Iteration of spreading models

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If E is a Banach space and $(x_n)_{n\in\mathbf{N}}$ is a bounded sequence in E, Brunel—Sucheston [2] showed that there exists a subsequence $(x'_n)_{n\in\mathbf{N}}$ of $(x_n)_{n\in\mathbf{N}}$ such that the limit of $\|a_1 x'_{n_1}+\ldots+a_k x'_{n_k}\|$ exists, when $n_1<n_2<\ldots<n_k$ go to infinity, for all finite sequences of scalars $a_1, \ldots, a_k$.

A sequence $(x'_n)_{n\in\mathbf{N}}$ satisfying this property will be called a good sequence.

If $(e_n)_{n\in\mathbf{N}}$ denotes the canonical basis of the space S of finite sequences of scalars, we denote by $|a_1 e_1+\ldots+a_k e_k|$ the previous limit. It is a norm on this space provided that the sequence $(x_n)_{n\in\mathbf{N}}$ has no norm-convergent subsequence. Let F be the completion of S under this norm: F will be called a spreading model of E, built on the sequence $(x_n)_{n\in\mathbf{N}}$. The sequence $(e_n)_{n\in\mathbf{N}}$ will be called the fundamental sequence of F. This notion was first introduced and studied by A. Brunel and L. Sucheston (see for example [2] and [3]); it has been applied to the study of the Banach—Saks properties by the first named author in [1], in which the reader may find the proofs of all the statements we give here.

Our aim in this paper will be to investigate the following question, raised by H. P. Rosenthal: given a space E, a spreading model F_1, built on some bounded sequence $(x_n)_{n\in\mathbf{N}}$ of E, a spreading model F_2, built on some bounded sequence $(y_n)_{n\in\mathbf{N}}$ of F_1, is F_2 isomorphic to some spreading model F of E, built on some bounded sequence $(z_n)_{n\in\mathbf{N}}$ of E?

The isometric version of this question was answered negatively by the authors in [1]. We shall show here that the question in its full generality has also a negative answer: we shall present a Banach space E, a spreading model F_1 of E, a spreading model F_2 of F_1, such that F_2 is not isomorphic to any spreading model of E.

The tools we use for this purpose will be the connections between the Banach—Saks properties and the isomorphism of a spreading model to l_1, established by the first named author in [1]. More precisely, let us recall the following definition and theorem, given in [1]:

Definition. A Banach space E has property $(\mathscr{P}_1)$ if, for some $\delta>0$, one can find a bounded sequence $(x_n)_{n\in\mathbf{N}}$ in E such that, for all $k\in\mathbf{N}$, all $n_1<\ldots<n_k$, all

$\varepsilon_1 \dots \varepsilon_k = \pm 1$,

(1) $$\left\| \frac{1}{k} \sum_1^k \varepsilon_i x_{n_i} \right\|_E \geqq \delta.$$

Theorem ([1]).

a) *E has property* $(\mathscr{P}_1)$ *if and only if it has a spreading model, the fundamental sequence of which is equivalent to the* l_1*-basis.*

b) *If* $(\mathscr{P}_1)$ *holds, one can, for all* $\eta > 0$, *find a norm-one sequence* $(x_n)_{n\in\mathbf{N}}$ *in E which gives the estimates* (1) *with* δ *replaced by* $1-\eta$.

Concerning this theorem, let us observe that, as was noticed by J. T. Lapresté in [4], the model is isomorphic to l_1 if and only if its fundamental sequence is equivalent to the l_1-basis.

Let us now turn to the construction of our example. We shall build a space E which does not possess property $(\mathscr{P}_1)$ (that is, which has Alternate—Signs Banach—Saks property, according to [1]), but has a spreading model F_1 which contains l_1. This is obviously enough to answer negatively H. P. Rosenthal's question.

Let us consider the Orlicz function:

$$\varphi(t) = \frac{t}{1-\log t} \quad \text{if} \quad 0 < t \leqq 1$$

$$= 2t-1 \quad \text{if} \quad 1 \leqq t < \infty.$$

Let us denote by l_φ the Orlicz space associated with this function and by $\|(x(k))_{k\in\mathbf{N}}\|_\varphi$ the norm of a sequence $(x(k))_{k\in\mathbf{N}}$ in l_φ (as usually, $x(k)$ is the k-th term of the sequence x); this norm is, by definition, $\inf\left\{C;\ \sum_{k\in\mathbf{N}} \varphi\left(\frac{|x(k)|}{C}\right) \leqq 1\right\}$.

With such a choice of the function φ, the space l_φ has the following properties:

a) $l_1 \subset l_\varphi \subset l_p$ for all $p > 1$, with continuous injections,

b) the space l_φ contains l_1: one can find a sequence of consecutive blocks on the canonical basis which is equivalent to the l_1-basis,

c) there is a number δ_0, $0 < \delta_0 < 1$, such that if $(a(k))_{k\in\mathbf{N}}$, $(b(k))_{k\in\mathbf{N}}$, are finite sequences, disjointly supported, with $|a(k)| \leqq 1$, $|b(k)| \leqq 1\ \forall k$, if $\|(a(k))_{k\in\mathbf{N}}\|_\varphi \geqq \delta_0$ and $\|(b(k))_{k\in\mathbf{N}}\|_\varphi \geqq \delta_0$, then $\|(a(k)+b(k))_{k\in\mathbf{N}}\|_\varphi > 1$ (one checks immediately that $\delta_0 < 8/10$).

The space for our counterexample will be constructed along the same lines as the one given in [1], using admissible sets, but with the norm l_φ instead of the norm l_1. Let us recall that a finite subset A of the integers, $A = \{n_1, \dots, n_k\}$, with $n_1 < n_2 < \dots < n_k$, is called admissible if $k \leqq n_1$. We call $\mathscr{N}$ the set of admissible subsets of $\mathbf{N}$.

We consider the sequences of scalars $(x(k))_{k\in\mathbf{N}}$ such that

$$\sup_{A\in\mathcal{N}} \|(x(k))_{k\in A}\|_\varphi < \infty$$

and we call S_φ the closure of the finite sequences under the norm

$$\|(x(k))_{k\in\mathbf{N}}\|_{S_\varphi} = \sup_{A\in\mathcal{N}} \|(x(k))_{k\in A}\|_\varphi$$

This is a "Schreier type" space, which, for many properties, is analogous to the space introduced by J. Schreier in [6], and to the space studied by the first named author in [1].

We shall now turn to the study of this space. Let us first observe that the canonical basis $(e_n)_{n\in\mathbf{N}}$ of l_1 is also an unconditional basis for S_φ, which we also call the canonical basis of S_φ.

Proposition 1. *The spreading model built on the canonical basis of S_φ is isometric to l_φ.*

Proof. For a given finite sequence of scalars, $a_1, \dots, a_k$, let us compute $\|a_1 e_{n_1} + \dots + a_k e_{n_k}\|_{S_\varphi}$.

When n_1 is large enough (namely $n_1 \geqq k$), we have clearly $\|a_1 e_{n_1} + \dots + a_k e_{n_k}\|_{S_\varphi} = \|(a_i)\|_\varphi$, since the set $\{n_1, \dots, n_k\}$ is admissible. This proves the Proposition.

Proposition 2. *S_φ has no spreading model isomorphic to l_1.*

Proof. Assume on the contrary that some spreading model F of S_φ is isomorphic to l_1. Then, by the Theorem, S_φ must have property $(\mathscr{P}_1)$. Choose $\eta > 0$ with $\eta < \frac{1-\delta_0}{7}$. Let $(x_n)_{n\in\mathbf{N}}$ be the norm-one sequence which gives $(\mathscr{P}_1)$ with the estimates $1-\eta$ in (1). For all $i\in\mathbf{N}$, the i-th coordinates $(x_n(i))_{n\in\mathbf{N}}$ are bounded; therefore, one can, by a diagonal procedure, find a subsequence $(x'_n)_{n\in\mathbf{N}}$ such that the limits $\lim_{n\to\infty} x'_n(i)$ exists for all $i\in\mathbf{N}$. If we put $y_n = \frac{1}{2}(x'_{2n-1} - x'_{2n})$, then, for all i, $y_n(i) \to 0$ as $n\to\infty$. Of course, $\|y_n\|_{S_\varphi} \leqq 1 \,\forall n$, and $(y_n)_{n\in\mathbf{N}}$ also gives $(\mathscr{P}_1)$ with the same estimates.

Since the coordinate functionals are continuous in S_φ and since the finite sequences are dense in this space, we can find a subsequence $(y'_n)_{n\in\mathbf{N}}$ of $(y_n)_{n\in\mathbf{N}}$ and a sequence of consecutive blocks

$$z_n = \sum_{i=l_{n-1}+1}^{l_n} \alpha_i e_i,$$

with $\|z_n\|_{S_\varphi} \leqq 1$ and $\|z_n - y'_n\|_{S_\varphi} \leqq \eta$. The sequence $(z_n)_{n\in\mathbf{N}}$ also gives $(\mathscr{P}_1)$ with $1-2\eta$ in (1) and is unconditional:

$$(2) \qquad \begin{cases} \forall k, \ \forall n_1 < \dots < n_k, \ \forall \varepsilon_1 \dots \varepsilon_k = \pm 1 \\ \left\| \frac{1}{k} \sum_1^k \varepsilon_i z_{n_i} \right\|_{S_\varphi} = \left\| \frac{1}{k} \sum_1^k z_{n_i} \right\|_{S_\varphi} \geqq 1 - 2\eta. \end{cases}$$

Let us first consider the terms:

$$\frac{1}{2}\|z_1+z_n\|_{S_\varphi}, \quad n \geqq 2,$$

and let us call l the index of the last non-zero term in z_1.

Lemma 1. *One has, for* $n>1$

$$\frac{1}{2}\|z_1+z_n\|_{S_\varphi} = \sup\left\{\frac{1}{2}\left\|(z_1(k)+z_n(k))_{k\in A}\right\|_\varphi;\ A\in\mathcal{N},\ |A|\leqq l\right\}.$$

Proof. If an admissible set A is such that $\frac{1}{2}\|(z_1(k)+z_n(k))_{k\in A}\|\geqq 1-3\eta$, one cannot have $z_1(k)=0$ for all $k\in A$. Therefore the set A starts before l, and has at most l elements. This proves the Lemma.

Lemma 2. *For all* $i\geqq 2$, *each block* z_i *contains a sub-block* z_i' *with the following properties:*

a) $z_i'(k)\neq 0$ *for at most* l *integers* k,

b) $\|z_i'\|_\varphi\geqq 1-5\eta$.

(by sub-block we mean that $z_i'(k)=z_i(k)$ *for some* k*'s, and is* 0 *for the others).*

Proof. Assume on the contrary that one could find a z_i, $i\geqq 2$, such that for all sub-blocks z_i' of z_i, of length l, one had $\|z_i'\|_\varphi<1-5\eta$. But, if A is an admissible set with $|A|\leqq l$, one has:

$$\frac{1}{2}\left\|(z_1(k)+z_i(k))_{k\in A}\right\|_\varphi \leqq \frac{1}{2}\left(\|z_1\|_\varphi+\left\|(z_i(k))_{k\in A}\right\|_\varphi\right) \leqq \frac{1}{2}(1+1-5\eta).$$

But this contradicts (2) and Lemma 1.

z_i' being the sub-block given by Lemma 2, let us put $z_i''=z_i-z_i'$. We shall now show that the z_i'' give $(\mathcal{P}_1)$.

Lemma 3. *When* $k\to\infty$, $\sup\left\{\frac{1}{k}\left\|\sum_1^k z_{n_i}'\right\|_{S_\varphi};\ n_1<\ldots<n_k\right\}\to 0.$

Proof. It is enough to show that $\sup\left\{\frac{1}{k}\left\|\sum_1^k z_{n_i}'\right\|_\varphi;\ n_1<\ldots<n_k\right\}\to 0$ as $k\to\infty$. But for each i, one can write $z_i=\sum_{j=1}^l x_i^j$, where $x_i^j(k)$ is not zero for at most one k, $|x_i^j|\leqq 1$, and, $\forall j$, $\left\|\frac{1}{k}\sum_{i=1}^k x_{n_i}^j\right\|_\varphi\to 0$ as $k\to\infty$; the Lemma follows.

For $k\geqq k_0$, one has, consequently,

$$\frac{1}{k}\left\|\sum_1^k z_{n_i}'\right\|_{S_\varphi}\leqq \eta, \quad \text{for all} \quad n_1<\ldots<n_k,$$

and therefore:

$$\frac{1}{k}\left\|\sum_1^k z_{n_i}''\right\|_{S_\varphi}\geqq 1-3\eta. \tag{3}$$

Of course, since the (z_i'') are sub-blocks of the (z_i), one also has $\|z_i''\|_{S_\varphi}\leqq 1\,\forall i$.

Let us now consider terms of the following form:

$$\frac{1}{2k_0}\|z_1''+\ldots+z_{k_0}''+z_{n_1}''+\ldots+z_{n_{k_0}}''\|_{S_\varphi}, \quad k_0<n_1<\ldots<n_{k_0}.$$

From the estimates (3) follows exactly as is the proof of lemma 1, that:

$$\frac{1}{2k_0}\|z_1''+\ldots+z_{k_0}''+z_{n_1}''+\ldots+z_{n_{k_0}}''\|_{S_\varphi}$$

$$=\sup\left\{\frac{1}{2k_0}\left\|\left(z_1''(k)+\ldots+z_{k_0}''(k)+z_{n_1}''(k)+\ldots+z_{n_{k_0}}''(k)\right)_{k\in A}\right\|_\varphi;\ A\in\mathcal{N}, |A|\leqq l'\right\}$$

where l' is the index of the last non-zero term in z_{k_0}''.

Lemma 4. *There exists an i_0 such that, for all $i>i_0$, each block z_i'' contains a sub-block z_i''' with*

a) $z_i'''(k)\neq 0$ *for at most l' integers k,*
b) $\|z_i'''(k)\|_\varphi \geqq 1-7\eta$.

Proof. If this was not the case, then one could find a subsequence $(z_{k_i}'')_{i\in\mathbf{N}}$ such that each sub-block z_{n_i}''' of z_{n_i}'', of length l', would satisfy $\|z_{n_i}'''\|_\varphi<1-7\eta$. But then, for all $A\in\mathcal{N}$ with $|A|\leqq l'$,

$$\frac{1}{2k_0}\left\|\left(z_1''(k)+\ldots+z_{k_0}''(k)+z_{n_1}''(k)+\ldots+z_{n_{k_0}}''(k)\right)_{k\in A}\right\|_{S_\varphi}$$

$$\leqq\frac{1}{2k_0}\left(\|z_1''\|_{S_\varphi}+\ldots+\|z_{k_0}''\|_{S_\varphi}+\left\|\left(z_{n_1}''(k)\right)_{k\in A}\right\|_\varphi+\ldots+\left\|\left(z_{n_{k_0}}''(k)\right)_{k\in A}\right\|_\varphi\right)$$

$$\leqq\frac{1}{2k_0}\left(k_0+(1-7\eta)k_0\right),$$

contradicting (3). This proves the Lemma.

Therefore, for i large enough, z_i contains two sub-blocks: z_i' and z_i''', both with l_φ norms greater than $1-7\eta$. But they have bounded lengths (l for z_i', l' for z_i'''), and, consequently, for i large enough, one can find an admissible set A which covers z_i' and z_i''' at the same time. For such an i, one has:

$$\|z_i'+z_i'''\|_\varphi=\|z_i'+z_i'''\|_{S_\varphi}\leqq\|z_i\|_{S_\varphi}\leqq 1,$$

and this contradicts the property c) of the space l_φ, since $1-7\eta>\delta_0$.

Remarks.

1. It follows obviously from Proposition 2 that S_φ does not contain l_1.

2. It follows easily from the preceding discussion that, for every spreading model F of S_φ, the sequence $u_n=e_{2n}-e_{2n-1}$ in F (which is unconditional, as proved in [2]) is equivalent either to the canonical basis of l_φ or to the canonical basis of c_0.

If F is constructed on a good sequence $(x_n)_{n\in\mathbf{N}}$ in S_φ, the first case occurs when $\liminf_{n\to\infty} \|x_n - x_{n-1}\|_{c_0} > 0$, and the second when $\lim \|x_n - x_{n-1}\|_{c_0} = 0$.

3. In this proof, we have chosen the function $\varphi(t) = \frac{t}{1-\log t}$ $(0 < t \leqq 1)$, $= 2t-1$ $(t \geqq 1)$. One sees easily that the same results hold, for S_φ, when φ is an Orlicz function satisfying the Δ_2-condition at 0, i.e.:

$$\sup_{0<t<1} \frac{\varphi(2t)}{\varphi(t)} < \infty.$$

In fact, it follows from this condition (see [5] part I § 4) that there exist q and C such that

$$\delta^q \varphi(x) \leqq C \cdot \varphi(\delta x) \quad \text{for all} \quad \delta, x \in [0, 2].$$

This yields that there exist a number δ_0 $(0 < \delta_0 < 1)$ and an integer N such that if $(a_1(k))_{k\in\mathbf{N}}, \dots, (a_N(k))_{k\in\mathbf{N}}$ are disjointly supported finite sequences, with $j = 1 \dots N$, $|a_j(k)| \leqq 1 \,\forall k$ and $\|a_j\|_\varphi \geqq \delta_0$, then $\left\|\sum_{j=1}^N a_j\right\|_\varphi > 1$.

With this property, essentially the same proof works; the difference is that one has to repeat the argument N times, in order to construct N disjoint sub-blocks, of bounded lengths, in any block z_{n_i} of a subsequence.

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