A minimum modulus theorem and applications to ultradifferential operators

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In this work we give a minimum modulus theorem which enables us to prove the invertibility of a large class of ultradifferential operators.

It is known that the invertibility of convolution operators defined by ultradistributions S with compact support is equivalent to the existence of a certain lower estimation for the modulus of the Fourier transform of S (see [1], [3], [8], [9]). While usual differential operators with constant coefficients are all invertible even in the space of Schwartz's distributions, the following problem is still open:

Is every ultradifferential operator invertible in the corresponding ultradistributions space or at least in the "union" of all ultradistributions?

In [2] Ch. Ch. Chou positively solved this problem for elliptic ultradifferential operators. For the general case some results are given by the same author in [1]; unfortunately, the invertibility is proved under very restrictive conditions on the considered ultradistributions space.

The aim of this work is to give a general minimum modulus theorem, improving the well-known theorem of L. Ehrenpreis [7] and which yields to an invertibility result in ultradistributions spaces satisfying less restrictive conditions then those of Ch. Ch. Chou. In particular we prove that all ultradifferential operators of class $\{k!\,(\prod_{j=2}^{k} \ln j)^{\alpha}\}$ with $\alpha > 1$, are invertible, while Chou's result works only for $\alpha > 2$.

1. A minimum modulus theorem

Let f be an entire function with $f(0)=1$ and let $a_1, a_2, \dots$ be its zeros indexed such that $|a_1| \leqq |a_2| \leqq \dots$. We denote for each $r>0$

$$M_f(r) = \sup_{|z|=r} |f(z)|$$

and

$$n_f(r) = \text{the numbers of } a_k \text{ with } |a_k| \leqq r.$$

Then, by the Jensen formula (see [14]), we have

(1.1) $$n_f(r) \leqq \ln M_f(er), \quad r > 0.$$

The following result is a refinement of the minimum modulus theorem of L. Ehrenpreis from [7] (for other variants see [8] and [9]); in its proof we use techniques both from [7] and from [15], Section 23.

Theorem 1.1. *Let f be an entire function of finite exponential type with $f(0)=1$; then for every $r_0>0$ and $0<\tau<1/8e$, there is an r' with $r_0\leqq r'\leqq(1+\tau)r_0$, such that*

$$\inf_{|z|=r'} \ln|f(z)| \geqq -6\ln M_f(2er_0)\ln\frac{1}{\tau} - 8\sum_{j=1}^{+\infty}\frac{\ln M_f(2^j er_0)}{4^j}.$$

Proof. Let us define the entire function g by

$$g(z) = f(z)f(-z), \quad z\in \mathbf{C}.$$

If for $r>0$, $z_r\in\mathbf{C}$ is such that $|z_r|=r$ and $|f(z_r)|=\inf_{|z|=r}|f(z)|$, then we have

$$\inf_{|z|=r}|g(z)| \leqq |f(z_r)|\cdot|f(-z_r)| \leqq \inf_{|z|=r}|f(z)|\cdot M_f(r),$$

so that

(1.2) $$\inf_{|z|=r}\ln|f(z)| \geqq -\ln M_f(r) + \inf_{|z|=r}\ln|g(z)|, \quad r > 0.$$

If $a_1, a_2, \ldots$ are the zeros of f, indexed such that $|a_1|\leqq|a_2|\leqq\ldots$, by Hadamard's factorization theorem ([14], 8.22 and 8.24), we have

$$\sum_{k=1}^{+\infty}\frac{1}{|a_k|^2} < +\infty \quad \text{and} \quad g(z) = \prod_{k=1}^{\infty}\left(1-\frac{z^2}{a_k^2}\right), \quad z\in\mathbf{C}.$$

Let $r_0>0$ and $0<\tau<1/8e$ be fixed and denote

$$n' = n_f((1-\tau)r_0), \quad n'' = n_f(2r_0).$$

We define the entire functions g_1, g_2, g_3 by

$$g_1(z) = \prod_{k\leqq n'}\left(1-\frac{z^2}{a_k^2}\right),$$

$$g_2(z) = \prod_{n'<k\leqq n''}\left(1-\frac{z^2}{a_k^2}\right),$$

$$g_3(z) = \prod_{k\geqq n''}\left(1-\frac{z^2}{a_k^2}\right).$$

Then

$$g = g_1g_2g_3.$$

Let r with $r_0 \leqq r \leqq (1+\tau)r_0$ and put $n=n_f(r)$; let further $z\in\mathbf{C}$ such that $|z|=r$. We can write

$$g_1(z) = \frac{z^{2n}}{a_1^2 \dots a_n^2} \cdot \frac{a_{n'+1}^2 \dots a_n^2}{z^{2(n-n')}} \prod_{k\leqq n'} \left(\frac{a_k^2}{z^2} - 1\right).$$

As for $1 \leqq k \leqq n$, $|a_k| \leqq |z|$, we get

$$\left|\frac{z^{2n}}{a_1^2 \dots a_n^2}\right| \geqq 1. \tag{1.3}$$

Next, since for $k>n'$, $|a_k|>(1-\tau)r_0$, we have

$$\left|\frac{a_{n'+1}^2 \dots a_n^2}{z^{2(n-n')}}\right| \geqq \left(\frac{1-\tau}{1+\tau}\right)^{2(n-n')}.$$

Using the inequality

$$\frac{1-u}{1+u} \geqq u, \quad 0 < u \leqq \sqrt{2}-1,$$

it follows that

$$\left|\frac{a_{n'+1}^2 \dots a_n^2}{z^{2(n-n')}}\right| \geqq \tau^{2(n-n')}. \tag{1.4}$$

Finally, for $1 \leqq k \leqq n'$, we have

$$\left|\frac{a_k^2}{z^2} - 1\right| \geqq \frac{(r-|a_k|)^2}{r^2} \geqq \frac{(r-(1-\tau)r_0)^2}{r^2} \geqq \frac{(r-(1-\tau)r)^2}{r^2} = \tau^2,$$

hence

$$\left|\prod_{k\leqq n'} \left(\frac{a_k^2}{z^2} - 1\right)\right| \geqq \tau^{2n'}. \tag{1.5}$$

By (1.3), (1.4) and (1.5)

$$|g_1(z)| \geqq \tau^{2n'} \geqq \tau^{2n_f(2r_0)},$$

consequently

$$\ln |g_1(z)| \geqq -2n_f(2r_0) \ln \frac{1}{\tau}. \tag{1.6}$$

Let us now estimate $|g_3(z)|$, for $|z|=r$, $r_0 \leqq r \leqq (1+\tau)r_0$. We have

$$\ln |g_3(z)| \geqq \sum_{k>n'} \ln \left(1 - \left|\frac{z}{a_k}\right|^2\right) = \sum_{j=2} \sum_{n_f(2^{j-1}r_0) < k \leqq n_f(2^j r_0)} \ln \left(1 - \left|\frac{z}{a_k}\right|^2\right).$$

Using the inequality

$$\ln (1-u) \geqq -\frac{u}{1-u}, \quad 0 < u < 1,$$

we obtain

$$\ln|g_3(z)| \geqq -\sum_{j=2} \sum_{n_f(2^{j-1}r_0)<k\leqq n_f(2^j r_0)} \frac{\left|\frac{z}{a_k}\right|^2}{1-\left|\frac{z}{a_k}\right|^2}.$$

Since for $j\geqq 2$ and $n_f(2^{j-1}r_0)<k\leqq n_f(2^j r_0)$ we successively have

$$\left|\frac{z}{a_k}\right| \leqq \frac{r_0(1+\tau)}{2^{j-1}r_0} \leqq \frac{1+\frac{1}{8}}{2^{j-1}} = \frac{9}{2^{j+2}},$$

$$\frac{\left|\frac{z}{a_k}\right|^2}{1-\left|\frac{z}{a_k}\right|^2} \leqq \frac{\frac{81}{4^{j+2}}}{1-\frac{81}{4^{j+2}}} = \frac{81}{4^{j+2}-81} \leqq \frac{8}{4^j},$$

then

(1.7) $$\ln|g_3(z)| \geqq -8\sum_{j=2}^{+\infty} \frac{n_f(2^j r_0)}{4^j}.$$

Finally we observe that

(1.8) $$g_2(z) = \frac{\prod_{n'<k\leqq n''}(a_k^2-z^2)}{a_{n'+1}^2 \dots a_{n''}^2}$$

and we apply the Boutroux—Cartan theorem (see [15], Section 2.?) to the polynomial in the numerator of the fraction from (1.8). Thus for $z\in\mathbf{C}$ outside of $2(n''-n')$ circles with the sum of their diameters less then $8er_0\tau^2$, we have

$$\left|\prod_{n'<k\leqq n''}(a_k^2-z^2)\right| \geqq \left(\frac{2er_0\tau^2}{e}\right)^{2(n''-n')} = (2r_0\tau^2)^{2(n''-n')}$$

so that

(1.9) $$|g_2(z)| \geqq \tau^{4(n''-n')} \geqq \tau^{4n''}.$$

Since $8er_0\tau^2<r_0\tau$, there exists r' with $r_0\leqq r'\leqq(1+\tau)r_0$ such that every $z\in\mathbf{C}$, $|z|=r'$, is outside of the above $2(n''-n')$ circles.

By (1.9), for $|z|=r'$ holds

(1.10) $$\ln|g_2(z)| \geqq -4n_f(2r_0)\ln\frac{1}{\tau}.$$

Introducing the estimates (1.6), (1.7) and (1.10) in (1.2), we find

$$\inf_{|z|=r'} \ln|f(z)| \geqq -\ln M_f(2r_0) - 6n_f(2r_0)\ln\frac{1}{\tau} - 8\sum_{j=2}^{+\infty}\frac{n_f(2^j r_0)}{4^j}.$$

Using (1.1) in the above inequality, the proof of the theorem is complete. Q.e.d.

If we take in the above theorem, for example $\tau=e^{-4}$, then we reobtain the minimum modulus theorem of L. Ehrenpreis ([4], Th. 6). In our result, unlike those of similar kind of O. von Grudzinski from [8] and [9], the dependence of the minimum of the modulus on the parameter τ has a more simple form which is very useful in the applications. This is illustrated by the following

Corollary 1.2. *Let $\alpha\colon (0, +\infty)\mapsto(0, +\infty)$ be an increasing function with*

$$\int_1^{+\infty}\frac{\alpha(r)}{r^2}\,dr<+\infty \quad \textit{and} \quad \int_1^{+\infty}\frac{\alpha(r)}{r^2}\ln\frac{r}{\alpha(r)}\,dr<+\infty.$$

Then there exists an increasing function $\beta\colon (0, +\infty)\mapsto(0, +\infty)$ such that

$$\int_1^{+\infty}\frac{\beta(r)}{r^2}\,dr<+\infty,$$

and with the property:

If f is an entire function satisfying

$$\ln|f(z)| \leqq c\alpha(|z|)+c', \quad z\in\mathbf{C}, \tag{1.11}$$

for some $c, c'>0$, then there exist $d, d'>0$ such that for every $r_0>0$

$$\sup_{r_0\leqq r\leqq r_0+d\beta(r_0)}\ \inf_{|z|=r}\ln|f(z)| \geqq -d\beta(r_0)-d'.$$

Proof. Since $\lim_{r\to+\infty}\alpha(r)/r=0$, (see [12]), there exists $c_0>0$ such that

$$\frac{c_0\alpha(2er)}{1+r}<\frac{1}{8e}, \quad r>0.$$

For $r>0$, let us put

$$\beta(r)=6\alpha(2er)\ln\frac{1+r}{c_0\alpha(2er)}+8\sum_{j=1}^{\infty}\frac{\alpha(2^jer)}{4^j}.$$

Then $\beta\colon (0, +\infty)\mapsto(0, +\infty)$ is an increasing function such that

$$\beta(r)\geqq\alpha(2er), \quad r>0. \tag{1.12}$$

By the assumptions on the function α, we successively have

$$\int_1^{+\infty}\frac{\alpha(2er)}{r^2}\ln\frac{1+r}{c_0\alpha(2er)}\,dr<+\infty,$$

$$\int_1^{+\infty}\left[\sum_{j=1}^{+\infty}\frac{\alpha(2^jer)}{4^jr^2}\right]dr\leqq e\left(\sum_{j=1}^{+\infty}\frac{1}{2^j}\right)\int_1^{+\infty}\frac{\alpha(r)}{r^2}\,dr<+\infty,$$

so that

$$\int_1^{+\infty} \frac{\beta(r)}{r^2}\,dr < +\infty.$$

Let f be an entire function satisfying (1.11) and $r_0>0$; we can obviously suppose that $f(0)=1$. Apply Theorem 1.1 to f, for $\tau=c_0\alpha(2er_0)/(1+r_0)$; then, by (1.12), there is an r' with $r_0\leqq r'\leqq r_0+c_0\beta(r_0)$, such that

$$\inf_{|z|=r'} \ln|f(z)| \geqq -c\beta(r_0)-6c'(\ln 8+2).$$

Then the Corollary results with $d=\max(c_0, c)$ and $d'=6c'(\ln 8+2)$. Q.e.d.

2. Invertible ultradifferential operators

Let $0<t_1\leqq t_2\leqq \dots$ be such that $t_1<+\infty$ and $\sum_{k=1}^{\infty} 1/t_k<+\infty$. For $r>0$ we define

the distribution function of the sequence $\{t_k\}$

$$n(r) = n_{\{t_k\}}(r) = \text{the number of } t_k \text{ with } t_k \leqq r;$$

the associated function of the sequence $\{t_k\}$

$$N(r) = N_{\{t_k\}}(r) = \ln\max\left\{1, \sup_{k\geqq 1} \frac{r^k}{t_1 \dots t_k}\right\}.$$

We further define the entire function of exponential type zero $\omega=\omega_{\{t_k\}}$ by

$$\omega(z) = \prod_{k=1}^{\infty}\left(1+\frac{iz}{t_k}\right), \quad z\in \mathbf{C}.$$

We note that (see [12], Ch. I. or [13], Ch. II, Section 1) n, N and ω satisfy the non-quasianalyticity conditions

$$(2.1)\quad \int_1^{+\infty} \frac{n(r)}{r^2}\,dr < +\infty, \quad \int_1^{+\infty} \frac{N(r)}{r^2}\,dr < +\infty, \quad \int_1^{+\infty} \frac{\ln|\omega(r)|}{r^2}\,dr < +\infty.$$

Moreover, we have

$$(2.2)\qquad \lim_{r\to+\infty} \frac{n(r)}{r} = 0, \quad \lim_{r\to+\infty} \frac{N(r)}{r} = 0, \quad \lim_{r\to+\infty} \frac{\ln|\omega(r)|}{r} = 0.$$

As

$$(2.3)\qquad N(r) = \int_0^r \frac{n(\lambda)}{\lambda}\,d\lambda,$$

it follows that

(2.4) $$n(r) \leqq N(er).$$

Obviously

(2.5) $$N(r) \leqq \ln |\omega(r)|.$$

On the other hand, for every $r>0$

$$\ln |\omega(r)| = \frac{1}{2}\int_0^{+\infty} \ln\left(1+\frac{r^2}{\lambda^2}\right) dn(\lambda)$$

$$= \frac{1}{2} n(\lambda) \ln\left(1+\frac{r^2}{\lambda^2}\right)\Bigg|_{\lambda=0}^{\lambda=+\infty} + \int_0^{+\infty} n(\lambda)\frac{r^2}{\lambda(\lambda^2+r^2)}\, d\lambda$$

$$= \int_0^{2r} n(\lambda)\frac{r^2}{\lambda(\lambda^2+r^2)}\, d\lambda + \sum_{j=2}^{+\infty}\int_{2^{j-1}r}^{2^j r} n(\lambda)\frac{r^2}{\lambda(\lambda^2+r^2)}\, d\lambda$$

$$\leqq \int_0^{2r} \frac{n(\lambda)}{\lambda}\, d\lambda + \sum_{j=2}^{+\infty}\int_{2^{j-1}r}^{2^j r} \frac{n(\lambda)}{\lambda}\cdot\frac{1}{4^{j-1}}\, d\lambda$$

$$\leqq \sum_{j=1}^{+\infty}\frac{1}{4^{j-1}}\int_0^{2^j r}\frac{n(\lambda)}{\lambda}\, d\lambda,$$

hence by (2.3), we finally get

(2.6) $$\ln |\omega(r)| \leqq 4\sum_{j=1}^{+\infty}\frac{N(2^j r)}{4^j}.$$

We shall give now the

Lemma 2.1. *Let n, N, ω be as above. Then the following statements are equivalent:*

(i) $\sum_{k=1}^{+\infty}\dfrac{\ln\frac{t_k}{k}}{t_k} < +\infty;$

(ii) $\int_1^{+\infty}\dfrac{n(r)}{r^2}\ln\dfrac{r}{n(r)}\, dr < +\infty;$

(iii) $\int_1^{+\infty}\dfrac{N(r)}{r^2}\ln\dfrac{r}{N(r)}\, dr < +\infty;$

(iv) $\int_1^{+\infty}\dfrac{\ln|\omega(r)|}{r^2}\ln\dfrac{r}{\ln|\omega(r)|}\, dr < +\infty.$

Proof. (i)⇔(ii). Let $k \geqq 1$ integer; integrating by parts, we get:

$$\int_{t_k}^{t_{k+1}} \frac{n(r)}{r^2} \ln \frac{r}{n(r)}\, dr = \int_{t_k}^{t_{k+1}} \frac{k}{r^2} \ln \frac{r}{k}\, dr$$

$$= k \int_{t_k}^{t_{k+1}} \ln r\, d\left(-\frac{1}{r}\right) - k \ln k \int_{t_k}^{t_{k+1}} \frac{dr}{r^2}$$

$$= k\left(\frac{\ln t_k}{t_k} - \frac{\ln t_{k+1}}{t_{k+1}}\right) + (k - k \ln k) \int_{t_k}^{t_{k+1}} \frac{dr}{r^2}$$

$$= k\left(\frac{\ln t_k}{t_k} - \frac{\ln t_{k+1}}{t_{k+1}}\right) + (k - k \ln k)\left(\frac{1}{t_k} - \frac{1}{t_{k+1}}\right).$$

Hence

$$\int_1^{+\infty} \frac{n(r)}{r^2} \ln \frac{r}{n(r)}\, dr$$

$$= \sum_{k=1}^{+\infty} \left[k\left(\frac{\ln t_k}{t_k} - \frac{\ln t_{k+1}}{t_{k+1}}\right) + (k - k \ln k)\left(\frac{1}{t_k} - \frac{1}{t_{k+1}}\right)\right]$$

$$= \sum_{k=1}^{+\infty} \left[\frac{\ln t_k}{t_k} + \frac{1 - \ln \dfrac{k^k}{(k-1)^{k-1}}}{t_k}\right]$$

$$= \sum_{k=1} \left[\frac{\ln \dfrac{t_k}{k}}{t_k} + \frac{1 + \ln k - \ln \dfrac{k^k}{(k-1)^{k-1}}}{t_k}\right].$$

Using the Stirling formula, it is easy to get

$$\lim_{k \to +\infty} \left[1 + \ln k - \ln \frac{k^k}{(k-1)^{k-1}}\right] = 0,$$

hence there exists $c > 0$ such that

$$\left|1 + \ln k - \ln \frac{k^k}{(k-1)^{k-1}}\right| \leqq c, \quad k \geqq 1.$$

This yields the desired equivalence.

Let us further examine the condition (iv). By the inequality

$$a^2 \ln\left(1 + \frac{u^2}{c^2}\right) \geqq \ln(1 + u^2), \quad a, u > 0,$$

we have

$$a^2 \ln\left|\omega\left(\frac{t}{a}\right)\right| \geqq \ln|\omega(t)|, \quad a, t > 0. \tag{2.7}$$

Using (2.6) and (2.7), we obtain

$$\int_1^{+\infty}\frac{\ln|\omega(r)|}{r^2}\ln\frac{r}{\ln|\omega(r)|}\,dr$$

$$=4\sum_{j=1}^{+\infty}\frac{1}{4^j}\int_1^{+\infty}\frac{N(2^jr)}{r^2}\ln\frac{r}{\ln|\omega(r)|}\,dr$$

$$=4\sum_{j=1}^{+\infty}\frac{1}{4^j}\cdot 2^j\int_{2^j}^{+\infty}\frac{N(t)}{t^2}\ln\frac{t}{2^j\ln\left|\omega\left(\frac{t}{2^j}\right)\right|}\,dt$$

$$\leqq 4\sum_{j=1}^{+\infty}\frac{1}{2^j}\int_1^{+\infty}\frac{N(t)}{t^2}\ln\frac{2^ja^2t}{4^ja^2\ln\left|\omega\left(\frac{t}{2^j}\right)\right|}\,dt$$

$$\leqq 4\sum_{j=1}^{+\infty}\frac{1}{2^j}\int_1^{+\infty}\frac{N(t)}{t^2}\ln\frac{2^ja^2t}{\ln|\omega(at)|}\,dt$$

$$=4\left(\sum_{j=1}^{+\infty}\frac{1}{2^j}\right)\int_1^{+\infty}\frac{N(t)}{t^2}\ln\frac{t}{\ln|\omega(at)|}\,dt+4\left(\sum_{j=1}^{+\infty}\frac{\ln 2^ja^2}{2^j}\right)\int_1^{+\infty}\frac{N(t)}{t^2}\,dt.$$

Thus, by (2.1), (iv) holds if

$$\int_{t_1}^{+\infty}\frac{N(t)}{t^2}\ln\frac{t}{\ln|\omega(at)|}\,dt<+\infty\quad\text{for some}\quad a>0. \tag{2.8}$$

(iii)$\Rightarrow$(iv). This implication results immediately from (2.5) which imply (2.8) with $a=1$.

(ii)$\Rightarrow$(iv). By (2.4) and (2.5), we have

$$\ln|\omega(er)|\geqq n(r).$$

Hence, from (2.8) for $a=e$, it is enough to prove that

$$\int_{t_1}^{+\infty}\frac{N(t)}{t^2}\ln\frac{t}{n(t)}\,dt<+\infty. \tag{2.9}$$

Let $\lambda>t_1$ and $n=n(\lambda)$. Then

$$\int_\lambda^{+\infty}\frac{1}{t^2}\ln\frac{t}{n(t)}\,dt=\int_\lambda^{t_{n+1}}\frac{1}{t^2}\ln\frac{t}{n(t)}\,dt+\sum_{k\geqq n+1}\int_{t_k}^{t_{k+1}}\frac{1}{t^2}\ln\frac{t}{n(t)}\,dt$$

$$=\int_\lambda^{t_{n+1}}\frac{1}{t^2}\ln\frac{t}{n}\,dt+\sum_{k\geqq n+1}\int_{t_k}^{t_{k+1}}\frac{1}{t^2}\ln\frac{t}{k}\,dt.$$

Since

$$\int \frac{1}{t^2} \ln \frac{t}{c}\, dt = -\frac{1+\ln \frac{t}{c}}{t} + \text{constant}, \quad c, t > 0,$$

we deduce

$$\int_\lambda^{+\infty} \frac{1}{t^2} \ln \frac{t}{n(t)}\, dt = \left(\frac{1}{\lambda} - \frac{1}{t_{n+1}}\right) + \sum\nolimits_{k \geqq n+1} \left(\frac{1}{t_k} - \frac{1}{t_{k+1}}\right)$$

$$+ \left(\frac{\ln \frac{\lambda}{n}}{\lambda} - \frac{\ln \frac{t_{n+1}}{n}}{t_{n+1}}\right) + \sum\nolimits_{k \geqq n+1} \left(\frac{\ln \frac{t_k}{k}}{t_k} - \frac{\ln \frac{t_{k+1}}{k}}{t_{k+1}}\right)$$

$$= \frac{1}{\lambda} + \frac{\ln \frac{\lambda}{n}}{\lambda} + \sum\nolimits_{k \geqq n+1} \frac{\ln \frac{k-1}{k}}{t_k},$$

consequently

$$\int_\lambda^{+\infty} \frac{1}{t^2} \ln \frac{t}{n(t)}\, dt \leqq \frac{1}{\lambda} + \frac{\ln \frac{\lambda}{n(\lambda)}}{\lambda}. \tag{2.10}$$

Further, by (2.3) and (2.10) we have

$$\frac{N(t)}{t^2} \ln \frac{t}{n(t)}\, dt = \int_{t_1}^{+\infty} \frac{1}{t^2} \left(\int_{t_1}^{t} \frac{n(\lambda)}{\lambda} d\lambda\right) \ln \frac{t}{n(t)}\, dt$$

$$= \int_{t_1}^{+\infty} \frac{n(\lambda)}{\lambda} \left(\int_\lambda^{+\infty} \frac{1}{t^2} \ln \frac{t}{n(t)}\, dt\right) d\lambda$$

$$\leqq \int_{t_1}^{+\infty} \frac{n(\lambda)}{\lambda^2} d\lambda + \int_{t_1}^{+\infty} \frac{n(\lambda)}{\lambda^2} \ln \frac{\lambda}{n(\lambda)}\, d\lambda.$$

Therefore (2.9) holds.

(iv)$\Rightarrow$(ii). By (2.7), (2.5) and (2.4), we have

$$e^2 \ln |\omega(r)| \geqq \ln |\omega(er)| \geqq n(r), \quad r > 0,$$

so that

$$\frac{n(r)}{r} \leqq e^2 \frac{\ln |\omega(r)|}{r}.$$

The function $t \to t \ln 1/t$ is increasing for $0 < t < e^{-1}$, so that by (2.2)

$$\frac{n(r)}{r^2} \ln \frac{r}{n(r)} \leqq e^2 \cdot \frac{\ln |\omega(r)|}{r^2} \cdot \ln \frac{r}{e^2 \ln |\omega(r)|}$$

if r is sufficiently large; this yields the desired implication.

By a similar reasoning we deduce that (iv)$\Rightarrow$(iii) and this ends the proof of the lemma. Q.e.d.

We can now give our main result:

Theorem 2.2. *Let* $0 < t_1 \leqq t_2 \leqq \dots$ *be such that*

$$t_1 < +\infty, \quad \sum_{k=1}^{+\infty} \frac{1}{t_k} < +\infty \quad \text{and} \quad \sum_{k=1}^{+\infty} \frac{\ln \frac{t_k}{k}}{t_k} < +\infty.$$

Then there exists $0 < s_1 \leqq s_2 \leqq \dots$, $s_1 < +\infty$, $\sum_{k=1}^{+\infty} 1/s_k < +\infty$, *with the property:*

If f is an entire function with

$$|f(z)| \leqq c_0 |\omega_{\{t_k\}}(|z|)|^{n_0} \tag{2.11}$$

for some $c_0 > 0$, $n_0 \geqq 1$ *integer, then there exist a,* $M > 0$ *such that*

$$\sup_{|\eta| \leqq a \ln M |\omega_{\{s_k\}}(|\xi|)|} \ln |f(\xi+\eta)| \geqq -a \ln M |\omega_{\{s_k\}}(|\xi|), \quad \xi \in \mathbf{C},$$

where the supremum can be taken over $\mathbf{R}$, *when* $\xi \in \mathbf{R}$.

Proof. Let us denote $\alpha(r) = \ln |\omega(r)|$, $r > 0$. Then by Lemma 2.1, the conditions from Corollary 1.2 are fullfilled for the functoin α. Hence there exists an increasing function $\beta: (0, +\infty) \mapsto (0, +\infty)$ with $\int_1^{+\infty} \beta(r)/r^2 dr < +\infty$ such that for every function satisfying (2.11) there are $d, d' > 0$ such that for each $r_0 > 0$

$$\sup_{r_0 \leqq r \leqq r_0 + d\beta(r_0)} \inf_{|z|=r} |f(z)| \geqq -d\beta(r_0) - d'.$$

But from the above inequality we easily get

$$\sup_{|\eta| \leqq d\beta(|\xi|)} \ln |f(\xi+\eta)| \geqq -d\beta(|\xi|) - d', \quad \xi \in \mathbf{C}, \ \xi \neq 0, \tag{2.12}$$

where for real ξ, the supremum can be taken over $\eta \in \mathbf{R}$, $|\eta| \leqq d\beta(|\xi|)$, as a simple reasoning shows.

Further, by a result of O. I. Izonemcev and V. A. Marcenko [11], for the function β there is a sequence $0 < s_1 \leqq s_2 \leqq \dots$, $\sum_{k=1}^{+\infty} 1/s_k < +\infty$, such that

$$\beta(t) \leqq \ln |\omega_{\{s_k\}}(t)| + d, \quad t > 0$$

for some constant $d > 0$. Then the statement results from (2.12) for suitable positive constants a and M, enabling us also to remove the restriction $\xi \neq 0$. Q.e.d.

We shall give now the

Definition 2.3. *Let* $0 < t_1 \leqq t_2 \leqq \dots$, $t_1 < +\infty$, $\sum_{k=1}^{+\infty} 1/t_k < +\infty$ *and* $\omega = \omega_{\{t_k\}}$. *An entire function f is called* ω*-slowly decreasing if there are a and* $M > 0$ *such that*

$$\sup_{\substack{|\eta| \leqq a \ln M |\omega(\xi)| \\ \eta \in \mathbf{R}}} |f(\xi+\eta)| \geqq M^{-a} |\omega(\xi)|^{-a}, \quad \forall \xi \in \mathbf{R}.$$

Let us remark that all kinds of slowly decreasing functions considered in [1], [8] and [9] are ω-slowly decreasing for some suitable function ω, as the above mentioned result on entire majorants from [11] shows.

From Theorem 2.2 we immediately get

Corollary 2.4. *Let $\omega_{\{t_k\}}$, $\omega_{\{s_k\}}$ and f be as in Theorem 2.2; then f is $\omega_{\{s_k\}}$-slowly decreasing.*

In order to apply the above results to ultradifferential operators we recall some facts about the ω-ultradistributions considered in [3], [4], [5] and [6].

For $K \subset \mathbf{R}$ compact, we define the functions space:

$$\mathcal{D}_\omega(K) = \left\{ \varphi \in C_0^\infty(K); \begin{array}{l} p_{L,n}(\varphi) = \sup_{t \in \mathbf{R}} |\hat{\varphi}(t)\omega(Lt)^n| < +\infty \\ \text{for every } L > 0,\ n \geqq 1 \text{ integer, } \operatorname{supp}\varphi \subset K \end{array} \right\}$$

($\hat{\varphi}$ denotes the Fourier transform of φ)

$$\mathcal{D}_\omega = \varinjlim_{K \subset \mathbf{R}} \mathcal{D}_\omega(K).$$

The elements of the dual $\mathcal{D}'_\omega$ are called ω-ultradistributions. The "union" over ω of all ω-ultradistributions coïncide with the "union" of all Roumieu (or Beurling) ultradistributions (see [6]); the parametrization over ω presents a series of advantages among which we mention the stability under ultradifferential operators of each $\mathcal{D}'_\omega$.

As for distributions, one can define the notion of support for ω-ultradistributions. If $S \in \mathcal{D}'_\omega$ has a compact support, then its Fourier transform $\hat{S}$ is defined by $\hat{S}(z) = \langle S, e^{-izt} \rangle$, $z \in \mathbf{C}$.

Definition 2.5. *We say that the ω-ultradistribution S with compact support is invertible in $\mathcal{D}'_\omega$ if $S * \mathcal{D}'_\omega = \mathcal{D}'_\omega$.*

In [3] the following result is proved:

S is invertible in $\mathcal{D}'_\omega$ iff S is ω-slowly decreasing.

In [5], [6] we called ω-ultradifferential operators all linear operators on $\mathcal{D}'_\omega$ preserving the support.

Among the ω-ultradifferential operators we distinguish a particular class defined by

Definition 2.6. *An operator of the form*

$$f(D) = \sum_{k=1}^{+\infty} c_k D^k, \quad c_k \in \mathbf{C},$$

is called an ω-ultradifferential operator with constant coefficients if there are $c_0, n_0 > 0$ such that

$$|f(z)| \leqq c_0 |\omega(|z|)|^{n_0}, \quad z \in \mathbf{C}.$$

(We mention that in [5], [6] we called ω-ultradifferential operators with constant coefficients a more general class of operators, but in applications those from the above definition are more significant). ω-ultradifferential operators with constant coefficients have the property that the series $\sum_{k=1}^{+\infty} c_k D^k$ converges in $\mathscr{L}(\mathscr{D}'_\omega)$, without requiring additional properties for the function ω (as the condition (M.2) of stability under ultradifferential operators in the frame of Roumieu or Beurling ultradistributions in [10]). Moreover, every ultradifferential operator of class (M_k) or $\{M_k\}$ (see [10]) is an ω-ultradifferential operator for a suitable function ω. With the above considerations we directly get from Theorem 2.2:

Proposition 2.7. *Let* $0<t_1\leqq t_2\leqq\dots$, *be such that*

$$t_1<+\infty, \quad \sum_{k=1}^{+\infty}\frac{1}{t_k}<+\infty \quad \text{and} \quad \sum_{k=1}^{+\infty}\frac{\ln\frac{t_k}{k}}{t_k}<+\infty. \tag{2.13}$$

Then there exists $0<s_1\leqq s_2\leqq\dots$, $s_1<+\infty$, $\sum_{k=1}^{+\infty} 1/s_k<+\infty$ *such that every* $\omega_{\{t_k\}}$*-ultradifferential operator with constant coefficients is invertible in* $\mathscr{D}'_{\omega\{s_k\}}$.

We shall apply this invertibility result in some particular cases.

I. It is clear that the Gevrey sequence $t_k=k^\alpha$, $\alpha>1$, satisfies conditions (2.13).

II. Let $t_k=k(\ln k)^\alpha$, $\alpha>1$; an easy computation shows that also this sequence satisfies the conditions (2.13). So we can positively answer the question asked in [1] concerning the invertibility of ultradifferential operators of class $\{k!(\prod_{j=2}^{k}\ln j)^\alpha\}$, for arbitrary $\alpha>1$. In [1] only the case $\alpha>2$ was solved.

III. Finally we remark that the above proposition improve Theorem III 2—3., from [1].

Namely, let $\{M_k\}\in\mathscr{M}$, where $\mathscr{M}$ is the space of sequences from [1]. If there is $\{Q_k\}\in\mathscr{M}$ such that the associated functions $M(r)$ and $Q(r)$, to the sequences $t_k=M_k/M_{k-1}$ and $s_k=Q_k/Q_{k-1}$, satisfy

i) $r\mapsto\dfrac{M(2r)}{Q(r)}$ is decreasing and $\displaystyle\int_r^{+\infty}\frac{M(2t)}{tQ(t)}\,dt=O\left(\frac{Q(r)}{r}\right)$,

ii) $\dfrac{Q(r)}{rM(r)}e^{\frac{Q(r)}{M(r)}}\geqq 1$,

then by Theorem III. 2—3. from [1], each $\{M_k\}$-ultradifferential operator with constant coefficients is invertible in a $\mathscr{D}'_{\{R_k\}}$, for a suitable sequence $\{R_k\}\in\mathscr{M}$.

As for r sufficiently large $Q(r)\leqq r$, from ii) we get

$$e^{\frac{Q(r)}{M(r)}}\geqq\frac{rM(r)}{Q(r)}\geqq M(r),$$

so that

$$M(r)\ln M(r)\leqq Q(r).$$

Consequently

$$\int_1^{+\infty}\frac{M(r)\ln M(r)}{r^2}\,dr\leqq\int_1^{+\infty}\frac{Q(r)}{r^2}\,dr<+\infty, \tag{2.14}$$

(we used the non-quasianalyticity of the sequences from $\mathscr{M}$). By Theorem 4, Ch. II, § 2 from [13], (2.14) is equivalent to

$$\sum_{k=1}^{+\infty} \frac{\ln t_k}{t_k} < +\infty. \tag{2.15}$$

It is obvious that condition (2.15) implies condition (2.13) and thus Proposition 2.7 extends Chou's result. Let us still remark that for the sequence $\{t_k\}$ from II, (2.15) holds only for $\alpha > 2$; the case $1 < \alpha \leqq 2$ can be solved only in the frame of the less restrictive condition (2.13).

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