

$$\left|\left[\sum_{k=1}^M(-1)^{k-1}\sum_{j=1}^{(2^J-1)^{k-1}}\hat{A}_N(f^{x_1,1}_{k,j},\dots,f^{x_N,N}_{k,j})\right]-A_N(x_1,\dots,x_N)\right|$$
$$=\left|\sum_{j=1}^{(2^J-1)^M}A_N(x^1_{1M,j},\dots,x^N_{NM,j})\right|,$$

where $x_1,\dots,x_N\in l^2((\mathbf{Z}^+)^K)$ are arbitrary. It follows from (2.5.4) that

$$\left|\sum_{j=1}^{(2^J-1)^M}A_N(x^1_{1M,j},\dots,x^N_{NM,j})\right|\leqq[(1+\delta)^J-1]^M\prod_{\alpha=1}^N\|x_\alpha\|_2.$$

Since $[(1+\delta)^J-1]^M\to 0$ as $M\to\infty$ the assertion follows. □

Remark. The proof of 2.5 can be modified and expanded to yield the following more concise statement.

Lemma 2.5'. *There are maps, $\Phi_1,\dots,\Phi_N$, from $l^2((\mathbf{Z}^+)^K)$ into $L^\infty(\Omega^K)$ with the following properties:*

(1) $A_N(x_1,\dots,x_N)=\hat{A}_N(\Phi_1(x_1),\dots,\Phi_N(x_N))$, *for all* $x_1,\dots,x_N\in l^2((\mathbf{Z}^+)^K)$.

(2) *There is* $0<C<\infty$ *so that for each* $1\leqq\alpha\leqq N$ *and all* $x\in l^2((\mathbf{Z}^+)^K)$

$$\|\Phi_\alpha(x)\|_\infty\leqq C\|x\|_2.$$

For our purposes here, we prefer the statement in 2.5 since part (2) of 2.5 offers precision that appears to be lost in part (2) of 2.5'.

Proof of Theorem 2.4. In order to simplify the notation, we proceed to prove a special case of Theorem 2.4. The argument in the general case is identical. Let $J=N\geqq 2$ be arbitrary and $K=2$. Let

$$S_\alpha = \begin{cases} (\alpha, \alpha+1) & \text{for} \quad 1 \leqq \alpha \leqq N-1 \\ (N, 1) & \text{for} \quad \alpha = N. \end{cases}$$

Let $(x_i^1)_{i=1}^\infty, \dots, (x_i^N)_{i=1}^\infty$ be arbitrary sequences of elements in the unit ball of $l^2((\mathbf{Z}^+)^2)$. Suppose that $g\in C_{E^N}(\Omega^N)$ is a trigonometric polynomial given by

$$g(\omega_1, \dots, \omega_N) = \sum_{i_1, \dots, i_N} a_{i_1 \dots i_N} r_{i_1}(\omega_1) \dots r_{i_N}(\omega_N).$$

We require the following elementary fact whose proof is left to the reader.

Sublemma. *Let $s^1=(s_i^1)_{i=1}^\infty, \dots, s^N=(s_i^N)_{i=1}^\infty$ be arbitrary elements in l^∞. Then,*

$$\left|\sum_{i_1, \dots, i_N} a_{i_1 \dots i_N} s_{i_N}^1 \dots s_{i_N}^N\right| \leqq 2^N \|g\|_\infty \prod_{j=1}^N \|s^j\|_\infty .$$

By (1) of Lemma 2.5, we write

$$\left|\sum_{i_1, \dots, i_N} a_{i_1 \dots i_N} A_N(x_{i_1}^1, \dots, x_{i_N}^N)\right|$$

$$=\left|\sum_{i_1, \dots, i_N} a_{i_1 \dots i_N} \sum_{k=1}^\infty (-1)^{k-1} \sum_{k=1}^{(2N-1)^{k-1}} \int_{\Omega^N} f_{k,j}^{x_{i_1}^1, 1}(t_1, t_2) \dots f_{k,j}^{x_{i_N}^N, N}(t_N, t_1)\, dt_1 \dots dt_N\right|$$

$$\leqq \sum_{k=1}^\infty \sum_{j=1}^{(2N-1)^{k-1}} \int_{\Omega^N} \left|\sum_{i_1, \dots, i_N} a_{i_1 \dots i_N} f_{k,j}^{x_{i_1}^1, 1}(t_1, t_2) \dots f_{k,j}^{x_{i_N}^N, N}(t_N, t_1)\right| dt_1 \dots dt_N .$$

Applying the sublemma for (almost) all $(t_1, \dots, t_N)\in\Omega^N$ and integrating over Ω^N, we obtain

$$\left|\sum_{i_1, \dots, i_N} a_{i_1 \dots i_N} A_N(x_{i_1}^1, \dots, x_{i_N}^N)\right|$$

$$\leqq \Big(\sum_{k=1}^\infty \sum_{j=1}^{(2N-1)^{k-1}} 2^N \sup_{i_1, \dots, i_N} \prod_{\alpha=1}^N \|f_{k,j}^{x_{i_\alpha}^\alpha, \alpha}\|_\infty\Big) \|g\|_\infty$$

$$\leqq \Big(\sum_{k=1}^\infty (2\beta)^N [(1+\delta)^N - 1]^{k-1}\Big) \|g\|_\infty \quad \text{(by (2) of Lemma 2.5)}$$

$$\leqq \|g\|_\infty (2\beta)^N / [2-(1+\delta)^N]. \qquad \square$$

3. Proof of Theorem 1.3.

($\Leftarrow$):

Lemma 3.1. *Let φ be a pointwise limit of a sequence $(\varphi_n)\subset l^\infty((\mathbf{Z}^+)^J)$. If $\sup_n \eta_{A_N,\varphi_n} \leqq C < \infty$, then $\eta_{A_N,\varphi} \leqq C$.*

Lemma 3.2. *Let $\Lambda\subset\Gamma$ and $\lambda\in B(\Lambda)$. Then, for every $\varepsilon>0$ there is $(\lambda_n)_{n=1}^\infty\subset A(\Lambda)$ so that (λ_n) converges pointwise on Λ to λ and* $\sup_n \|\lambda_n\|_{A(\Lambda)} = \sup_n \|\lambda_n\|_{B(\Lambda)} \leqq (1+\varepsilon)\|\lambda\|_{B(\Lambda)}$.

The proof of 3.1 follows a standard line and is based on the fact that if ψ is a pointwise limit of $(\psi_n)_{n=1}^\infty \subset B(E^N)$ so that $\sup_n \|\psi_n\|_{B(E^N)} \leqq C < \infty$, then $\|\psi\|_{B(E^N)} \leqq C$. The proof of 3.2 is also standard and is based on the existence of approximate identities in $M(G)$ whose transforms have finite support in Γ.

We now observe that if $\varphi\in l^\infty((\mathbf{Z}^+)^J)$ is given by

$$\varphi(i) = \varphi_1(P_1(i)) \dots \varphi_N(P_N(i))$$

for all $i\in(\mathbf{Z}^+)^J$, where $\varphi_1, \dots, \varphi_N \in l^\infty((\mathbf{Z}^+)^K)$, then (by Theorem 2.4)

$$\eta_{A_N,\varphi} \leqq \max_{1\leqq\alpha\leqq N} \|\varphi_\alpha\|_\infty \eta_{A_N}.$$

Therefore, if $\varphi\in l^\infty((\mathbf{Z}^+)^J)$ is given by

$$\varphi(i) = \sum_{j=1}^\infty \lambda_j \varphi_{1j}(P_1(i)) \dots \varphi_{Nj}(P_N(i)), \tag{3.1}$$

where $\varphi_{\alpha j} \in l^\infty((\mathbf{Z}^+)^K)$, $\|\varphi_{\alpha j}\|_\infty \leqq 1$ for all $\alpha=1, \dots, N$ and $j=1, \dots,$ and $\sum_j |\lambda_j| < \infty$, then

$$\eta_{A_N,\varphi} \leqq \left(\sum |\lambda_j|\right)\eta_{A_N}.$$

In view of the preceding remark, Lemmas 3.1 and 3.2, to prove the 'only if' direction of Th. 1.3, it suffices to prove that every $\varphi\in A(E_N)$ can be written, for a given $\varepsilon>0$, in the form of (3.1) where

$$(1+\varepsilon)\|\varphi\|_{A(E_N)} \geqq \sum |\lambda_j|.$$

To this end, we recall a general (and not difficult to prove) result from [1]. Let $D\subset G$ be a dense and countable subgroup of (metrizable) G. Let

$$\tau_D = \tau\colon \Gamma \to \hat{D}$$

be the canonical injective map given by $(\tau(\gamma), d) = (\gamma, d)$. For $\Lambda\subset\Gamma$, denote by $B_d(\Lambda)$ the restrictions of Fourier Stieltjes transforms of discrete measures to Λ.

Theorem (Corollary 1 in [1]). *Let D and τ be as above. Let $\Lambda\subset\Gamma$ be so that $\tau(\Lambda)^-$ is a countable set (closure in $\hat{D}$) so that*

$$\partial(\tau(\Lambda)^-)\cap\tau(\Lambda) = \emptyset$$

$(\partial(\tau(\Lambda)^-)\equiv\tau(\Lambda)^-\setminus\tau(\Lambda))$. *Then, $A(\Lambda)$ is (canonically and isometrically) a closed subalgebra of $B_d(\Lambda)$. That is, for every $\varphi\in A(\Lambda)$ and $\varepsilon>0$ there is a discrete measure $\mu\in M(G)$ so that*

$$\varphi(\gamma) = \hat{\mu}(\gamma)$$

for all $\gamma\in\Lambda$ and

$$(1+\varepsilon)\|\varphi\|_{A(\Lambda)} \geqq \|\mu\|_M.$$

We apply the above theorem to our current setting. Let $D_1=\oplus\mathbf{Z}_2\subset\otimes\mathbf{Z}_2$ $(=\Omega)$. Clearly, D_1 is dense in $\otimes\mathbf{Z}_2$. Furthermore, observe that $\tau_{D_1}(E)$ in $\hat{D}_1=\Omega$ is a countable set with 0 in $\hat{D}_1$ as its only accumulation point (recall that the topology on Ω is that of coordinate-wise convergence and note that the n^{th} Rademacher function is carried by τ_{D_1} to the point $(0, \ldots, \underset{n^{\text{th}}\text{ slot}}{\underset{\uparrow}{1}}, 0, \ldots)$ in $\hat{D}_1$).

Next, let $D=(D_1)^N\subset\Omega^N$ and observe, by virtue of the preceding remark, that $E_N=E_{(S_\alpha)_{\alpha=1}^N}\subset E^N$ has the property that $\tau_D(E_N)^-$ is countable and that

$$\partial(\tau_D(E_N)^-)\cap\tau_D(E_N) = \emptyset.$$

Therefore, given any $\varphi\in A(E_N)$ and $\varepsilon>0$, there is a discrete measure

$$\mu = \sum_{j=1}^{\infty}\lambda_j\delta_{\gamma_j}\in M(\Omega^N)$$

so that

$$\hat{\mu}\big((r_{P_1(i)}, \ldots, r_{P_N(i)})\big) = \varphi(i)$$

for all $i\in(\mathbf{Z}^+)^K$, and

$$(1+\varepsilon)\|\varphi\|_{A(F_N)} \geqq \sum|\lambda_j|.$$

But, $\gamma_j=(\gamma_{1j}, \ldots, \gamma_{Nj})$, where $\gamma_{\alpha j}\in\Omega$, $1\leqq\alpha\leqq N$. Therefore,

$$\varphi(i) = \sum_{j=1}^{\infty}\lambda_j\gamma_{1j}(r_{P_1(i)}) \ldots \gamma_{Nj}(r_{P_N(i)}).$$

Therefore, $\varphi\in A(E_N)$ can be written in the form of (3.1); the proof of the 'only if' direction of Theorem 1.3 is complete.

$(\Rightarrow)$: The idea for the argument that follows was indicated to us by N. Varopoulos. Fix a one-one onto mapping

$$\theta\colon (\mathbf{Z}^+)^K \to \mathbf{Z}^+,$$

and define $(x_n)\subset l^2((\mathbf{Z}^+)^K)$ by

$$x_n(i) = \begin{cases} 1 & \text{if } \theta(i) = n \\ 0 & \text{otherwise.} \end{cases}$$

Let $\varphi\in l^\infty((\mathbf{Z}^+)^J)$ be so that $A_{N,\varphi}$ is projectively bounded. Then, there is $\mu\in M(\Omega^N)$ with the property

$$\hat{\mu}((r_{j_1}, \ldots, r_{j_N})) = A_{N,\varphi}(x_{j_1}, \ldots, x_{j_N}) \tag{3.2}$$

for all $j_1, \dots, j_N \in \mathbf{Z}^+$. But, by the definition of $A_{N,\varphi}$ and $(x_n)_{n=1}^\infty$ above, (3.2) implies that

$$\hat{\mu}((r_{P_1(i)}, \dots, r_{P_N(i)})) = \varphi(i)$$

for all $i \in (\mathbf{Z}^+)^J$. This completes the proof of the theorem. □

4. The existence of projectively unbounded N-linear forms, $N>2$

To display projectively unbounded multilinear forms on a Hilbert space, we appeal to the notion of Sidonicity. We recall

Definition 4.1. $\Lambda \subset \Gamma$ is said to be a *Sidon set* if

$$l^\infty(\Lambda) = B(\Lambda),$$

i.e., there is $C \geqq 1$ so that for all $\varphi \in l^\infty(\Lambda)$

$$C\|\varphi\|_\infty \geqq \|\varphi\|_{B(\Lambda)}.$$

The 'smallest' C for which the above holds is the Sidon constant of Λ.

The archetypical example of a Sidon set is $\{r_n\}_{n=1}^\infty = E \subset \hat{\Omega}$.

In view of the ($\Rightarrow$) direction of Theorem 1.3 (the easy direction), to check for existence of projectively unbounded forms we test for Sidonicity of $E_{(S_\alpha)_{\alpha=1}^N} = E_N \subset \hat{\Omega}^N$. We do this through the following.

Theorem (Th. 5.7.7 in [11]). *Let $\Lambda \subset \Gamma$ be a Sidon set with a Sidon constant C. Then, for all $g \in L^2_\Lambda(G)$ and $2 < p < \infty$*

$$C\sqrt{p}\,\|g\|_2 \geqq \|g\|_p. \tag{4.1}$$

Proposition 4.2. *E_N is a Sidon set if and only if $J=K$.*

Proof. ($\Leftarrow$): If $J=K$, E_N can be written as

$$E_N = \{r_j, \dots, r_j)\}_{j=1}^\infty$$

which is a Sidon set.

($\Rightarrow$): Let $n>0$ be arbitrary. Let

$$V_n = \{i = (i_1, \dots, i_J) \in (\mathbf{Z}^+)^J \colon 1 \leqq i_1, \dots, i_J \leqq n\}.$$

Define $g \in L^2_{E_N}(\Omega^N)$ by

$$g = \textstyle\sum_{i \in V_n} (r_{P_1(i)}, \dots, r_{P_N(i)}).$$

Clearly,

$$\|g\|_2 = n^{J/2}. \tag{4.2.1}$$

Next, let

$$U_n = \{j = (j_1, \dots, j_K) \in (\mathbf{Z}^+)^K \colon 1 \leqq j_1, \dots, j_K \leqq n\},$$

and h be the Riesz product in $M(\Omega^N)$ defined by

$$h(\omega_1, \dots, \omega_N) = [\Pi_{j \in U_n} (1 + r_j(\omega_1))] \dots [\Pi_{j \in U_n} (1 + r_j(\omega_N))].$$

As usual, $\|h\|_1 = \hat{h}(0) = 1$, and an easy estimate yields

$$\|h\|_2 \leqq \|h\|_\infty \leqq 2^{Nn^K}.$$

Therefore, for any $1 < p < 2$ we have

$$\|h\|_p \leqq 2^{Nn^K/q} \tag{4.2.2}$$

$\left(\frac{1}{p} + \frac{1}{q} = 1\right)$. Also, observe that the spectral analysis of h yields

$$\hat{h} = 1 \quad \text{on} \quad \{(r_{P_1(i)}, \dots, r_{P_N(i)})\}_{i \in V_n}. \tag{4.2.3}$$

Combining (4.2.2) and (4.2.3), we obtain

$$n^J = h * g(0) \leqq \|h\|_p \|g\|_q \leqq 2^{Nn^K/q} \|g\|_q.$$

Recalling (4.2.1) and letting $q = n^K$, we deduce

$$n^{K/2} n^{(J/2 - K/2)} \|g\|_2 \leqq 2^N \|g\|_{n^K}. \tag{4.2.4}$$

Since n was arbitrary, (4.2.4) implies that unless $J = K$ (4.1) is violated. The proof of the proposition is complete. □

The estimates in (4.1) and (4.2.4) yield the following.

Corollary 4.3. *For every $n > 0$, the Sidon constant of*

$$\{(r_{P_1(i)}, \dots, r_{P_N(i)})\}_{i \in V_n} \geqq 2^{-N} (n^{(J-K)/2}).$$

Combining Proposition 4.2 and the ($\Rightarrow$) direction of Theorem 1.3, we deduce

Corollary 4.4. *For every $N \geqq 3$, there are bounded N-linear forms on a Hilbert space which are projectively unbounded.*

5. Extensions of the Von Neumann inequality

A classical inequality due to J. Von Neumann (1951) states that if T is a contraction on a Hilbert space, and p is any complex polynomial, then

$$\|p(T)\| \leqq \sup_{|z| \leqq 1} |p(z)|.$$

(Throughout this section, $\|\cdot\|$ will denote the usual Hilbert space operator norm.) The extension of this inequality to two variables, due to T. Ando (1963) asserts that if T_1 and T_2 are commuting contractions on a Hilbert space and p is any complex polynomial in two variables, then

$$\|p(T_1, T_2)\| \leqq \sup_{|z_1|, |z_2| \leqq 1} |p(z_1, z_2)|$$

(for proofs of these inequalities, see Ch. 1 of [8]).

The question whether the Von Neumann inequality could be extended to the case of $n \geqq 3$ commuting contractions on a Hilbert space was answered in the negative by N. Varopoulos.

Theorem (Th. 1. in [14]). *For every $K>0$ there exist $T_1, \dots, T_n$ ($n \geqq 3$) commuting contractions on a Hilbert space, and p a complex polynomial homogeneous of degree 3 in n variables so that*

$$\|p(T_1, \dots, T_n)\| \geqq K \sup_{\substack{|z_i| \leqq 1 \\ i=1, \dots, n}} |p(z_1, \dots, z_n)|.$$

The crucial part in Varopoulos' proof of the above theorem was an intricate argument based on the probabilistic Kahane—Salem—Zygmund estimates (see Prop. 4.1 and Prop. 4.2 in [13]) showing the existence of projectively unbounded 3-linear forms on a Hilbert space. The existence of such forms is a direct consequences of Proposition 4.2 (Corollary 4.4). Estimates on the deterioration to infinity of $\|p(T_1, \dots, T_n)\|$ are carried out in [5] where, as in [13], use is made of the Kahane—Salem—Zygmund inequalities (Lemma 3.1 in [5]). The same estimates can be obtained by making use of estimates on Sidon constants of finite subsets in $E_{(S_\alpha)_{\alpha=1}^N}$ (Corollary 4.3). For a complete discussion, we refer the reader to [13] and [5].

In the other direction, we give an application of a particular instance of Theorem 2.4, and deduce polynomial inequalities for operators in the Hilbert—Schmidt class. Consider $\mathcal{S}=l^2((\mathbf{Z}^+)^2)$ as the algebra of Hilbert—Schmidt operators on a Hilbert space, where the algebra norm of $T \in \mathcal{S}$ is given by

$$\|T\|_{\mathcal{S}} = \left(\sum_{i,j} |T(i, j)|^2\right)^{1/2},$$

and the algebra multiplication is given by operator composition. Let $J=N \geqq 2$ be arbitrary and $K=2$. For $1 \leqq \alpha \leqq N-1$ let

$$S_\alpha = (\alpha, \alpha+1) \quad \text{and} \quad S_N = (N, 1).$$

Observe that for $T_1, \dots, T_N$ and $U \in \mathcal{S}$,

$$(T_1 \dots T_N, U) = A_{(S_\alpha)_{\alpha=1}^N}(T_1, \dots, T_N \cdot U)$$

($(\cdot, \cdot)$ denotes the scalar product in $\mathcal{S}$).

Theorem 5.1. *There is $C>0$ with the following property: Let $L>0$ be arbitrary and $T_1, \dots, T_L$ be commuting operators in $\mathscr{S}$, $\|T_1\|_{\mathscr{S}}, \dots, \|T_L\|_{\mathscr{S}} \leqq 1$. Then, for all complex polynomials in L variables that are homogeneous of degree N*

$$\|p(T_1, \dots, T_L)\|_{\mathscr{S}} \leqq C^{N \ln N} \sup_{\substack{|z_i| \leqq 1 \\ i=1,\dots,L}} |p(z_1, \dots, z_L)|.$$

Proof. First, observe that a given polynomial p in L variables and homogeneous of degree N can be written as

$$p(z_1, \dots, z_L) = \sum_{i_1,\dots,i_N=1}^{L} a_{i_1 \dots i_N} z_{i_1} \dots z_{i_N}.$$

Next, recall (2.1 in [4]) that

$$\|\tilde{p}\|_\infty \leqq (2e)^N \sup_{\substack{|z_i| \leqq 1 \\ i=1,\dots,L}} |p(z_1, \dots, z_L)|, \tag{5.1.2}$$

where $\tilde{p} \in C_{E^N}(\Omega^N)$ is given by

$$\tilde{p}(\omega_1, \dots, \omega_N) = \sum_{i_1,\dots,i_N=1}^{L} a_{i_1 \dots i_N} r_{i_1}(\omega_1) \dots r_{i_N}(\omega_N).$$

Let $T_1, \dots, T_L$ be arbitrary elements in the unit ball of $\mathscr{S}$, and let U in the unit ball of $\mathscr{S}$ be so that

$$\|p(T_1, \dots, T_L)\|_{\mathscr{S}} = \left|\sum_{i_1,\dots,i_N} a_{i_1 \dots i_N} (T_{i_1} \dots T_{i_N}, U)\right|. \tag{5.1.3}$$

By the remark preceding the statement of the theorem, (5.1.3) can be rewritten as

$$\|p(T_1, \dots, T_L)\|_{\mathscr{S}} = \left|\sum_{i_1,\dots,i_N} a_{i_1 \dots i_N} A_{(S_\alpha)_{\alpha=1}^N}(T_{i_1}, \dots, T_{i_N} \cdot U)\right|.$$

Therefore, by Theorem 2.4,

$$\|p(T_1, \dots, T_L)\|_{\mathscr{S}} \leqq \|\tilde{p}\|_\infty [2\beta_{E^2}(\delta)]^N / (2-(1+\delta)^N). \tag{5.1.4}$$

Let $C_2>0$ be so that

$$\beta_{E^2}(\delta) \leqq C_2/\delta \tag{5.1.5}$$

(Corollary 2.3). Choosing $\delta = \dfrac{1}{2N}$, we have $2-(1+\delta)^N > \frac{1}{2}$ and by combining (5.1.4) and (5.1.5) we obtain

$$\|p(T_1, \dots, T_L)\|_{\mathscr{S}} \leqq \|p\|_\infty 2[4C_2 \cdot N]^N.$$

Finally, by (5.1.2), there is $C>0$ so that

$$\|p(T_1, \dots, T_L)\| \leqq C^{N \ln N} \sup_{\substack{|z_i| \leqq 1 \\ i=1,\dots,L}} |p(z_1, \dots, z_L)|. \qquad \square$$

Remark. Suppose that the growth (in N) of the constants in Theorem 5.1 were bounded by C^N for some fixed $C>0$. Then, we would conclude that there is $\delta>0$ and $M>0$ so that for all $L>0$ whenever $T_1, \dots, T_L$ are commuting elements in $\mathscr{S}$, $\|T_1\|_{\mathscr{S}}, \ldots, \|T_L\|_{\mathscr{S}} \leqq \delta$, then

$$\|p(T_1, \ldots, T_L)\|_{\mathscr{S}} \leqq M \sup_{\substack{|z_i| \leqq 1 \\ i=1, \ldots, L}} |p(z_1, \ldots, z_L)|$$

where p is any polynomial in L variables without a constant term (see 3.1 and 3.3 in [4]).

Problem. Can the (factorial) growth of constants in 5.1 be improved?

6. Absolutely summing operators from $C_{E^N}(\Omega^N)$ into a Hilbert space

The notion of one-absolutely summing operators was introduced by *A.* Grothendieck (e.g., [8]). An accessible introduction and development of the subject can be found in [9].

Definition 6.1. Let X and Y be Banach spaces. An operator $T\colon X \to Y$ is said to be one-absolutely summing if there is $0 < C < \infty$ so that whenever $(x_i) \subset X$ satisfies

$$\sup_{\omega \in \Omega} \left\| \sum_i x_i r_i(\omega) \right\|_X \leqq 1$$

then

$$\sum_i \|Tx_i\|_Y \leqq C.$$

In this context, Grothendieck's classical inequality (the instance $N=2$ in Theorem 2.4) is equivalent to the fact that every bounded operator from $C_E(\Omega)$ $(=l^1)$ into l^2 is one-absolutely summing (see Th. 4.1 in [9], for example). We are thus led to the natural problem of determining all the one-absolutely summing operators from $C_{E^N}(\Omega^N)$ into a Hilbert space. The aim of this section is to exploit Theorem 1.3 and display classes of operators from $C_{E^N}(\Omega^N)$ into l^2 which are and which are not one-absolutely summing.

The fact that there are bounded operators from $C_{E^2}(\Omega^2)$ into l^2 which are not one-absolutely summing was pointed out to us (private communication) by A. Pelczynski whose demonstration relied on Dvoretzky's theorem ([6]). Below, we prove this fact by employing arguments different from Pelczynski's and appealing to the 'necessity' direction of 1.3.

Proposition 6.2. *There is a bounded operator*

$$T\colon C_{E^2}(\Omega^2) \to l^2$$

which is not one-absolutely summing.

Proof. Let $J=N=3$ and $K=2$. Let

$$S_1 = (1, 2), \quad S_2 = (2, 3), \quad \text{and} \quad S_3 = (3, 1).$$

By Prop. 4.2, there is $\varphi \in l^\infty((\mathbf{Z}^+)^3) \setminus B(E_3)$ (as usual, $E_3 = E_{(S_\alpha)_{\alpha=1}^3}$). Therefore, $A_{3,\varphi}$ is not projectively bounded and we can find $(x_k)_{k=1}^\infty, (y_n)_{n=1}^\infty, (z_m)_{m=1}^\infty$ in the unit ball of $l^2((\mathbf{Z}^+)^2)$ so that for *no* measure $\mu \in M(\Omega^3)$

$$\hat{\mu}((r_k, r_n, r_m)) = A_{3,\varphi}(x_k, y_n, z_m).$$

Therefore, there is $g \in C_{E^3}(\Omega^3)$, given by

$$g \sim \sum_{k,n,m} a_{knm}(r_k, r_n, r_m) \tag{6.2.1}$$

and $\|g\|_\infty = 1$ so that

$$\left|\sum_{k,n,m} a_{knm} A_{3,\varphi}(x_k, y_n, z_m)\right| = \infty. \tag{6.2.2}$$

Define an operator

$$T\colon C_{E^2}(\Omega^2) \to l^2((\mathbf{Z}^+)^2)$$

by

$$T((r_k, r_n))(i_1, i_3) = \sum_{i_2} \varphi(i_1, i_2, i_3) x_k(i_1, i_2) y_n(i_2, i_3)$$

for all i_1 and $i_3 \in \mathbf{Z}^+$. We now verify that T is a bounded operator. Let $h \in C_{E^2}(\Omega^2)$ be given by

$$h \sim \sum_{k,n} b_{kn}(r_k, r_n).$$

Let z be in the unit ball of $l^2((\mathbf{Z}^+)^2)$ so that

$$\|T(h)\|_2 = |(T(h), z)| = \left|\sum_{k,n} b_{kn} A_{3,\varphi}(x_k, y_n, z)\right|.$$

But, $A_{3,\varphi}(\cdot, \cdot, z)$ is a bounded *bilinear* form on $l^2((\mathbf{Z}^+)^2)$. Therefore, it is projectively bounded (Grothendieck's inequality) and there is $C>0$ so that

$$\|T(h)\|_2 \leqq C\|h\|_\infty.$$

Finally, to show that T is not one-absolutely summing, define $(f_m)_{m=1}^\infty \subset C_{E^2}(\Omega^2)$ by

$$f_m \sim \sum_{k,n} a_{knm}(r_k, r_n).$$

By (6.2.1)

$$\sup_{\omega\in\Omega} \left\|\sum_m f_m r_m(\omega)\right\|_\infty \leqq 1,$$

and by (6.2.2)

$$\left|\sum_{m=1}^N (T(f_m), z_m)\right| = \left|\sum_{m=1}^N \sum_{k,n} a_{knm} A_{3,\varphi}(x_k, y_n, z_m)\right| \to \infty$$

as $N \to \infty$. □

We proceed now to construct classes of one-absolutely summing operators from $C_{E^N}(\Omega^N)$ into a Hilbert space. Let $J \geqq K$, $N>0$ and $(S_\alpha)_{\alpha=1}^N$ be given. Let $\varphi \in B(E_N) \subset l^\infty((\mathbf{Z}^+)^J)$.

Let $(x_i^1)_{i=1}^\infty, \ldots, (x_i^N)_{i=1}^\infty$ be arbitrary but fixed sequences of elements in the unit ball of $l^2((\mathbf{Z}^+)^K)$. Define

$$T_\varphi\colon C_{E^N}(\Omega^N) \to l^2((\mathbf{Z}^+)^K)$$

in the following way: Let $\bigcup_{\alpha=1}^{N-1} S_\alpha = \{m_1, \ldots, m_M\}$. Write

$$T_\varphi((r_{i_1}, \ldots, r_{i_N}))(j_{N_1}, \ldots, j_{N_K}) \tag{6.1}$$

$$= \sum_{j_{m_1}, \ldots, j_{m_M}=1}^\infty \varphi(j_1, \ldots, j_J) x_{i_1}^1(j_{1_1}, \ldots, j_{1_K}) \ldots x_{i_N}^N(j_{N_1}, \ldots, j_{N_K})$$

for all $j_{N_1}, \ldots, j_{N_K} \in \mathbf{Z}^+$ (recall that $S_\alpha = (\alpha_1, \ldots \alpha_K)$). The projective boundedness of $A_{N,\varphi}$ yields the following theorem the details of whose proof are left to the reader (it follows the outline of 4.1 in [9]).

Theorem 6.3. *T_φ (defined by (6.1)) is one-absolutely summing.*

Addendum

S. Kaijser at Uppsala University solved affirmatively the problem in Section 5: The constants' growth in Theorem 5.1 is $\mathcal{O}(C^N)$. A Tonge independently gave the same solution to the problem (A. Tonge, "The Von Neumann inequality for polynomials in several Hilbert—Schmidt operators," preprint.).

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