Solvability and alternative theorems for a class of non-linear functional equations in Banach spaces

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0. Introduction

In a preceding paper [4], we proved the existence of a minimum for mappings $F\colon B\to\mathbf{R}$ from a reflexive Banach space B into the reals under the following assumptions (we present only a special case):

(0.1) *F is lower semi-continuous in the weak topology.*
(0.2) *F is bounded from below.*
(0.3) *F is convex (resp. satisfies a surrogate convexity).*
(0.4) *F is semi-coercive, i.e.*

$$F(u) \geqq c\|u\|^p - K\|Qu\|^p - K$$

with constants $c, K, p>0$ and a linear projection Q onto a finite dimensional subspace.

$$F(u+tv) \quad \text{is a polynomial in} \quad t\in\mathbf{R}. \tag{0.5}$$

Furthermore, we obtained a Fredholm alternative theorem for the existence of minima of $F(u)+\langle g, u\rangle$, $g\in B^*$.

Note that condition (0.4) frequently occurs in the theory of partial differential equations. It is well-known that condition (0.5) can be deleted if "full" coercivity $F(u)\geqq c\|u\|^p-K$ holds.

In this paper, we present a non-variational analogue of the above theorem for continuous mappings T from a Banach space B into its dual B^*. In particular we shall show that equ. $Tu=0$ is solvable if the following conditions hold:

$$\langle Tu-Tv, u-v\rangle \geqq 0, \quad u, v\in B \tag{0.6}$$

$$\liminf \langle Tu, u\rangle/\|u\| \geqq 0 \quad (\|u\| \to\infty) \tag{0.7}$$

$$\langle Tu, u\rangle \geqq c\|u\|^p - K\|Qu\|^p - K \tag{0.8}$$

with c, K, p, Q *as in* (0.4)

(0.9) $\langle T(u+tv), w\rangle$ *is a polynomial in* $t\in\mathbf{R}$.

The difference between this result and the classical one is that we do not assume the "full" coerciveness $\langle Tu, u\rangle \geqq c\|u\|^p - K$. Again, condition (0.8) is natural for applications involving partial differential equations, however condition (0.9) may not be deleted in this case.

Our method of proof yields the following alternative theorem: Under the above conditions — without the asymptotic non-negativity (0.7) — the linear hull of the range $R(T)$ of T has finite codimension and equ. $Tu=f$ is solvable if and only if

$$f-T(0)\perp(R(T)-T(0))^{\perp}$$

i.e. $R(T-T(0))$ *is a linear closed subspace of* B.

Alternative theorems with linear principal part have been obtained by Kačurovskii [7], [8], Hess [6] and Petryshyn. Our conditions allow polynomial growth of the mapping T. The alternative theorems of Pohodjayev [12], Nečas [10] and Petryshyn [11], Theorem 2, are of a different type since they treat only the surjectivity of T.

1. The finite dimensional case

We study continuous mappings $T\colon \mathbf{R}^n\to\mathbf{R}^n$ with the following properties

(1.1) *"Polynomial behaviour". If for some pair* $v, w\in\mathbf{R}^n$ $\limsup |(T(w+tv), v)|<\infty$ $(t\to\infty)$ *then* $(T(w+tv), v)$ *is constant in* $t\in\mathbf{R}$. Here, $(.\,,.)$ denotes the Euclidean scalar product.

(1.2) *"Even polynomial behaviour". If for some pair* $v, w\in\mathbf{R}^n$ *we have*

(i) $$\liminf |t|^{-1}\varphi(t)\geqq 0 \quad (|t|\to\infty)$$

(ii) $$\limsup |t|^{-1}\varphi(t)\geqq 0 \quad (|t|\to\infty),$$

where

$$\varphi(t)=(T(w+tv), w+tv),$$

then

$$t^{-1}\varphi(t)\to 0 \quad (|t|\to\infty)$$

(1.3) *"Asymptotic monotonicity". For any fixed* $v\in\mathbf{R}^n$

$$\liminf |u-v|^{-1}(Tu-Tv, u-v)\geqq 0 \quad (|u|\to\infty)$$

(1.4) *"Asymptotic non-negativity."*

$$\liminf |u|^{-1}(Tu, u)\geqq 0 \quad (|u|\to\infty).$$

Property (1.2) holds if the components of T are polynomials in n variables. Then $\varphi(t)$ is a polynomial in t and condition (i) implies that φ is an even polynomial. Condition (ii) implies that φ is at most linear (for this special pair v, w) and, being even, must be constant. But then, $t^{-1}\varphi(t)\to 0$ $(|t|\to\infty)$.

Theorem 1.1. *Let* $T\colon \mathbf{R}^n\to\mathbf{R}^n$ *be a continuous mapping which satisfies the conditions* (1.1)—(1.4). *Then the equation* $Tu=0$ *is solvable.*

For the proof of Theorem 1.1 and, later, Theorem 1.2, we need the following technical

Lemma 1.1. *Let* $T\colon \mathbf{R}^n\to\mathbf{R}^n$ *be a continuous mapping which satisfies* (1.1)—(1.4). *If for some* $v\in\mathbf{R}^n$ *we have*

$$\sup\{(Tw, v)\mid w\in\mathbf{R}^n\}<\infty,$$

then $v\perp R(T)$.

Here $R(T)$ denotes the range of T.

Proof. Let $t\in\mathbf{R}$. We insert $w+tv$ for w in (1.5) and obtain

$$g(t):=\big(T(w+tv), v\big)\leqq K,\quad t\in\mathbf{R}. \tag{1.6}$$

We show that $g(t)$ is bounded from below for fixed $w\in\mathbf{R}^n$. By (1.3)

$$\liminf |t|^{-1}\big(T(w+tv)-Tw, tv\big)\geqq 0\quad (t\to\infty)$$

and hence there exist constants $C(w)$ and t_0 such that

$$\big(T(w+tv), v\big)\geqq -C(w),\quad t\geqq t_0.$$

Thus, for fixed $w\in\mathbf{R}^n$, $g(t)$ is bounded from above and below and hence, by condition (1.1)

$$\big(T(w+tv), v\big) = \text{const} := (Tw, v),\quad t\in\mathbf{R}. \tag{1.7}$$

Now, let

$$\varphi(t) = \big(T(w+tv), w+tv\big),\quad t\in\mathbf{R}. \tag{1.8}$$

By (1.3)

$$\liminf |t|^{-1}\big(T(w+tv)-T(2w), -w+tv\big)\geqq 0\quad (|t|\to\infty).$$

This yields in view of (1.7)

$$\limsup |t|^{-1}\big(T(w+tv), w\big)\leqq C(w),\quad |t|\to\infty,\quad t\in\mathbf{R}$$

with some constant $C(w)$. Using (1.7) again we obtain

$$\limsup |t|^{-1}\varphi(t)<\infty\quad (|t|\to\infty)$$

From this, condition (1.4), and (1.2) we conclude

$$|t|^{-1}\varphi(t)\to 0\quad (|t|\to\infty). \tag{1.9}$$

Finally, for fixed $s \in \mathbf{R}$, we have in view of (1.3)

$$\liminf |t|^{-1}(T(w+tv)-T(sw), (1-s)w+tv) \geqq 0 \quad (|t| \to \infty). \tag{1.10}$$

Using (1.7), (1.8), and (1.10)

$$\liminf |t|^{-1}[(1-s)\varphi(t)+s(Tw, tv)-(T(sw), (1-s)w+tv)] \geqq 0 \quad (|t| \to \infty)$$

Passing to the limit $t \to \pm\infty$ and using (1.9) we find the inequality

$$\pm s(Tw, v) \mp (T(sw), v) \geqq 0$$

from which

$$s(Tw, v) = (T(sw), v)$$

and, in view of (1.5)

$$s(Tw, v) \leqq K, \quad s \in \mathbf{R}.$$

Passing to the limit $s \to \pm\infty$ we obtain

$$(Tw, v) = 0, \quad w \in \mathbf{R}^n$$

q.e.d.

Proof of Theorem 1.1: Set $T_\varepsilon u = Tu + \varepsilon u$, $\varepsilon > 0$. In view of (1.4) the mapping T_ε is coercive, i.e. $(T_\varepsilon u, u)/|u| \to \infty$ as $|u| \to \infty$. Thus there exists a solution u_ε of the equation $T_\varepsilon u = 0$ (cf. e.g. [3]). If the sequence (u_ε) is bounded as $\varepsilon \to 0$ it has a clusterpoint u^* which solves $Tu^* = 0$. Hence we may assume that for a sequence Λ_0 of numbers $\varepsilon \to 0$ we have $|u_\varepsilon| \to \infty$, $|u_\varepsilon| \neq 0$. Selecting a subsequence $\Lambda \subset \Lambda_0$ we may assume that $|u_\varepsilon|^{-1} u_\varepsilon \to v$ $(\varepsilon \in \Lambda,\ \varepsilon \to 0)$ for some $v \in \mathbf{R}^n$ with $|v| = 1$. We show $(Tw, v) = 0$ for all $w \in \mathbf{R}^n$. By condition (1.3)

$$\liminf (T_\varepsilon u_\varepsilon - T_\varepsilon w, u_\varepsilon - w)/|u_\varepsilon - w| \geqq 0 \quad (\varepsilon \to 0,\ \varepsilon \in \Lambda). \tag{1.11}$$

Using $T_\varepsilon u_\varepsilon = 0$ and then passing to the limit $\varepsilon \to 0$ we obtain from (1.11)

$$-(Tw, v) \geqq 0, \quad w \in \mathbf{R}^n$$

and by Lemma 1.1

$$(Tw, v) = 0, \quad w \in \mathbf{R}^n. \tag{1.12}$$

In the case $n=1$ this gives us the solvability of $Tu=0$. For $n \geqq 2$, we proceed by induction: Let $\langle v \rangle$ be the one dimensional subspace spanned by v and $V = \langle v \rangle^\perp$ its orthogonal complement. Then the restriction T_V of T to V maps V into itself and satisfies the conditions (1.1)—(1.4). By induction hypothesis, there is a $u^* \in V$ such that $(Tu^*, z) = 0$ for all $z \in V$. Using (1.12) it follows that $Tu^* = 0$ which proves the theorem.

With the method of the proof of Theorem 1 one can obtain the following "alternative theorem".

Theorem 1.2. *Let* $T: \mathbf{R}^n \to \mathbf{R}^n$ *be a continuous mapping which satisfies the conditions* (1.1)—(1.3) *and let* $T(0)=0$. *Then the equation* $Tu=f$ *is solvable if and only if* $f \perp R(T)^\perp$, *i.e.* $R(T)$ *is a linear subspace of* $\mathbf{R}^n$.

In the simplest case of a monotone mapping T with polynomials as components the above theorem yields that *the equation* $Tu=f$ *is solvable if and only if* $f-T(0)$ *is orthogonal to* $R(T-T(0))^\perp$.

We first prove

Lemma 1.2. *Let* $v \in \mathbf{R}^n$, $v \neq 0$, $v \perp R(T)$, $V=\langle v\rangle^\perp$ *and* $z \in \mathbf{R}^n$ *such that* $z \perp T(V)$. *Then, under the assumptions of Theorem* 1.2, $z \perp R(T)$.

Here $\langle v\rangle^\perp$ denotes the orthogonal complement of the space spanned by v.

Proof. Let $z=z_1+\zeta v$, $z_1 \in V$, $\zeta \in \mathbf{R}$, and $w \in V$, $\alpha \in \mathbf{R}$. By (1.3) and the orthogonality $v \perp R(T)$

$$\liminf |t|^{-1}\big(T(w+tz_1+\alpha tv)-T(w\pm 2tz_1),\ \pm tz_1\big) \geqq 0 \quad (t \to \infty).$$

We have $z_1 \perp T(V)$ and $w \pm 2tz_1 \in V$. Thus $\big(T(w\pm 2tz_1), z_1\big)=0$ and

$$\lim\big(T(w+tz_1+\alpha tv), z_1+\alpha v\big) = \lim\big(T(w+tz_1+\alpha tv), z_1\big) = 0 \quad (t \to \infty).$$

By (1.1) thence $\big(T(w+tz_1+\alpha tv), z_1+\alpha v\big)=0$ for all t or $\big(T(w+tz_1+\alpha tv), z_1\big)=0$ for all $t \in \mathbf{R}$, $\alpha \in \mathbf{R}$, $w \in V$. Setting $t=1$, the lemma follows.

Proof of Theorem 1.2. The "only if" — part of the theorem is trivial: If f is not orthogonal to $R(T)^\perp$, then there is a $w \in \mathbf{R}^n$ such that $(f, w) \neq 0$ and $(w, Tx)=0$, $x \in \mathbf{R}^n$. But then equ. $Tu=f$ cannot be solvable.

Since $T(0)=0$, we conclude from (1.3) the asymptotic nonnegativity (1.4) and the coercitivity of the mapping $T_\varepsilon=\varepsilon\,\mathrm{Id}+T$. If u_ε remains bounded as $\varepsilon \to 0$, a clusterpoint u^* of (u_ε) exists and is a solution of $Tu=f$. Thus we may assume that (u_ε) is unbounded and that for a subsequence Λ we have the convergence $|u_\varepsilon| \to \infty$ and $|u_\varepsilon|^{-1}u_\varepsilon \to v$ $(\varepsilon \to 0,\ \varepsilon \in \Lambda)$ with $|v|=1$. By (1.3)

$$\liminf |u_\varepsilon-w|^{-1}(T_\varepsilon u_\varepsilon - Tw, u_\varepsilon - w) \geqq 0 \quad (\varepsilon \to 0,\ \varepsilon \in \Lambda)$$

for every $w \in \mathbf{R}^n$ and hence

$$(f-Tw, v) \geqq 0, \quad w \in \mathbf{R}^n. \tag{1.11}$$

From Lemma 1.1 we then conclude

$$(v, Tw) = 0, \quad w \in \mathbf{R}^n. \tag{1.12}$$

By hypothesis, $f \perp R(T)^\perp$, and thus

$$(f, v) = 0. \tag{1.13}$$

If $n=1$, it follows from (1.13) that $f=0$ and from (1.12) that $Tw=0$, $w\in\mathbf{R}^n$, i.e. $Tu=f$ is solvable. If $n\geqq 2$ we conclude from (1.12) that

$$T\colon V\to V$$

where

$$V:=\langle v\rangle^{\perp}.$$

Let $z\perp T(V)$. By Lemma 1.2 we conclude $z\perp R(T)$ and hence $f\perp z$ by hypothesis. Therefore, we have $f\perp (T(V))^{\perp}$ and, by (1.13), $f\in V$. Applying the induction hypothesis for the dimension $n-1$ to the mapping $T\colon V\to V$ we obtain the theorem.

2. The infinite dimensional case

In this section we want to generalize the results of section 1 to the case of *regular* mappings $T\colon B\to B^*$ from a reflexive real Banach space B into its dual B^*.

We call a mapping $T\colon B\to B^*$ *regular* if for every bounded closed convex set $\mathbf{K}$ and any $f\in B^*$ the variational inequality

$$\langle Tu-f, u-v\rangle \leqq 0, \quad v\in\mathbf{R}$$

has a solution $u\in\mathbf{K}$.

Monotone or pseudomonotone continuous mappings are regular (see [2], [3]). We shall deal with the following conditions

(2.1) *"Polynomial behaviour". If for some pair $v, w\in B$*

$$\limsup |\langle T(w+tv), v\rangle| < \infty \quad (t\to\infty)$$

then $\langle T(w+tv), v\rangle$ is constant in $t\in\mathbf{R}$.

(2.2) *"Even polynomial behaviour". If for some pair $v, w\in B$ we have*

(i) $$\liminf |t|^{-1}\varphi(t) \geqq 0 \quad (|t|\to\infty)$$

(ii) $$\limsup |t|^{-1}\varphi(t) < \infty \quad (|t|\to\infty)$$

where

$$\varphi(t) = \langle T(w+tv), w+tv\rangle,$$

then

$$t^{-1}\varphi(t)\to 0 \quad (|t|\to\infty)$$

(2.3) *"Asymptotic monotonicity". For every $v\in B$*

$$\liminf \|u-v\|^{-1}\langle Tu-Tv, u-v\rangle \geqq 0 \quad (u\in B,\ \|u\|\to\infty)$$

(2.4) *"Asymptotic non-negativity".*

$$\liminf \|u\|^{-1}\langle Tu, u\rangle \geqq 0 \quad (u\in B,\ \|u\|\to\infty)$$

(2.5) *"Semi-coercitivity". There exists a finite dimensional subspace* $V \subset B$ *with bounded linear projection* $Q: B \to V$ *and a constant* C *such that*

$$\|u\| \leqq C\|Qu\|+C \quad \text{for all } u \text{ with } \langle Tu, v\rangle \leqq 0.$$

For Theorem 2.2 we need a stronger condition

(2.5'). *There exists a finite dimensional subspace* $V \subset B$ *with bounded linear projection* $Q: B \to V$ *such that for every* $K \in \mathbf{R}$

$$\sup \{\|u\|/(\|Qu\|+1) \mid u \in B, \ \|u\|^{-1}\langle Tu, u\rangle \leqq K\} < \infty.$$

Remark. Condition (2.1) and (2.2) have been explained in section 1. Condition (2.5) is satisfied if the following "Garding"-type inequality holds:

$$\langle Tu, u\rangle \geqq c\|u\|^p - \lambda\|Qu\|^p - \lambda$$

with constants $\lambda, c, p > 0$ resp. $p > 1$ in the case (2.5').

Theorem 2.1. *Let* $T: B \to B^*$ *be a regular mapping from a real reflexive Banach space* B *into its dual* B^*, *which satisfies* (2.1)—(2.5). *Then the equation* $Tu=0$ *has a solution.*

We first prove

Lemma 2.1. *Let* $V_0 \subset B$ *be a linear subspace such that* $V_0 \perp R(T)$. *Then, under the assumptions of Theorem* 2.1,

$$\dim V_0 \leqq \dim V.$$

Proof. We argue that the assumption of the existence of a space V_0 with $\dim V_0 = n+1$, $n := \dim V$, and $V_0 \perp R(T)$ leads to a contradiction. Let $z_i \in V_0$ be $n+1$ linearly independent vectors. The $n+1$ vectors $Qz_i \in V$ must be linearly dependent, thus there exist numbers λ_i such that $\sum_i |\lambda_i| \neq 0$ and $\sum_i \lambda_i Qz_i = 0$ $(i=1, \ldots, n+1)$. Let $z = \sum_i \lambda_i z_i$ $(i=1, \ldots, n+1)$. Then $z \neq 0$ and $Qz=0$. By hypothesis

$$\langle T(tz), tz\rangle = 0, \quad t \in \mathbf{R}.$$

On account of the semi-coercitivity (2.5)

$$\|tz\| \leqq C\|tQz\|+C = C$$

which, as $t \to \infty$, results in a contradiction.

Proof of Theorem 2.1. We may assume $\dim B = \infty$ and suppose that equ. $Tu=0$ is *not* solvable. By induction we then construct linearly independent elements $z_i \in B$, $i=1, 2, 3, \ldots$, such that $z_i \perp R(T)$ which contradicts Lemma 2.1. Assume that z_j, $j=1, 2, \ldots, i-1$, have been constructed. Let W be a closed linear complement to the space spanned by the elements $z_1, \dots, z_{i-1}$. For $i=1$ set $V_i=\{0\}$. Since T is regular, the variational inequality

(2.6) $$\langle Tu, u-x\rangle \leqq 0, \quad x\in B_R\cap W, \quad B_R = \{x\in B \mid \|x\| \leqq R\},$$

has a solution $u_R\in B_R\cap W$. If u_R lies in the algebraic interior of $B_R\cap W$ for some $R>0$ then $Tu_R \perp W$ and hence $Tu_R \perp W\oplus V_i=B$ by the induction hypothesis. This leads to the contradiction $Tu_R=0$. Therefore, we may assume $u_R\in\partial B_R$ and $\|u_R\|=R$. Setting $x=0$ in (2.6) we obtain

$$\langle Tu_R, u_R\rangle \leqq 0$$

and from (2.5)

(2.7) $$\|u_R\| \leqq C\|Qu_R\|+C$$

and

(2.8) $$\|Qu_R\| \to\infty \quad (R\to\infty).$$

By (2.7), (2.8), and the boundedness of Q

(2.9) $$\|Qu_R\| \leqq K\|u_R\| \leqq 2CK\|Qu_R\|, \quad R\geqq R_0.$$

From (2.9) and the asymptotic monotonicity (2.3) we conclude for any $w\in B$

(2.10) $$\liminf \|Qu_R\|^{-1}\langle Tu_R-Tw, u_R-w\rangle \geqq 0 \quad (R\to\infty).$$

Let $w=w_1+w_2$, $w_1\in W$, $w_2\in V_i$. Since u_R satisfies the variational inequality (2.6), $w_1\in W\cap B_R$ for $R\geqq R'$, and $w_2\perp R(T)$, we obtain from (2.10)

$$\liminf \|Qu_R\|^{-1}\langle -Tw, u_R\rangle \geqq 0 \quad (R\to\infty).$$

By (2.7), the elements $\|Qu_R\|^{-1}u_R$ remain bounded uniformly as $R\to\infty$, and since B is reflexive there exists a subsequence Λ and an element $z\in W$ such that

(2.11) $$\|Qu_R\|^{-1}u_R \rightharpoonup z \quad \text{weakly} \quad (R\to\infty,\ R\in\Lambda)$$

and

(2.12) $$\langle -Tw, z\rangle \geqq 0, \quad w\in B.$$

Furthermore, since Q maps B onto a finite dimensional space we conclude from (2.11) that $\|Qz\|=1$ and $z\neq 0$.

Now, let V_w be the space spanned by w and z which we equip with some scalar product $(.\,,.)$. Let $T_w\colon V_w\to V_w$ be the mapping defined by

$$(T_w x, y) = \langle Tx, y\rangle, \quad x, y\in V_w.$$

T_w satisfies the assumptions of the mapping T in Lemma 1.1. Hence, by (2.12)

(2.13) $$\langle Tw, z\rangle = 0, \quad w\in B.$$

Since $z \in W$, $z \neq 0$, we have that $z \notin V_i$ and the element $z_i := z$ is linearly independent of $z_1, \ldots, z_{i-1}$, but orthogonal to $R(T)$. This completes the construction of the z_i and we obtain a space $V_0 \perp R(T)$ with $\dim V_0 = \infty$ which contradicts Lemma 2.1. The theorem is proved.

Theorem 2.2. *Let $T: B \to B^*$ be a regular mapping from a real reflexive Banach space into its dual B^*, which satisfies (2.1)—(2.3), (2.5) and the condition $T(0)=0$. Then the equation $Tu=f \in B^*$ is solvable if and only if $f \perp R(T)^{\perp}$, i.e. $R(T)$ is a linear and closed subspace of B. Furthermore,*

$$\dim R(T)^{\perp} \leqq \dim V (< \infty).$$

Proof. We note first that also condition (2.4) holds on account of (2.3) and $T(0)=0$. The "only if-part" of the theorem is trivial, cf. Theorem 1.2. For the "if-part" we may assume $\dim B = \infty$ and suppose that the equation $Tu=f$ has no solution where $f \perp R(T)^{\perp}$. Similarly to the proof of Theorem 2.1 we construct linearly independent elements $z_i \in B$, $i=1, 2, 3, \ldots$ such that $z_i \perp R(T)$ and $z_i \perp f$, which contradicts Lemma 2.1. Assume that z_j, $j=1, 2, \ldots, i-1$ have been constructed. Let W be a closed linear complement to the space V_i spanned by the elements $z_1, \ldots, z_{i-1}$. Set $V_1 = \{0\}$. Since T is regular, there exists an $u_R \in B_R \cap W$ such that

$$\langle Tu_R - f, u_R - x \rangle \leqq 0, \quad x \in B_R \cap W. \tag{2.14}$$

If u_R lies in the algebraic interior of $B_R \cap W$ for some $R>0$, then $Tu_R - f \perp W$ and hence $Tu_R - f \perp W \oplus V_i = B$ since f, $Tu_R \perp V_i$ by induction hypothesis. This yields the contradiction $Tu_R - f = 0$. Therefore, we may assume $u_R \in \partial B_R$ and $\|u_R\| = R$. From (2.14)

$$\limsup \|u_R\|^{-1} \langle Tu_R, u_R \rangle < \infty \quad (R \to \infty)$$

and from (2.5′)

$$\|u_R\| \leqq C \|Qu_R\| + C, \quad (R \to \infty), \tag{2.15}$$

with some constant C.
Hence

$$\|Qu_R\| \to \infty \quad (R \to \infty). \tag{2.16}$$

Similarly as in the proof of Theorem 2.1 we conclude from the asymptotic monotonicity condition (2.3) for any $w \in B$

$$\liminf \|Qu_R\|^{-1} \langle Tu_R - Tw, u_R - w \rangle \geqq 0 \quad (R \to \infty). \tag{2.17}$$

From the variational inequality (2.14) and the orthogonality $V_i \perp R(T)$ and $V_i \perp f$, we know for $w = w_1 + w_2 \in B$, $w_1 \in W \cap B_R$, $w_2 \in V_i$,

$$\langle Tu_R - f, \; u_R - w \rangle \leqq 0.$$

From (2.17), we obtain

$$\liminf \|Qu_R\|^{-1}\langle f-Tw,\ u_R-w\rangle \geqq 0 \quad (R\to\infty)$$

for all $w\in B$.

With the same argument as in the proof of Theorem 2.1 we obtain a subsequence Λ and an element $z\in W$, $z\neq 0$, such that

$$\|Qu_R\|^{-1}u_R \rightharpoonup z \quad \text{weakly} \quad (R\to\infty,\ R\in\Lambda)$$

and

$$\langle f-Tw, z\rangle \geqq 0, \quad w\in B.$$

Using the mapping T_w of the proof of Theorem 2.1 we obtain with aid of Lemma 1.1

$$\langle Tw, z\rangle = 0, \quad w\in B$$

and by hypothesis,

$$\langle f, z\rangle = 0.$$

Setting $z_i=z$ this completes the construction of the z_j (Cf. the last lines of the proof of Theorem 2.1).

The inequality $\dim R(T)^{\perp} \leqq \dim V$ follows from Lemma 2.1. The theorem is proved.

The following simple lemma gives some insight into the "linear" structure of the mapping T occurring in Theorem 2.2. For this we need the stronger condition

(2.18) *If for some triple* $v, w, z\in B$, $v\neq 0$,

$$\lim |t|^{-1}\langle T(w+tv), z\rangle = 0 \quad (t\to\pm\infty)$$

then $\langle T(w+tv), z\rangle$ *is constant in* $t\in\mathbf{R}$.

Condition (2.18) is satisfied if $\langle T(w+tv), z\rangle$ is a polynomial in t.

Lemma 2.2. *Let* T: $B\to B^*$ *be a mapping which satisfies the asymptotic monotonicity* (2.3) *and condition* (2.18). *Let* $v\in R(T)^{\perp}$, $v\neq 0$. *Then for every* $w\in B$

$$T(w+tv) \quad \textit{is constant in} \quad t\in\mathbf{R}.$$

Proof. By (2.3) and the orthogonality $v\perp R(T)$ we have for any $w, z\in B$

$$\liminf |t|^{-1}\langle T(w+tv)-T(w-z), z\rangle \geqq 0 \quad (t\to\pm\infty).$$

and hence

$$\liminf |t|^{-1}\langle T(w+tv), z\rangle \geqq 0 \quad (t\to\pm\infty).$$

Replacing z by $-z$ we conclude

$$\lim |t|^{-1}\langle T(w+tv), z\rangle = 0 \quad (t\to\pm\infty)$$

and by (2.18) that $\langle T(w+tv), z\rangle$ is constant in $t\in\mathbf{R}$. The lemma follows.

3. Applications

Let Ω be a bounded domain of $\mathbf{R}^n$ whose boundary satisfies the cone property cf. [1], pp. 11, Def. 2.1. Let $[H^{m,p}]^r$ be the space of r-vector-functions with components in the Sobolev space $H^{m,p}(\Omega)$, $p>1$, cf. [9], and let W be a closed subspace of $[H^{m,p}(\Omega)]^r$. As usual we define

$$\|u\|_p = \left(\int |u|^p dx\right)^{1/p}, \quad \|u\|_{m,p} = \sum_j \|\nabla^j u\|_p + \|u\|_p, \quad (j = 1, \ldots, m).$$

We consider formal differential operators

$$\sum_\alpha (-1)^{|\alpha|} \partial^\alpha A_\alpha(x, u, \nabla u, \ldots, \nabla^m u) \quad (|\alpha| \leqq m)$$

and mappings $T\colon W \to W^*$ defined by

$$\langle Tu, v\rangle := \sum_\alpha \int_\Omega A_\alpha(x, u, \nabla u, \ldots, \nabla^m u)\, \partial^\alpha v\, dx.$$

Here, we have used the usual notation with multi-indices α, and the A_α are functions with values in $\mathbf{R}^r$ which satisfy the following conditions.

(3.1) $A_\alpha(x, \eta)$ is measurable in $x \in \Omega$ and continuous in η.

(3.2) $|A_\alpha(x, \eta)| \leqq K(1 + |\eta|^{p-1})$

(3.3) $\langle Tu, u\rangle \geqq c\|u\|_{m,p}^p - K\|u\|_p^p - K$

(3.4) $\sum_\alpha (A_\alpha(x, \eta) - A_\alpha(x, \zeta))(\eta_\alpha - \zeta_\alpha) > 0, \quad \eta \neq \zeta, \quad |\alpha| = m.$

(3.5) $\sum_\alpha (A_\alpha(x, \eta) - A_\alpha(x, \zeta))(\eta_\alpha - \zeta_\alpha) \geqq -K, \quad |\alpha| \leqq m.$

(3.6) $A_\alpha(x, \eta)$ is a polynomial in η, $|\alpha| \leqq m$.

Condition (3.5) may be replaced by the asymptotic monotonicity condition

$$\liminf \|u - w\|^{-1} \langle Tu - Tw, u - w\rangle \geqq 0 \quad (\|u\| \to \infty),$$

condition (3.6) by the more general condition (2.1)—(2.2).

Theorem 3.1. *Under the assumptions* (3.1)—(3.6), *the equation*

$$Tu = f \in W^*$$

has a solution if and only if

$$f - T(0) \perp (R(T) - T(0))^\perp.$$

Furthermore, $R(T)$ *has finite codimension in* W^*.

Concerning the solvability of $Tu=0$ we need an asymptotic non-negativity condition of type (2.4), say

(3.7) $\sum_\alpha A_\alpha(x, \eta)\eta_\alpha \geqq -K$

Theorem 3.2. *Under the assumptions* (3.1)—(3.7), *the equation* $Tu=0$ *has a solution.*

Proof of Theorem 3.1 *and* 3.2. The continuity and pseudo-monotonicity follow from (3.1), (3.2) and (3.4). (A trick from [5] is used in order to obtain pseudo-monotonicity). Condition (2.1) and (2.2) of Theorem 2.1 and 2.2 follow from (3.6), condition (2.3) from (3.5). (2.5) resp. (2.5′) follow from (3.3) since $p>1$. Rellich's Lemma in L^p is used, cf. [4], § 3, to obtain the finite dimensional projection Q in condition (2.5) resp. (2.5′). Finally, (2.4) is a consequence of (2.7). The results of section 2 then complete the proof.

Example. Let $P_j\colon \mathbf{R}^s\to\mathbf{R}$, $j=1, \ldots, s$, be polynomials such that

(i) $$|P_j(\zeta)| \leqq K+K|\zeta|^{p-1}$$

(ii) $$\sum_j P_j(\zeta)\zeta_j \geqq c|\zeta|^p-K$$

(iii) $$\sum_j (P_j(\zeta)-P_j(\xi))(\zeta_j-\xi_j) \geqq 0 \quad (j = 1, \ldots, s)$$

with constants $K, c>0$ and $p>1$.

(iv) $$P_j(0) = 0, \quad j = 1, \ldots, s.$$

Let L_j be second order uniformly elliptic operators defined by

$$L_j u = \sum_{ik} a_{ik}^{(j)} \partial_i \partial_k u, \quad (i, k = 0, \ldots, n)$$

where ∂_0=identity. Assume $\partial\Omega\in C^{2+\alpha}$, $a_{ik}^{(j)}\in C^\alpha$. Let $W=H_0^{1,p}\cap H^{2,p}$ and $T\colon W\to W^*$ be defined by

$$\langle Tu, v\rangle = \sum_j \int_\Omega P_j(L_1u, \ldots, L_su)L_jv\, dx \quad (j = 1, \ldots, s).$$

Then the equation $Tu=f\in W^*$ *has a solution if and only if* $f\perp R(T)^\perp$. (Note that one may replace (3.4) by (iii).)

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