Bases and biorthogonal systems in the spaces C and L^1

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0. Introduction

The main result of the present paper is

Theorem A. *Let $(S, \mathcal{B}, m)$ be a probability space. Let $(f_n, g_n)_{n=1}^{\infty}$ be a biorthogonal system of measurable functions on S such that*

$1°$ $\sup_n \|g_n\|_\infty < \infty$

$2°$ *the functions f_n are equiintegrable.*

Then

$$\sup_n \int_S \int_S \left|\sum_{i=1}^n f_i(s) g_i(t)\right| m(ds)\, m(dt) = \infty.$$

As corollaries from Theorem A we obtain

Corollary B. *No normalized basis of $L^1(0, 1)$ consists of equiintegrable functions; equivalently, every normalized basis of $L^1(0, 1)$ contains a subsequence, which is equivalent to the unit vector basis of l^1.*

Corollary C. *If (g_n) is a normalized basis of $C(0, 1)$, then, for some increasing sequence (n_k) of positive integers, the map $\sum c_n g_n \to (c_{n_k})_{k=1}^{\infty}$ takes $C(0, 1)$ onto c_0.*

Corollaries B and C answer the questions stated in [11], Ch. II, p. 296, Problem 7.1 (see also [9], p. 36, problem (v)).

Remark 0. Particular cases of the Corollaries are the known results on non-existence of

$1°$ a uniformly bounded orthonormal system, which is a basis of L^1 or C (Olevskiĭ, [9], Ch. I, § 2, Theorems 2 and 9)

$2°$ a normalized basis of L^1, which is bounded in order ([4], Th. 1)

$3°$ a Besselian basis of C or a Hilbertian basis of L^1 ([12], Th. A and Th. B); even p-Besselian basis of C for any $p < \infty$ and q-Hilbertian basis of L^1 for any $q > 1$.

The paper consists of four sections.

In section 1 we prove our main results. We start with the proof of Proposition 1.A, which is a stronger, 'local' version of Theorem A. The main idea of the proof, especially the inductive construction at the end of it, is due to Olevskiĭ (cf. the proof of Theorem 1, Ch. I, § 1, [9]). We derive also Corollaries B and C from Theorem A. Finally we sketch briefly, how the facts stated in Remark 0 follow from the Corollaries.

In the next two sections we give some generalizations and strengthenings of the main results. Corollaries 2.B and 2.C of Section 2 generalize Corollaries B and C to the case of $\mathscr{L}^1$- and $\mathscr{L}^\infty$-spaces respectively. Section 3 contains Theorem 3.C, which is a 'pointwise' analogue of Corollary C. It generalizes Olevskiĭ's result:

Given uniformly bounded orthonormal system on [0, 1] *there exists a continuous function on* [0, 1], *whose Fourier series with respect to this system diverges at some point*

(see Theorem 3, [9], Ch. I, § 2).

Finally, Section 4 contains a number of remarks and open problems.

Terminology and notation for classical Banach spaces used in this paper is standard (see e.g. [7]). All facts, which admit real and complex versions, hold in both cases together with their proofs.

The author would like to thank Professor A. Pełczyński for suggesting the problem; him and also Professor L. Tzafriri for helpful discussions. In particular Lemmas 2.1 and 2.2 together with their proofs were communicated to the author by Professor A. Pełczyński.

1. Proof of the main results

We start with some definitions. Let $(S, \mathscr{B}, m)$ be a measure space. Let Z be a set of measurable functions on S. We define the modulus of integrability of the set Z as the smallest concave function $\eta(Z, \cdot)\colon (0, \infty)\to(0, \infty)$ such that

$$\int_A |f|\, dm \leqq \eta(Z, m(A))$$

for all $f\in Z$ and for all $A\in\mathscr{B}$. If $Z=\{f\}$, we write $\eta(f, \cdot)$ instead of $\eta(\{f\}, \cdot)$. By definition, a set Z is equiintegrable iff $\eta(Z, 0^+)=0$.

Now we are ready to state

Proposition 1.A. *Let* $\eta\colon \mathbf{R}^+\to\mathbf{R}^+$ *be a function satisfying*
0° $\eta(0^+)=0$, η *is increasing and continuous.*

Then there exists a sequence of positive numbers $c_N \uparrow \infty$, *depending only on* η, *such that if* $(S, \mathscr{B}, m)$ *is a probability space and* $(f_i, g_i)_{n=1}^N$ — *a biorthogonal system of measurable functions on* S *(i.e.* $\int_S f_i g_j \, dm = \delta_{ij}$*) satisfying*

1° $\|g_i\|_\infty \leqq 1$ for $i=1, 2, \ldots, N$

2° $\eta(f_i, \cdot) \leqq \eta$ for $i=1, 2, \ldots, N$

then

$$\max_{1 \leqq n \leqq N} \int_S \int_S \left| \sum_{i=1}^n f_i(s) g_i(t) \right| m(ds)\, m(dt) \geqq c_N.$$

Clearly Theorem A follows from Proposition 1.A.

Proof of Prop. 1.A. In the sequel we shall denote by $(T, \sum, v)$ the measure space $(S \times S, \mathscr{B} \otimes \mathscr{B}, m \otimes m)$; $\sum_{i=k+1}^{k+n} f_i(s) g_i(t)$ by $F_{k,n}(s,t)$ and $F_{0,n}$ by F_n. In this notation the conclusion of Prop. 1.A may be expressed as

$$\max_{1 \leqq n \leqq N} \int_T |F_n| \, dv \geqq c_N. \tag{1}$$

We start with several lemmas.

Lemma 1.1. *Under assumptions* 0°—2° *of Prop. 1.A there exists* $\gamma > 0$, *depending only on* η, *such that*

$$\|g_i - g_j\|_1 \geqq \gamma \tag{2}$$

$$\|g_i\|_1 \geqq \gamma \tag{3}$$

for all i, j, $1 \leqq i \neq j \leqq N$.

Proof. Choose $\gamma > 0$ so small that $\eta(2\gamma\eta(1)) \leqq 4^{-1}$ (it is possible by 0°). Fix i, j and denote $\Gamma = \{|f_i| > (2\gamma)^{-1}\}$. Then we have

$$\int_\Gamma |f_i| \, dm \leqq \eta(f_i, m(\Gamma)) \leqq \eta(m(\Gamma)) \quad \text{(by 2°)} \tag{4}$$

$$\int_\Gamma |f_i| \, dm \geqq (2\gamma)^{-1} m(\Gamma). \tag{5}$$

Combining (4) and (5) and applying the fact that η is increasing we obtain

$$m(\Gamma) \leqq 2\gamma\eta(m(\Gamma)) \leqq 2\gamma\eta(1). \tag{6}$$

Now, using successively (4), (6), the fact that η is increasing and the choice of γ, we get

$$\int_\Gamma |f_i| \, dm \leqq \eta(m(\Gamma)) \leqq \eta(2\gamma\eta(1)) \leqq 4^{-1}. \tag{7}$$

Finally

$$1=\int_S (g_i-g_j)f_i\,dm \leqq \int_S |g_i-g_j|\cdot|f_i|\,dm = \int_\Gamma |g_i-g_j|\cdot|f_i|\,dm$$

$$+\int_{S\setminus\Gamma}|g_i-g_j|\cdot|f_i|\,dm \leqq 2\int_\Gamma |f_i|\,dm+(2\gamma)^{-1}\int_{S\setminus\Gamma}|g_i-g_j|\,dm$$

$$\leqq,\ \text{by (7)},\ \frac{1}{2}+(2\gamma)^{-1}\|g_i-g_j\|_1,$$

whence (2) follows. Proof of (3) is similar: we replace everywhere g_i-g_j by g_i.

Lemma 1.2. *Let* $\delta\in(0,1)$. *Let* $T\colon L^2\to L^2$ *be a Hilbert—Schmidt operator (i.e.* $hs(T)=(trT^*T)^{1/2}<\infty$*). Denote by* B *the unit ball of* L^2. *Then* $T(B)$ *admits a* δ*-net of cardinality not exceeding* $c(\delta)^{hs(T)^2}$, *where* $c(\delta)=4^{\delta^{-2}}$.

Proof of Lemma 1.2 is standard and we omit it.

The following lemma is crucial.

Lemma 1.3. *Under assumptions* 0°—2° *of Proposition 1.A there exist* $\alpha,\beta>0$ *(depending only on* η*) such that, for any integers* k, n *with* $0\leqq k<k+n\leqq N$,

$$\int_H |F_{k,n}|\,dv>\alpha,$$

where $H=H(k,n)$ *is defined by* $H=\{|F_{k,n}|>\beta(\ln n)^{\frac{1}{2}}\}$.

Proof. We prove Lemma 1.3 with $\alpha=\gamma/4$ and $\beta=(2\ln c(\gamma/4))^{-\frac{1}{2}}$, where γ and $c(\cdot)$ are the same as in Lemma 1.1 and Lemma 1.2 respectively. Since the case $n=1$ follows immediately from (3) and the fact that $\|f_{k+1}\|_1\geqq 1$, we can assume $n\geqq 2$. Fix k,n and denote $F^+=F_{k,n}\chi_H$, $F^-=F_{k,n}\chi_{CH}$. Then the conclusion of Lemma 1.3 may be expressed as

(8) $$\int_T |F^+|\,dv\geqq\gamma/4.$$

Denote operators $g\to\int_S F^+(s,\cdot)g(s)m(ds)$ and $g\to\int_S F^-(s,\cdot)g(s)m(ds)$ by P^+ and P^- respectively (we do not specify function spaces, which are domain and range of P^+ and P^- as far). Clearly

(9) $$(P^++P^-)g_i=g_i \quad\text{for}\quad i=k+1,\dots,k+n.$$

Since $|F^-|\leqq\beta(\ln n)^{\frac{1}{2}}$, the Hilbert—Schmidt norm of P^- (considered as an operator on $L^2(m)$) does not exceed $\beta(\ln n)^{\frac{1}{2}}$ (more precisely, $hs(P^-)=(\int_T|F^-|^2dv)^{\frac{1}{2}}$). Hence, by Lemma 1.2, $P^-(B)$ (B is the unit ball in $L^2(m)$) admits a $\gamma/4$-net of cardinality at most $c(\gamma/4)^{\beta^2\ln n}=n^{\frac{1}{2}}<n$ (remember that $n\geqq 2$). Com-

bining this with the estimate $\|g_i\|_2 \leqq \|g_i\|_\infty \leqq 1$ we get that there exist two distinct integers $i, j \in \{k+1, \ldots, k+n\}$ such that, when we denote $g = g_i - g_j$, then

$$\|P^- g\|_1 \leqq \|P^- g\|_2 = \|P^- g_i - P^- g_j\|_2 \leqq \gamma/2.$$

Now (2), (9) and the preceding inequality show that

$$\|P^+ g\|_1 \geqq \|g\|_1 - \|P^- g\|_1 \geqq \gamma/2. \tag{10}$$

In other words

$$\int_S \left| \int_S F^+(s, t) g(s) m(ds) \right| m(dt) \geqq \gamma/2.$$

Hence

$$\gamma/2 \leqq \int_S \left| \int_S F^+(s, t) g(s) m(ds) \right| m(dt) \leqq \int_S \left(\int_S |F^+(s, t)| m(ds) \cdot \|g\|_\infty \right) m(dt)$$
$$\leqq 2 \int_T |F^+| \, dv.$$

whence (8) and therefore Lemma 1.3 follow.

Lemma 1.4. *Let* $L > 0$. *Then, under assumptions* 0°—2° *of Proposition 1.A, we have, for any integers* k, n *with* $0 \leqq k < k+n \leqq N$,

$$\int_V |F_{k,n}| \, dv \leqq n\eta(n\eta(1) L^{-1}),$$

where $V = V(k, n, L)$ *is defined by* $V = \{|F_{k,n}| > L\}$.

Proof. Denote $V_t = \{s\colon (s, t) \in V\}$ for $t \in S$. Then we have, by 1° and 2°, for each $t \in S$,

$$\int_{V_t} |F_{k,n}(s, t)| m(ds) \leqq \sum_{i=k+1}^{k+n} \int_{V_t} |f_i(s)| \, m(ds) \leqq \sum_{i=k+1}^{k+n} \eta(f_i, m(V_t)) \leqq n\eta(m(V_t)).$$

Hence, for all $t \in S$,

$$m(V_t) \leqq n\eta(m(V_t)) L^{-1} \leqq n\eta(1) L^{-1},$$

because η is increasing. By the same reason, combining two preceding inequalities, we get for all $t \in S$

$$\int_{V_t} |F_{k,n}(s, t)| \, m(ds) \leqq n\eta(n\eta(1) L^{-1}).$$

Integrating the above inequality with respect to m over $t \in S$ we obtain the desired estimate.

Now we return to the proof of Proposition 1.A. We deduce (1) from Lemmas 1.3 and 1.4, using the assumption 0° of the proposition only. In other words, if η satisfies 0°, then there exists $c_N \uparrow \infty$ such that if, for some N and for some sequence of measurable function $(F_n)_{n=0}^N$, $F_0=0$, on some measure space $(T, \sum, \nu)$, a corresponding family of functions $F_{k,n}$ defined by

$$F_{k,n} = F_{k+n} - F_k \quad \text{for} \quad 0 \leqq k \leqq k+n \leqq N$$

satisfies the conclusions of Lemmas 1.3 and 1.4, then (1) holds.

We define by induction two sequences of nonnegative real numbers (A_j), (B_j) and a sequence of positive integers (d_j) (all for $j=1, 2, \dots$). These sequences will depend on the function η only.

Put $d_1=1$. Suppose that for some r we have defined A_j and B_j for $j \leqq r-1$ and d_j for $j \leqq r$. Then we put

$$A_r = \beta (\ln d_r)^{\frac{1}{2}}$$

$$B_r = \min \{L > 0\colon\ d_r \cdot \eta(d_r \eta(1) L^{-1}) \leqq 2^{-3}\alpha\},$$

where α and β are the same as in Lemma 1.3. Since η is continuous and $\eta(0^+)=0$ (by 0°), the set in the definition of B_r is nonempty, closed and therefore this definition is correct. Since $d_r \geqq 1$, the definition of A_r is also correct.

Now put

$$d_{r+1} = [\exp((18 r B_r/\beta)^2)] + 1.$$

Clearly all three sequences increase to infinity and

(11) $$A_{j+1} \geqq 18 j B_j \quad \text{and} \quad B_j > A_j \quad \text{for} \quad j = 1, 2, \dots .$$

Given N we set

(12) $$q = q_N = \max \{r\colon\ \textstyle\sum_{j=1}^{2r} d_j \leqq N\}.$$

Observe that we have, by Lemmas 1.3 and 1.4,

(13) $$\int_{\{|F_{k,d_j}| > A_j\}} |F_{k,d_j}|\, d\nu > \alpha$$

(14) $$\int_{\{|F_{k,d_j}| > B_j\}} |F_{k,d_j}|\, d\nu \leqq 2^{-3}\alpha$$

for $j=1, 2, \dots, 2q$ and $k \leqq N-d_j$.

We shall prove Proposition 1.A with (c_N) defined by

(15) $$c_N = 2^{-3}\alpha q_N \quad \text{for} \quad N = 1, 2, \dots .$$

Clearly $c_N \uparrow \infty$.

We define by induction a sequence of integers N_j: $N_0=0$, $N_{j+1}=N_j$ or $N_{j+1}=N_j+d_{2q-j}$ (for $j=0, 1, 2, \dots, q$) and a sequence of measurable functions G_j on T, $G_0=0$, such that if we set

$$E_j = \left\{|G_j| > \frac{1}{2} A_{2q-j+1}\right\} \quad \text{for} \quad j = 0, 1, 2, \dots, q,$$

then

(16)$_j$ $$\int_{E_j} |G_j|\,dv - \int_T |G_j - F_{N_j}|\,dv \geqq 2^{-3}\alpha j$$

(17)$_j$ $$\int_T |G_j - F_{N_j}|\,dv \leqq 2^{-2}\alpha j$$

for $j=0, 1, 2, \dots, q$.

Suppose we have done this. Then $(16)_q$ and (15) imply

$$\int_T |F_{N_q}|\,dv \geqq 2^{-3}\alpha q = c_N.$$

Since, by (12), $N_q \leqq \sum_{j=q+1}^{2q} d_j \leqq N$, this proves Proposition 1.A.

The inductive construction

$(16)_0$ and $(17)_0$ hold trivially.

Suppose that for some $j<q$ we have defined N_i and G_i to satisfy $(16)_i$ and $(17)_i$ for all $i \leqq j$. Set

$$U_j = \left\{\frac{1}{2} A_{2q-j} < |G_j| \leqq \frac{1}{2} A_{2q-j+1}\right\}$$

and consider separately two cases.

Case 1.

$$\int_{E_j \cup U_j} |G_j|\,dv - \int_T |G_j - F_{N_j}|\,dv \geqq 2^{-3}\alpha(j+1).$$

We define $N_{j+1}=N_j$, $G_{j+1}=G_j$. Then $E_{j+1}=E_j \cup U_j$; $(16)_{j+1}$ and $(17)_{j+1}$ are clearly satisfied.

Case 2. We have the converse inequality. Combining it with $(16)_j$ we obtain

(18) $$\int_{U_j} |G_j|\,dv < 2^{-3}\alpha;$$

combining with $(17)_j$ —

$$\int_{E_j} |G_j|\,dv < 2^{-3}\alpha(j+1) + 2^{-2}\alpha_j < \frac{3}{8}\alpha(j+1).$$

Since, by definition, $|G_j| > \frac{1}{2} A_{2q-j+1}$ on E_j, the preceding estimate implies

(19) $$v(E_j) = \int_{E_j} dv \leqq \frac{2}{A_{2q-j+1}} \int_{E_j} |G_j|\,dv < \frac{3\alpha(j+1)}{4A_{2q-j+1}}.$$

We define $N_{j+1}=N_j+d_{2q-j}$ and $G_{j+1}=G_j\chi_{CU_j}+F_{N_j,d_{2q-j}}\chi_{CV_j}$, where $V_j=\{|F_{N_j,d_{2q-j}}|>B_{2q-j}\}$.

Observe now that if $x\in E_j$ (and hence $x \notin U_j$), then

$$|G_{j+1}(x)| = |G_j(x)+F_{N_j,d_{2q-j}}(x)\chi_{CV_j}(x)| \geqq |G_j(x)|-B_{2q-j}$$
$$\geqq \frac{1}{2} A_{2q-j+1} - B_{2q-j} > \frac{1}{2} A_{2q-j}$$

(the latest inequality follows from (11)). This shows that

(20) $$E_{j+1} \supset E_j.$$

Similarly

(21) $$\int_{E_j} |G_{j+1}|\, dv \geqq \int_{E_j} (|G_j| - B_{2q-j})\, dv = \int_{E_j} |G_j|\, dv - B_{2q-j} v(E_j).$$

Set $W_j = \{A_{2q-j} < |F_{N_j, d_{2q-j}}| \leqq B_{2q-j}\}$. Then $W_j \subset CV_j$.

Let $x \in W_j \setminus E_j$. Then, similarly as in the proof of (20),

$$|G_{j+1}(x)| = |G_j(x)\chi_{CU_j}(x) + F_{N_j, d_{2q-j}}(x)\chi_{CV_j}(x)| \geqq |F_{N_j, d_{2q-j}}(x)| - \frac{1}{2} A_{2q-j} \geqq \frac{1}{2} |F_{N_j, d_{2q-j}}(x)| > \frac{1}{2} A_{2q-j}.$$

Hence $x \in E_{j+1}$. Combining this with (20) we obtain

(22) $$E_{j+1} \supset E_j \cup W_j.$$

Similarly we get, using (13) and (14),

(23) $$\int_{W_j \setminus E_j} |G_{j+1}|\, dv \geqq \frac{1}{2} \int_{W_j \setminus E_j} |F_{N_j, d_{2q-j}}|\, dv > \frac{1}{2} \int_{W_j} |F_{N_j, d_{2q-j}}|\, dv - \frac{1}{2} v(W_j \setminus E_j) B_{2q-j} \geqq \frac{7}{16}\alpha - \frac{1}{2} v(E_j) B_{2q-j}.$$

Now, using (22), (21), (23) and (19) consecutively, we obtain

$$\int_{E_{j+1}} |G_{j+1}|\, dv \geqq \int_{E_j} |G_{j+1}|\, dv + \int_{W_j \setminus E_j} |G_{j+1}|\, dv \geqq \int_{E_j} |G_j|\, dv + \frac{7}{16}\alpha - \frac{3}{2} v(E_j) B_{2q-j} \geqq \int_{E_j} |G_j|\, dv + \frac{7}{16}\alpha - \frac{9}{8}\alpha \frac{B_{2q-j}(j+1)}{A_{2q-j+1}}.$$

Since $j \leqq q-1$ we have, by (11),

$$\frac{B_{2q-j}(j+1)}{A_{2q-j+1}} \leqq \frac{j+1}{18(2q-j)} \leqq \frac{1}{18},$$

which combined with the previous estimate gives

(24) $$\int_{E_{j+1}} |G_{j+1}|\, dv \geqq \int_{E_j} |G_j|\, dv + \frac{3}{8}\alpha.$$

On the other hand

$$\int_T |G_{j+1} - F_{N_{j+1}}|\, dv \leqq \int_T |G_j - F_{N_j}|\, dv + \int_{U_j} |G_j|\, dv + \int_{V_j} |F_{N_j, d_{2q-j}}|\, dv \leqq \int_T |G_j - F_{N_j}|\, dv + \frac{\alpha}{4},$$

by (18) and (14).

Now to get $(16)_{j+1}$ or $(17)_{j+1}$ it suffices to combine the above inequality with (24) and $(16)_j$ or $(17)_j$ respectively. Thus the induction is completed.

This ends the proof of Proposition 1.A and completes the proof of Theorem A.

Now we derive Corollaries B and C from Theorem A.

Let us recall two basic facts on equiintegrable sets. The first of them is well-known, the second one is due to Kadec and Pełczyński (cf. [3], Theorem 6). Let $(S, \mathcal{B}, m)$ be a measure space.

1) *If m is finite, then a set Z of m-measurable functions is equiintegrable iff $Z \subset L^1(m)$ and Z is relatively weakly compact in $L^1(m)$.*

2) *A bounded set $Z \subset L^1(m)$ is relatively weakly compact iff no sequence of elements of Z is equivalent to the unit vector basis of l^1.*

Recall that two sequences (x_n) and (y_n) of elements of Banach spaces X and Y respectively are said to be equivalent if, for some $c \in (0, \infty)$,

$$c^{-1} \|\sum t_n x_n\|_X \leqq \|\sum t_n y_n\|_Y \leqq c \|\sum t_n x_n\|_X$$

for every sequence of scalars (t_n) with finite number of nonzero elements.

The above facts explain, in particular, equivalence of the two statements contained in Corollary B.

Proof of Corollary B. Let us assume the converse. Let $(f_n)_{n=1}^\infty$ be a normalized basis of $L^1(0, 1)$ such that the functions f_n are equiintegrable. Denote by (g_n), $g_n \in L^\infty(0, 1)$, a corresponding sequence of basis functionals. Since (f_n) is a normalized basis, $\sup_n \|g_n\|_\infty < \infty$. Thus $(f_n, g_n)_{n=1}^\infty$ satisfy the assumptions of Theorem A and hence

$$\sup_n \int_0^1 \int_0^1 \left|\sum_{i=1}^n f_i(s) g_i(t)\right| ds\, dt = \infty. \tag{25}$$

On the other hand, since (f_n) is a basis, the norms of the partial sums operators $P_n \colon L^1 \to L^1$, $P_n(\sum_{i=1}^\infty c_i f_i) = \sum_{i=1}^n c_i f_i$, are uniformly bounded (say, by $K < \infty$). It is a wellknown fact that each P_n may be written as

$$(P_n f)(s) = \int_0^1 \left(\sum_{i=1}^n f_i(s) g_i(t)\right) f(t)\, dt$$

and hence, for any positive integer n,

$$K \geqq \|P_n\| = \operatorname*{supess}_{t \in (0,1)} \int_0^1 \left|\sum_{i=1}^n f_i(s) g_i(t)\right| ds \geqq \int_0^1 \int_0^1 \left|\sum_{i=1}^n f_i(s) g_i(t)\right| ds\, dt$$

which contradicts (25). This proves Corollary B.

Proof of Corollary C. Denote by (f_n) the sequence of basis functionals corresponding to the basis (g_n). Then there exists some regular Borel probability measure m such that all f_n belong to $L^1(m) \subset C(0, 1)^*$. The same argument as in the proof of Corollary B shows that the functions f_n are not equiintegrable and consequently, by the Kadec—Pełczyński result, some subsequence $(f_{n_k})_{k=1}^{\infty}$ of (f_n) is equivalent to the unit vector basis of l^1.

Let us consider two operators:

$$T\colon C\to c_0 \quad \text{defined by } T\left(\sum t_n g_n\right)=(t_{n_k})_{k=1}^{\infty} \quad \text{and}$$

$$S\colon l^1\to C^* \quad \text{defined by } S[(t_k)]=\sum t_k f_{n_k}.$$

Clearly $S=T^*$ and S is a isomorphic embedding. Hence T is onto. This proves Corollary C.

Now we show, how the facts stated in Remark 0 follow from Corollaries B and C.

3° Recall that a sequence (x_n) of elements of a Banach space is said to be q-Hilbertian (resp. p-Besselian) iff there exists a constant c such that, for any sequence of scalars (t_n),

$$\left\|\sum t_n x_n\right\|^q \leqq c^q \sum |t_n|^q \|x_n\|^q$$

(resp.
$$c^p\left\|\sum t_n x_n\right\|^p \geqq \sum |t_n|^p \|x_n\|^p).$$

Let (f_n) (resp. (g_n)) be a q-Hilbertian sequence in L^1 for some $q>1$ (resp. a p-Besselian sequence in C for some $p<\infty$), which is a basis of L^1 (resp. C). We can assume that (f_n) (resp. (g_n)) is normalized.

By Corollary B, some subsequence (f_{n_k}) of (f_n) is equivalent to the unit vector basis of l^1; in other words, there exists a constant c' such that

$$\left\|\sum t_k f_{n_k}\right\|_1 \geqq c' \sum |t_k|$$

for all $(t_k)\in l^1$. On the other hand, since (f_n) is q-Hilbertian, we have

$$\left\|\sum t_k f_{n_k}\right\|_1 \leqq c\left(\sum |t_k|^q\right)^{1/q},$$

which contradicts the previous estimate.

Similarly, by Corollary C, there exists a function $g=\sum t_n g_n\in C$ such that $(t_n)\in c_0\setminus l^p$. On the other hand, since (g_n) is p-Besselian, we have

$$\sum |t_n|^p \leqq c^p\left\|\sum t_n g_n\right\|^p = c^p\|g\|^p<\infty,$$

a contradiction.

2° Every subset of L^1, which is bounded in order, is equiintegrable and hence, by facts 1) and 2), no sequence of its elements is equivalent to the unit vector basis of l^1. Now we obtain the desired conclusion from Corollary B.

1° It is easy to see that any uniformly bounded orthonormal system is 2-Besselian in C and 2-Hilbertian in L^1. Therefore, by 3°, it is no a basis of any of these spaces.

2. The $\mathcal{L}^p$-spaces

In this section we generalize Corollaries B and C to the cases of $\mathcal{L}^1$-spaces and $\mathcal{L}^\infty$-spaces respectively (for definitions and basic properties see [5] and [6]). More precisely, we prove

Corollary 2.B. *Let X be a $\mathcal{L}^1$-space. Let $(f_n)_{n=1}^\infty$ be a normalized basis of X. Then some subsequence of (f_n) is equivalent to the unit vector basis of l^1.*

Corollary 2.C. *Let Y be a $\mathcal{L}^\infty$-space. Let $(g_n)_{n=1}^\infty$ be a normalized basis of Y. Then, for some increasing sequence of positive integers $(n_k)_{k=1}^\infty$, the map $\sum c_n g_n \to (c_{n_k})_{k=1}^\infty$ takes Y onto c_0.*

Proof of Corollary 2.B. Let Z be a Banach space, Z_0 its subspace. We shall say that Z_0 is locally complemented in Z iff, for some constant c, every finite dimensional subspace E of Z_0 is contained in another finite dimensional subspace F of Z_0 such that there exists a projection P_F from Z onto F with $\|P_F\| \leqq c$. We have the following

Lemma 2.1. *Let X be a $\mathcal{L}^1$-space. Then X is isomorphic to a locally complemented subspace of some $L^1(m)$-space.*

Proof. By [6], Theorem III (c), X is locally complemented in itself. Since each P_F^{**} (see the definition above) is then a projection from X^{**} onto F with $\|P_F^{**}\| = \|P_F\| \leqq c$, X is locally complemented in X^{**}. By [6], Theorem III(a), X^{**} is a $\mathcal{L}^1$-space. Hence, by [5], Corollary 1 to Theorem 7.1, X^{**} is isomorphic to a complemented subspace of a $L^1(m)$-space. This proves Lemma 2.1.

We return to the proof of Corollary 2.B.

We know, by Lemma 2.1, that X may be considered as a locally complemented subspace of a $L^1(m)$-space. Since X has a basis, and hence is separable, we can assume that m is a probability measure. Similarly as in the proof of Corollary B, it is enough to show that f_i, considered as elements of $L^1(m)$, are not equiintegrable.

Let us assume the converse. Denote by $(\tilde{g}_n)$ the seqence of corresponding basis functionals ($\tilde{g}_n \in X$). Set $M = \sup_n \|\tilde{g}_n\|$ $(<\infty)$. Given positive integer N put $E^N = \text{span}\{f_1, f_2, \dots, f_N\}$. Then there exist $F^N, E^N \subset F^N \subset X$, and a projection Q_N: $L^1(m) \xrightarrow{\text{onto}} F^N$, $\|Q_N\| \leqq c$. Denote $g_n = Q_N^*(g_{n|F^N}) \in L^\infty(m)$ for $n = 1, 2, \dots, N$, then $\|g_n\|_\infty \leqq Mc$.

Now we proceed similarly as in the proof of Corollary B. First we observe that the norms of the operators $P_n \circ Q_N$, where $P_n: X \to X$ is defined by $P_n(\sum_{i=1}^\infty t_i f_i) = \sum_{i=1}^n t_i f_i$, are uniformly bounded for $N = 1, 2, \dots$ and $n = 1, 2, \dots, N$. Then, applying Proposition 1.A to the biorthogonal system $(Mcf_n, (Mc)^{-1} g_n)_{n=1}^N$ and $\eta = \eta(Mcf_i, \cdot)$, we obtain that at least one of the numbers $\|P_n \circ Q_N\|$, $n = 1, 2, \dots, N$, is not less than c_N, where the sequence $c_N\uparrow\infty$ depends only on c, M and (f_n), a contradiction for large N. This proves Corollary 2.B.

Proof of Corollary 2.C. We start with the following

Lemma 2.2. *Let Y be a $\mathscr{L}^\infty$-space. Let (g_n) be a basis of Y, (f_n) — the sequence of corresponding basis functionals. Denote $X=\overline{\operatorname{span}\{f_n\}}\subset Y^*$. Then X is a $\mathscr{L}^1$-space.*

Proof. Set $Y_k=\operatorname{span}\{g_1,\dots,g_k\}$, $Y^k=\operatorname{span}\{g_{k+1},g_{k+2},\dots\}$, $X_k=\operatorname{span}\{f_1,\dots,f_k\}$ and $X^k=\operatorname{span}\{f_{k+1},f_{k+2},\dots\}$. To prove Lemma 2.2, it is enough to show that, for some constant λ, each X_k is contained in a finite dimensional subspace F^k of X with

$$d(F^k, l^1_{n(k)})\leqq\lambda\quad\bigl(n(k)=\dim F^k\bigr), \tag{26}$$

where d is the Banach—Mazur distance.

Denote by K the basis constant of (g_n). Fix k and denote by P_k the natural projection from Y onto Y_k. Then we have

$$\|P_k\|\leqq K,\quad \|Id_Y-P_k\|\leqq K+1 \tag{27}$$

and consequently

$$d(X_k, Y_k^*)\leqq K,\quad d(X^k, Y^{k*})\leqq K+1. \tag{28}$$

Since Y is a $\mathscr{L}^\infty_\mu$-space for some μ, Y_k is contained in some finite dimensional subspace E^k of Y with

$$d(E^k, l^\infty_{n(k)})\leqq\mu\quad\bigl(n(k)=\dim E^k\bigr). \tag{29}$$

Denote $\tilde{E}^k=(Id_Y-P_k)E^k$. Then, remembering that $Y_k=P_kE^k$, we obtain, by (27) and (29),

$$d\bigl(l^\infty_{n(k)}, (Y_k\oplus\tilde{E}^k)_2\bigr)\leqq 2(K+1)\mu.$$

Hence, by (28),

$$d\bigl(l^1_{n(k)}, (X_k\oplus\tilde{E}^{k*})_2\bigr)\leqq 2K(K+1)\mu. \tag{30}$$

By (29), there exists a projection Q_k from Y onto E^k with $\|Q_k\|\leqq\mu$ (one can also use Theorem III(c), [6]). Therefore $(Id_Y-P_k)Q_k$ is a projection from Y^k onto $\tilde{E}^k$ and hence Y^{k*} contains a subspace, whose Banach—Mazur distance from $\tilde{E}^{k*}$ does not exceed $\|(Id_Y-P_k)Q_k\|\leqq(K+1)\mu$. This shows, by (28), that some subspace $\tilde{F}^k$ of X^k satisfies $d(\tilde{F}^k, \tilde{E}^{k*})\leqq K(K+1)\mu$.

Since the basis constant of (f_n) is at most K, we have

$$d\bigl(X_k+\tilde{F}^k, (X_k\oplus\tilde{F}^k)_2\bigr)\leqq 2(K+1),$$

which combined with the previous estimate and (30) proves (26) with $F^k = X_k + \tilde{F}^k$ and $\lambda = 4K^2(K+1)^3\mu^2$, completing the proof of Lemma 2.2.

Now we return to the proof of Corollary 2.C. Since, by Lemma 2.2, (f_n) is a basis of the $\mathscr{L}^1$-space X, some subsequence of $(f_n/\|f_n\|)$ is, by Cor. 2.B, equivalent to the unit vector basis of l^1. We have clearly $1 \leqq \inf_n \|f_n\| \leqq \sup_n \|f_n\| < \infty$ and therefore (f_n) itself contains a subsequence, which is equivalent to the unit vector basis of l^1. This, similarly as in the proof of Corollary C, implies the desired conclusion.

3. The pointwise convergence

The main result of this section is the following generalization of Theorem 3, [9], Ch. I, § 2.

Theorem 3.C. *Let S be a compact topological space, m — a regular Borel measure on S. Let $(f_n, g_n)_{n=1}^{\infty}$, where $g_n \in C(S)$ and $f_n \in L^1(m) \subset C(S)$, be a biorthogonal (with respect to m) system of functions satisfying*

1° $\sup_n \|g_n\|_\infty < \infty$

2° the functions f_n are equiintegrable.

Then there exists a Borel set $D \subset S$, $m(D) > 0$, such that if $s \in D$ then, for some function $g \in C(S)$, the formal series of g

$$(+) \qquad \sum_{n=1}^{\infty} \left(\int_S f_n g \, dm \right) \cdot g_n$$

diverges unboundedly at the point s.

Remark 3.1. Theorem 3.C strengthens Corollary C. Indeed, Cor. C says that, under the assumptions of Th. 3.C, (g_n) is not a basis of $C(S)$; moreover, the proof of Cor. C shows that, for some $g \in C(S)$ the series $(+)$ is not uniformly bounded.

In the sequel the following concept will be useful. In the notation of Theorem 3.C we say that

$$L_n = L_n(t) = \int_S \left| \sum_{i=1}^{n} f_i(s) g_i(t) \right| m(ds) \quad (t \in S,\ n = 1, 2, \ldots)$$

is the n-th Lebesgue function of the system (f_n, g_n) (not necessarily satisfying the conditions 1° and 2°). It is a well-known fact that the series $(+)$ is uniformly bounded for every $g \in C(S)$ (and then uniformly convergent for every $g \in Y = \overline{\text{span}\{g_i\}}$) iff there exists a constant $K < \infty$ such that $L_n \leqq K$ for $n = 1, 2, \ldots$ (we used this in deducing Cor. C from Th. A). The series $(+)$ is bounded at the point t for every $g \in C(S)$ (and then convergent at t for every $g \in Y$) iff the sequence $(L_n(t))_{n=1}^{\infty}$ is bounded.

Proof of Theorem 3.C. By above, we must show that, when we denote $\Delta=\{t\in S: \sup_n L_n(t)=\infty\}$, then $m(\Delta)>0$. To do this, we need the following proposition, which generalizes Theorem 1, [9], Ch. I, § 2.

Proposition 3.A. *Under assumptions of Theorem 3.C there exists a sequence* $b_N\uparrow\infty$ *and a constant* $\varrho>0$ *such that*

$$m\big(\{\max_{1\leqq n\leqq N} L_n(t)\geqq b_N\}\big)\geqq\varrho \tag{31}$$

for $N=1,2,\ldots$. *Both* (b_N) *and* ϱ *depend only on the modulus of integrability of the set* $\{f_i\}$.

Suppose we have proved this and assume, to get a contradiction, that $m(\Delta)=0$. Denote $L(t)=\sup_n L_n(t)$ for $t\in S$. Since L is finite m - a.e., there exists $\lambda<\infty$ such that $m(\{L\geqq\lambda\})<\varrho$. Choose N to satisfy $b_N\geqq\lambda$. Then, by Proposition 3.A,

$$m\big(\{\max_{1\leqq n\leqq N} L_n(t)\geqq b_N\}\big)\leqq m(\{L\geqq\lambda\})<\varrho;$$

a contradiction. Thus it remains to prove Prop. 3.A.

Proof of Prop. 3.A. We can assume $\|g_i\|_\infty\leqq 1$. Then $\eta=\eta(\{f_i\},\cdot)$ and $(f_n, g_n)_{n=1}^N$ satisfy the assumptions of Prop. A.1 for every N. Hence, by Lemma 1.1, there exists $\gamma>0$, depending on $\eta(\{f_i\},\cdot)$ only, such that (2) and (3) hold. We prove Prop. 3.A with $\varrho=\gamma/8$.

Let $D\subset S$ be a Borel set with $m(D)\leqq\varrho$. Denote $\bar g_n(t)=g_n(t)\chi_{CD}(t)$ and $\bar F_n(s,t)=\sum_{i=1}^n f_i(s)\bar g_i(t)$ for $n=0,1,2,\ldots$ and $s,t\in S$. Our present goal is to prove that

$$\max_{1\leqq n\leqq N}\int_S\int_S \bar F_n(s,t)\,m(ds)\,m(dt)\geqq a_N \tag{1'}$$

for some sequence $a_N\uparrow\infty$, depending only on $\eta(\{f_i\},\cdot)$. To do this, by the observation at the beginning of the proof of Prop. 1.A, it is enough to show that the family of functions $F_{k,n}$ defined by

$$F_{k,n}=\bar F_{k+n}-\bar F_k \quad\text{for}\quad k,n\geqq 0$$

satisfies the conclusions of Lemmas 1.3 and 1.4 for $N=1,2,\ldots$.

The case of Lemma 1.4 is immediate. The proof of Lemma 1.3 needs slight modifications only. We have, instead of (9),

$$(P^++P^-)g_i=\bar g_i=g_i\chi_{CD}\quad\text{for}\quad i=k+1,\ldots,k+n \tag{9'}$$

and consequently, instead of (10),

$$(10') \qquad \|P^+g\|_1 \geqq \|g\chi_{CD}\|_1 - \|P^-g\|_1 \geqq \|g\|_1 - \|g\chi_D\|_1 - \|P^-g\|_1 \geqq \gamma - \gamma/4 - \gamma/2 = \gamma/4,$$

whence the conclusion of Lemma 1.3 (for our current $F_{k,n}$) follows with $\alpha=\gamma/8$ instead of $\alpha=\gamma/4$. This proves (1′).

We claim that (31) holds with $b_N=a_N/2$. Suppose not and apply the above procedure to $D=\{\max_{1\leqq n\leqq N} L_n(t)\geqq b_N\}$. Observe that (1′) may be rewritten as

$$\max_{1\leqq n\leqq N}\int_{CD} L_n(t)\,m(dt) \geqq a_N.$$

On the other hand we have, for $n=1, 2, \dots, N$,

$$\int_{CD} L_n(t)\,m(dt) \leqq \int_{CD}\max_{1\leqq n\leqq N} L_n(t)\,m(dt) \leqq b_N m(CD) < 2b_N = a_N,$$

a contradiction. This proves Prop. 3.A and completes the proof of Theorem 3.C.

4. Remarks and open problems

Remark 4.1. It is easy to prove Cor. 2.B in the special case $X=l^1$. Moreover, the conclusion of Cor. 2.B holds then for every normalized basic sequence $(f_n)_{n=1}^{\infty}$. To show this let us assume that, to the contrary, $\{f_n\}$ is a relatively compact set. Then f_n, being a basic sequence, tends weakly to 0. Hence $\|f_n\|\to 0$, because l^1 has the Schur property; a contradiction.

Remark 4.2. Cor. 2.B does not hold, if we replace 1 by any $p>1$, $p\neq 2$. Indeed, if $p\in(1, \infty)$, then the trigonometric system $(e^{2\pi in})_{n=-\infty}^{+\infty}$, which is a basis of $L^p(0, 1)$, clearly does not contain subsequences, which are equivalent to the unit vector basis of l^p for $p\neq 2$. To solve the case $p=\infty$ let us mention that, by [4], there exists a basis (g_n) of $C(0, 1)$ such that $1\leqq g_n(t)\leqq 2$ for all n and all t. Therefore no subsequence of (g_n) is equivalent to the unit vector basis of c_0 (contrary to the cases of classical Schauder, Haar and Franklin bases). On the other hand it is a well known fact that we can replace 1 by 2 in Cor. 2.B.

Remark 4.3. The case of unconditional bases is much simpler. Indeed, every normalized basis of a $\mathscr{L}^1$-space (resp. $\mathscr{L}^\infty$-space) is equivalent to the unit vector basis of l^1 (resp. c_0) (see e.g. [5], Theorem 6.1). Since, by [2], Theorem 5.1, every separable $\mathscr{L}^p$-space has a basis and hence, by [10], a conditional basis, this is not true, if we omit the unconditionality assumption. If $p\in(1, \infty)$, then, by [3], Theorem

4, every normalized unconditional basis of L^p contains a subsequence, which is equivalent to the unit vector basis of l^p.

Remark 4.4. By Remark 0, Corollaries B and C generalize most qualitative results of [4] and [12]. On the other hand, in [4] and [12] we always get a logarithmic order of growth of Lebesgue functions of investigated systems, while in the present paper we obtain much worse estimates.

In connections with the above remark the following problem seems natural.

Problem 4.5. What is "the best" correspondence between η and (c_N) in Proposition 1.A?

Problem 4.6. Suppose that the assumptions of Theorem 3.A are satisfied

a) Let (L_k) be the corresponding sequence of Lebesgue functions. Does $\frac{1}{N}\sum_{k=1}^{N}\|L_k\|_1$ tend to infinity with N? Cf. Lemma 3, [1], and Lemma B, [4].

b) Does there exists a function $f\in L^1(m)$ such that the series $\sum\left(\int_S g_n f\,dm\right)\cdot f_n$ diverges on a set of positive measure m? Cf. [1].

Remark 4.7. Let (f_n) be a normalized basis of L^p for some $p\in[1,\infty)$. It is a well-known fact that the Haar system (χ_n) (and hence the unit vector basis of l^p) is equivalent to some sequence, whose elements are spanned by disjoint blocks of (f_n) (see e.g. [9], Ch. III, § 1). Now let $p=1$. Normalize (χ_n) in L^1. In view of Corollary B it is natural to ask, whether (χ_n) is equivalent to some subsequence of (f_n). In general the answer is negative: by Cor. 4.6, [13], there exists a basis of L^1, which is p-Besselian for every $p>1$. Hence (χ_n), which is not p-Besselian for any $p<\infty$, is not equivalent to any subsequence of (f_n).

Problem 4.8. Does there exists a normalized basis of L^1, say (f_n), such that it may be represented as $\{f_n\}=Z_1\cup Z_2$, where Z_1 is relatively weakly compact and Z_2 is equivalent to the set of unit vectors of l^1? Of course the Haar basis does not posess this property.

Remark 4.9. Corollary B does not remain true after replacing $L^1(0, 1)$ by the Hardy space H^1. To see this notice first that H^1 has a basis and, by the Paley's theorem, contains a complemented subspace isomorphic to l^2. Hence, by Cor. 3.6, [13], H^1 has a normalized basis, which is weakly convergent to 0 (even p-Hilbertian for $p<2$).

Problem 4.10. Does Cor. C remain true after replacing $C(0, 1)$ by the space of analytic functions A?

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Received February 1, 1979

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