On a class of linear partial differential equations whose formal solutions always converge

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0. Introduction

The purpose of this paper is to present a sufficient condition on a linear partial differential operator P with holomorphic coefficients which guarantees that any formal power series solution of the equation $Pu=0$ is convergent. We also show that the obstruction against the solvability of the equation $Pu=f$ is the same in the convergent power series category and in the formal power series category. This problem was discussed by Oshima [4] for first order operators. A related problem was discussed by Baouendi—Sjöstrand [1] for a class of degenerate elliptic equations. It is rather surprising that such a basic problem has rarely been investigated for partial differential equations in spite of the fact that the comparison between formal power series solutions and convergent ones has been one of the central problems in the theory of ordinary differential equations.

We hope that the result in this paper can be used to obtain a more elementary proof of the fact that for holonomic systems with regular singularities, the cohomology groups, considered for formal power series and for convergent power series, are the same. This was proved by Kashiwara—Kawai [3].

In this paper we use the following notation:

$$z=(z_1, \dots, z_n)\in \mathbf{C}^n, \quad \alpha=(\alpha_1, \dots, \alpha_n)\in \mathbf{N}^n, \quad \mathbf{N}=\{0, 1, 2, \dots\}, \quad |\alpha|=\alpha_1+\dots+\alpha_n,$$

$$z^\alpha = z_1^{\alpha_1}\dots z_n^{\alpha_n}, \frac{\partial^{|\alpha|}}{\partial z^\alpha}=\left(\frac{\partial}{\partial z_1}\right)^{\alpha_1}\dots\left(\frac{\partial}{\partial z_n}\right)^{\alpha_n}.$$

Let $a_{\alpha\beta}(z)$, $\alpha, \beta\in\mathbf{N}^n$, $|\alpha|=|\beta|\leqq m$ be holomorphic in a neighborhood of $0\in\mathbf{C}^n$ and let

$$P=\sum\nolimits_{|\alpha|=|\beta|\leqq m} a_{\alpha\beta}(z)z^\alpha\frac{\partial^{|\beta|}}{\partial z^\beta}. \tag{0.1}$$

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Throughout this paper we shall assume that

$$\sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(0) z^{\alpha}\bar{z}^{\beta} \neq 0, \quad z \in \mathbf{C}^n \setminus 0, \tag{0.2}$$

and we observe that this condition is not invariant under changes of coordinates but depends on the choice of the hermitian metric on the tangent vector space of $\mathbf{C}^n$ at the origin. The object of the paper is to study how P acts on various power series.

Let S_l, $l \in \mathbf{N}$, be the space of homogeneous polynomials of degree l and let $\hat{\mathcal{O}}_0 = \{u = \sum_{j=0}^{\infty} u_j,\ u_j \in S_j\}$ be the ring of formal power series. Let $\mathcal{O}_0 \subset \hat{\mathcal{O}}_0$ be the ring of those power series which converge in some neighborhood of 0. We denote by $\mathfrak{m}$ the maximal ideal of $\mathcal{O}_0$. Note that $\mathfrak{m}^k$ (resp. $\mathfrak{m}^k\hat{\mathcal{O}}_0$) consists of convergent (resp. formal) power series of the form $u = \sum_{j \geqq k} u_j$, $u_j \in S_j$. The main result is

Theorem 0.1. *There exists $k_0 \in \mathbf{N}$, such that P induces isomorphisms $\mathfrak{m}^k \to \mathfrak{m}^k$, $\mathfrak{m}^k\hat{\mathcal{O}}_0 \to \mathfrak{m}^k\hat{\mathcal{O}}_0$ for all $k \geqq k_0$.*

In the next section we prove this result, and in Section 2 we prove several comparison theorems by using this theorem.

1. Proof of Theorem 0.1.

As a preliminary we first recall how convergence of formal power series can be expressed in terms of Sobolev norms. Let $H^m(S^{2n-1})$ denote the m-th order Sobolev space on the unit-sphere $S^{2n-1} = \{z \in \mathbf{C}^n;\ |z| = 1\}$ and let $C(S^{2n-1})$ denote the space of continuous functions on S^{2n-1}. We denote by $\|u\|_m$, $\|u\|_C$ the corresponding norms. If u is holomorphic in a neighborhood of $\{|z| \leqq 1\}$, we write $\|u\|_m$, $\|u\|_C$ for the corresponding norms of $u|_{S^{2n-1}}$. From now on m shall be fixed to be equal to the order of P.

Lemma 1.1. *For every $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ such that*

$$\|u\|_m \leqq C_\varepsilon (1+\varepsilon)^l \|u\|_0 \tag{1.1}$$

$$\|u\|_C \leqq C_\varepsilon (1+\varepsilon)^l \|u\|_0 \tag{1.2}$$

for all $u \in S_l$, $l \in \mathbf{N}$.

Proof. If $u \in S_l$, then u is harmonic, and we have

$$u(z) = \int_{|w|=1+\varepsilon} K_\varepsilon(z, w) u(w)\, d\mu(w) = (1+\varepsilon)^l \int_{|w|=1+\varepsilon} K_\varepsilon(z, w) u\left(\frac{w}{1+\varepsilon}\right) d\mu(w)$$

for $|z|<1+\varepsilon$, if K_ε is the Poisson kernel for the ball $|z|\leqq 1+\varepsilon$ and $d\mu$ is the standard measure on the sphere $|w|=1+\varepsilon$. Since $K_\varepsilon(z, w)$ is smooth when restricted to $|z|=1$, $|w|=1+\varepsilon$, the inequalities (1.1), (1.2) follow easily.

Corollary 1.2. *Let* $\|\ \|$ *denote any of the norms* $\|\ \|_m$, $\|\ \|_0$, $\|\ \|_C$. *Then the formal power series* $u=\sum_{j=0}^{\infty} u_j$, $u_j\in S_j$, *converges for* $|z|<1$ *if and only if for every* $\varepsilon>0$, *there exists* $C_\varepsilon>0$ *such that*

$$\|u_j\| \leqq C_\varepsilon(1+\varepsilon)^j, \quad j\in \mathbf{N}.$$

Now let $P_0=\sum_{|\alpha|=|\beta|\leqq m} a_{\alpha\beta}(0)z^\alpha \dfrac{\partial^{|\beta|}}{\partial z^\beta}$ and put

$$L = \sum_{j=1}^{n} z_j \frac{\partial}{\partial z_j}. \tag{1.3}$$

For every $j\in\{1, \dots, n\}$ we have a unique decomposition at every point of S^{2n-1}:

$$\frac{\partial}{\partial z_j} = \alpha_j L+v_j, \tag{1.4}$$

where α_j is a smooth function and v_j is a smooth holomorphic vector field, tangent to S^{2n-1}. (By "holomorphic vector field" we mean a complex vector field of the form $\sum a_j \dfrac{\partial}{\partial z_j}$ even if the coefficients a_j are not holomorphic.)

We write $L=M-iN$, where M, N are real vector fields. Then

$$M = \frac{1}{2}(L+\bar{L}) = \frac{1}{2}\sum_{j=1}^{n}\left(x_j \frac{\partial}{\partial x_j}+y_i\frac{\partial}{\partial y_j}\right) \tag{1.5}$$

if $z=(z_1, \dots, z_n)$, $z_j=x_j+iy_j$. Moreover, $N=\dfrac{1}{2i}(\bar{L}-L)$ is tangent to S^{2n-1}. If u is a holomorphic function defined near a point on S^{2n-1}, then from $\bar{L}u=0$ we obtain that $Lu=\dfrac{2}{i}Nu$ and hence that

$$\frac{\partial}{\partial z_i}u = \alpha_j\frac{2}{i}Nu+v_ju \quad \text{on} \quad S^{2n-1}. \tag{1.6}$$

From P_0 we obtain a differential operator Q_0 on S^{2n-1} by replacing every $\dfrac{\partial}{\partial z_j}$ by $\left(\alpha_j\dfrac{2}{i}N+v_j\right)$. Clearly,

$$(P_0u)|_{S^{2n-1}} = Q_0(u|_{S^{2n-1}}) \tag{1.7}$$

for every holomorphic function u. The condition (0.2) means that S^{2n-1} is non-characteristic with respect to P_0 and it follows easily that the principal symbol of

Q_0 is nonvanishing on the (real) characteristic variety of the induced Cauchy—Riemann equations $\bar{\partial}_{S^{2n-1}}u=0$. In other words, the operator $u \mapsto (Q_0 u, \bar{\partial}_{S^{2n-1}}u)$ is elliptic, so we have the *a priori* estimate

$$\|u\|_m \leqq C(\|Q_0 u\|_0+\|\bar{\partial}_{S^{2n-1}}u\|_{m-1}+\|u\|_0), \quad u\in H^m(S^{2n-1}). \tag{1.8}$$

Since N is a first order operator, we get (with a larger constant C)

$$\sum_{j=1}^m \left\|\left(\frac{2}{i}N\right)^{m-j}u\right\|_j \leqq C(\|Q_0 u\|_0+\|u\|_0), \tag{1.9}$$

when $u\in H^m(S^{2n-1})$ and $\bar{\partial}_{S^{2n-1}}u=0$. Now, if $u\in S_k$ we have

$$\left(\frac{2}{i}N\right)^{m-j}u = L^{m-j}u = k^{m-j}u,$$

so (1.9) gives

$$\sum_{j=0}^m k^{m-j}\|u\|_j \leqq C(\|P_0 u\|_0+\|u\|_0), \quad u\in S_k. \tag{1.10}$$

(Here we recall that all Sobolev norms are taken over S^{2n-1}. If k is sufficiently large, we can absorb the term $\|u\|_0$ to the left hand side, and we conclude that P_0: $S_k\to S_k$ is injective. Since S_k is finite-dimensional, P_0 is also surjective as a map from S_k to S_k when k is large enough.)

Summing up, we have the following

Lemma 1.3. *There exist constants $k_0\geqq 0$, $C_0\geqq 1$ such that P_0: $S_k\to S_k$ is bijective for $k\geqq k_0$ and such that*

$$\sum_{j=0}^m k^{m-j}\|u\|_j \leqq C_0\|P_0 u\|_0, \quad u\in S_k,\ k\geqq k_0. \tag{1.11}$$

Now let $a_{\alpha\beta}(z)$ be the functions in (0.1) and write $\alpha_{\alpha\beta}=\sum_{j=0}^\infty a_{\alpha\beta}^j$, $a_{\alpha\beta}^j\in S_j$. After a change of variables of the form $z\mapsto\lambda z$, $\lambda>0$, (which does not change P_0) we may assume that $a_{\alpha\beta}^j(z)$ are all bounded by some constant, independent of α, β, j when $|z|\leqq 1$. Let

$$P_j = \sum_{|\alpha|=|\beta|\leqq m} a_{\alpha\beta}^j(z)\, z^\alpha \frac{\partial^{|\beta|}}{\partial z^\beta}:\ S_k \to S_{k+j}. \tag{1.12}$$

Then, if C_0 is sufficiently large, we have

$$\|P_j u\|_0 \leqq C_0\|u\|_m, \quad u\in S_k, \quad k\in \mathbf{N}. \tag{1.13}$$

We first claim that P: $\mathfrak{m}^k\hat{\mathcal{O}}_0\to\mathfrak{m}^k\hat{\mathcal{O}}_0$ is injective when $k\geqq k_0$ for k_0 given by Lemma 1.3. In fact, let $u=\sum_{j\geqq k}^\infty u_j$, $u_j\in S_j$, and assume that $Pu=0$. Then

$$P_0 u_k = 0, \quad P_0 u_{k+1}+P_1 u_k = 0, \quad P_0 u_{k+2}+P_1 u_{k+1}+P_2 u_k = 0, \ldots$$

and Lemma 1.3 shows that $u_k=0$, $u_{k+1}=0, \ldots$. The injectivity of P on $\mathfrak{m}^k$ follows trivially.

Next we show that $P\colon \mathfrak{m}^k\hat{\mathcal{O}}_0 \to \mathfrak{m}^k\mathcal{O}_0$ is surjective for $k \geqq k_0$. Let $v=\sum_{j\geqq k} v_j \in \mathfrak{m}^k\hat{\mathcal{O}}_0$. The formal solution of $Pu=v$ is then $u=\sum_{j\geqq k} u_j$, where u_j are determined recursively by

$$P_0 u_j + \sum_{l=0}^{j-1} P_{j-l} u_l = v_j, \tag{1.14}$$

so it is clear that $P\colon \mathfrak{m}^k\hat{\mathcal{O}}_0 \to \mathfrak{m}^k\hat{\mathcal{O}}_0$ is surjective.

Now let $v\in \mathfrak{m}^k\mathcal{O}_0$ so that $\|v_j\|_0 \leqq D^{j+1}$ for some constant D. Let $C>0$ be so large that $C\geqq 2C_0^2 D$, $C\geqq 4C_0^2$. (Recall that $C_0\geqq 1$). Then from $P_0 u_k = v_k$ we get

$$\|u_k\|_m \leqq C_0\|v_k\|_0 \leqq C_0 D^{k+1} \leqq C^{k+1}.$$

Assume that

$$\|u_l\|_m \leqq C^{l+1}, \quad k \leqq l < j.$$

Then from (1.14), (1.13) and Lemma 1.3, we get

$$\|u_j\|_m \leqq C_0\big(D^{j+1} + \sum_{l=0}^{j-1} C_0 C^{l+1}\big) \leqq C_0^2\left(D^{j+1} + \frac{C(C^j-1)}{(C-1)}\right)$$

$$\leqq C_0^2(D^{j+1}+2C^j) \leqq \frac{1}{2}C^{j+1} + \frac{1}{2}C^{j+1} = C^{j+1}.$$

By iteration we get $\|u_j\|_m \leqq C^{j+1}$ for all j. Hence Corollary 1.2 shows that u is convergent. This completes the proof of Theorem 0.1.

2. Some Consequences

First note that P naturally induces operators

$$P_k'\colon \mathfrak{m}^k \to \mathfrak{m}^k$$

and

$$P_k''\colon \mathcal{O}_0/\mathfrak{m}^k \to \mathcal{O}_0/\mathfrak{m}^k.$$

Then we have the following commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & \mathfrak{m}^k \arrow[r] \arrow[d, "P_k'"] & \mathcal{O}_0 \arrow[r] \arrow[d, "P"] & \mathcal{O}_0/\mathfrak{m}^k \arrow[r] \arrow[d, "P_k''"] & 0 \\ 0 \arrow[r] & \mathfrak{m}^k \arrow[r] & \mathcal{O}_0 \arrow[r] & \mathcal{O}_0/\mathfrak{m}^k \arrow[r] & 0. \end{tikzcd}$$

Here the horizontal lines are exact. We then get the exact sequence

$$0 \to \operatorname{Ker} P_k' \to \operatorname{Ker} P \to \operatorname{Ker} P_k'' \to \operatorname{Coker} P_k' \to \operatorname{Coker} P \to \operatorname{Coker} P_k'' \to 0. \tag{2.1}$$

If $k \geqq k_0$, where k_0 is given in Theorem 0.1, then P'_k is bijective. Hence we have $\operatorname{Ker} P'_k = 0$, $\operatorname{Coker} P'_k = 0$. From (2.1) we then get $\operatorname{Ker} P \cong \operatorname{Ker} P''_k$, $\operatorname{Coker} P \simeq \operatorname{Coker} P''_k$. Let us write $\hat{P}$ when we consider P as an operator on $\hat{\mathcal{O}}_0$. The argument above works for formal power series as well and we get the following comparison theorem.

Theorem 2.1. *Let P be a linear differential operator which satisfies conditions* (0.1) *and* (0.2). *Let k_0 be given by Theorem* 0.1. *Then we have the isomorphisms*

$$\operatorname{Ker} \hat{P} \simeq \operatorname{Ker} P \simeq \operatorname{Ker} P''_k, \quad \operatorname{Coker} \hat{P} \simeq \operatorname{Coker} P \simeq \operatorname{Coker} P''_k$$

for every $k \geqq k_0$.

It will be interesting to examine whether such a comparison theorem holds for the pair of $\mathcal{O}_0$ and the formal completion along Y, $\hat{\mathcal{O}}_{X/Y,0}$, where Y is a subvariety of $X = \mathbf{C}^n$. Here $\hat{\mathcal{O}}_{X/Y} = \varprojlim \mathcal{O}_X/J^k$, where J is the defining Ideal of Y. As one of the simplest cases, we now discuss the case where Y is a nonsingular hypersurface. Our argument also works for a nonsingular submanifold Y. The needed modification is only that we should deal with some determined systems instead of scalar operators in § 1.

In order to prove such a comparison theorem, we first prepare the following

Theorem 2.2. *Let P be a linear differential operator satisfying the conditions* (0.1) *and* (0.2) *and let r_0 be a sufficiently small positive number. Assume that $Pu = v$, where $u \in \hat{\mathcal{O}}_0$, $v \in \mathcal{O}_0$ and v converges for $|z| < r$, $0 < r \leqq r_0$. Then $u \in \mathcal{O}_0$ and u converges for $|z| < r$.*

Proof. First note that Theorem 0.1 asserts that $u \in \mathcal{O}_0$. On the other hand, since S^{2n-1} is noncharacteristic with respect to P_0, the spheres $\{|z| = r\}$ $(0 < r \leqq r_0)$ are all noncharacteristic with respect to P for sufficiently small r_0. Then by a result of Zerner [5] and Bony—Schapira [2] we find that u extends across all spheres $\{|z| = r'\}$ $(0 < r' < r)$. Hence u converges for $|z| < r$. Q.E.D.

Now we denote $z = (x, y)$, $x \in \mathbf{C}^{n-1}$, $y \in \mathbf{C}$, so that Y is defined by $y = 0$. Then, $\hat{\mathcal{O}}_{X/Y}$ is nothing but the ring of formal power series in y whose coefficients are holomorphic functions in x. More precisely, an element in $\hat{\mathcal{O}}_{X/Y,0}$, the stalk of the sheaf $\hat{\mathcal{O}}_{X/Y}$ at 0, has the form $u = \sum_{k=0}^{\infty} u_k(x) y^k$, where $u_k(x)$ are power series which converge in some ball $\{|x| \leqq r(u)\}$ independent of k.

We now assume in addition to (0.2) that

(2.2) for every $u \in \hat{\mathcal{O}}_0$ and $k \in \mathbf{N}$, there exists $w \in \hat{\mathcal{O}}_0$ such that

$$P(y^k u(x, y)) = y^k w(x, y).$$

This means that we can write

$$P = \sum_{\nu=0}^{m} A_{m-\nu}\left(x, y, \frac{\partial}{\partial x}\right)\left(y\frac{\partial}{\partial y}\right)^{\nu} \tag{2.3}$$

where $A_{m-\nu}$ is of order $\leqq m-\nu$. We write

$$A_{m-\nu}\left(x, y, \frac{\partial}{\partial x}\right) = \sum_{j=0}^{\infty} A_{m-\nu}^{j}\left(x, \frac{\partial}{\partial x}\right) y^{j}, \tag{2.4}$$

where (after a change of variables: $z \mapsto \lambda z$) we may assume that (2.4) converges for $|x| \leqq 1$. $|y| \leqq 1$. The operators $Q_k\left(x, \frac{\partial}{\partial x}\right) = \sum_{\nu=0}^{m} A_{m-\nu}^{0} k^{\nu}$ are of the same type as (0.1) and satisfy (0.2) if we view them as operators on $\mathbf{C}^{n-1}$. Moreover, the principal part of Q_k is independent of k. Hence, if we choose Q_k as P in Theorem 2.2, Theorem 2.2 holds with r_0 independent of k.

Lemma 2.3. *If* $u \in \hat{\mathcal{O}}_0$, $v \in \hat{\mathcal{O}}_{X/Y,0}$, *and* $Pu = v$ *holds, then* $u \in \hat{\mathcal{O}}_{X/Y,0}$.

Proof. It follows from the definition of $\hat{\mathcal{O}}_{X/Y,0}$ that v has the form $\sum_{k=0}^{\infty} v_k(x) y^k$, where $v_k(x)$ converges for $|x| < r$, $0 < r \leqq r_0$. Write u as $\sum_{k=0}^{\infty} u_k(x) y^k$, where u_k is a formal power series in $(n-1)$-variables. Then we get

$$\sum_{\nu=0}^{m} \sum_{j=0}^{\infty} \sum_{k'=0}^{\infty} \left\{A_{m-\nu}^{j}\left(x, \frac{\partial}{\partial x}\right) u_{k'}\right\} (k')^{\nu} y^{j+k'} = \sum_{k=0}^{\infty} v_k(x)\, y^k$$

so that

$$\sum_{\nu=0}^{m} \sum_{k'+j=k} (k')^{\nu} A_{m-\nu}^{j}\left(x, \frac{\partial}{\partial x}\right) u_{k'}(x) = v_k(x). \tag{2.5}$$

Suppose that we have already shown that $u_{k'}$ converges for $|x| < r$ when $k' < k$. Then (2.5) shows that $Q_k u_k(x)$ converges for $|x| < r$ and Theorem 2.2 combined with the previous remark on Q_k entails that u_k also converges for $|x| < r$. Repeating this argument we see that $u \in \hat{\mathcal{O}}_{X/Y,0}$. Q.E.D.

Now we find the following

Theorem 2.4. *Assume that a linear differential operator* P *satisfies conditions* (0.1), (0.2) *and* (2.2). *Let* $\hat{P}_{X/Y}$: $\hat{\mathcal{O}}_{X/Y,0} \to \hat{\mathcal{O}}_{X/Y,0}$ *be given by* P. *Then there are natural isomorphisms*

$$\operatorname{Ker} P \simeq \operatorname{Ker} \hat{P} \simeq \operatorname{Ker} \hat{P}_{X/Y}, \tag{2.6}$$

$$\operatorname{Coker} P \simeq \operatorname{Coker} \hat{P} \simeq \operatorname{Coker} \hat{P}_{X/Y}. \tag{2.7}$$

Proof. We have $\operatorname{Ker} P \subset \operatorname{Ker} \hat{P}_{X/Y} \subset \operatorname{Ker} \hat{P}$ and since $\operatorname{Ker} \hat{P} = \operatorname{Ker} P$, (2.6) is clear. To prove (2.7) we consider the natural maps

$$\mathcal{O}_0/P\mathcal{O}_0 \xrightarrow{a} \hat{\mathcal{O}}_{X/Y,0}/\hat{P}_{X/Y}\hat{\mathcal{O}}_{X/Y,0} \xrightarrow{b} \hat{\mathcal{O}}_0/\hat{P}\hat{\mathcal{O}}_0.$$

We shall show that b is bijective. By Theorem 2.1 we know that ba is surjective. Thus b is surjective. On the other hand, the injectivity of b follows from Lemma 2.3 and the proof is complete. Q.E.D.

Remark. The following example shows that (2.7) does not hold in general without the condition (2.2) even if P satisfies conditions (0.1) and (0.2).

Example 2.5. Let $P = x\dfrac{\partial}{\partial x} + (x+y)\dfrac{\partial}{\partial y}$ $(x, y \in \mathbf{C})$. Then b: $\operatorname{Coker} \hat{P}_{X/Y} \to \operatorname{Coker} \hat{P}$ is not injective.

Proof. Since P satisfies conditions (0.1) and (0.2), $\operatorname{Ker} P = \operatorname{Ker} \hat{P}$ holds (Theorem 2.1). Therefore, in order to show that b is not injective, it suffices to show that there exists $u \in \hat{\mathcal{O}}_0$ which does not belong to $\hat{\mathcal{O}}_{X/Y,0}$ such that $Pu = f$ holds for some $f \in \hat{\mathcal{O}}_{X/Y,0}$. In fact, if b is injective, we can find $v \in \hat{\mathcal{O}}_{X/Y,0}$ such that $Pv = f$ holds for such f. Then $P(u-v) = 0$ holds, and hence $u - v$ is a convergent power series in (x, y). Hence u itself must be contained in $\hat{\mathcal{O}}_{X/Y,0}$. This is a contradiction. Therefore, b is not injective.

We now try to find such u and f. First choose nonconvergent formal power series $u_0(x)$ and $u_1(x)$ so that they satisfy

$$\frac{\partial}{\partial x} u_0 + u_1 = 0. \tag{2.8}$$

Set $f_0 = 0$. We define u_j $(j \geqq 2)$ and f_j $(j \geqq 1)$ successively by setting f_j to be the constant term of $\left(x\dfrac{\partial}{\partial x} + j\right) u_j$ and $u_{j+1} = \left(f_j - \left(x\dfrac{\partial}{\partial x} + j\right) u_j\right) / (j+1)x$ for $j \geqq 1$. Clearly, $Pu = f$ holds in $\hat{\mathcal{O}}_0$, $\sum_{j=0}^{\infty} f_j y^j$ belongs to $\hat{\mathcal{O}}_{X/Y,0}$ and $\sum_{j=0}^{\infty} u_j y^j$ does not belong to $\hat{\mathcal{O}}_{X/Y,0}$. Thus we have seen that b is not injective.

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