Computing 2-summing norm with few vectors

Nicole Tomczak-Jaegermann

Recall that if $u\colon X\to Y$ is a linear operator (X, Y-normed spaces) then $\pi_2(u)$, the 2-absolutely summing norm of u, is defined by

$$\pi_2(u)^2 = \sup \sum_{x\in F} \|ux\|^2,$$

where the supremum is taken over all finite subsets $F\subset X$ such that $\sum_{x\in F} |(x^*, x)|^2 \leqq \|x^*\|^2$ for every $x^*\in X^*$.

In the present paper we show that if u is an operator of rank n then $\pi_2(u)$ is essentially determined by the subsets F of cardinality $\leqq n$. "Essentially" means up to a universal factor T with $\frac{1}{2}\leqq T\leqq \cos \pi/12$. This answers a question of T. Figiel [6] and enables to obtain as corollaries analogous facts for the type 2 and cotype 2 constants of n-dimensional normed spaces (cf. [1] p. 85).

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We use standard notation and terminology (cf. [4] and [5]).

Let X, Y be Banach spaces and let $u\colon X\to Y$ be a linear operator. For each positive integer k we define $\pi_2^{(k)}(u)$ as the smallest number satisfying the inequality

$$\Big(\sum_{j=1}^k \|ux_j\|^2\Big)^{1/2} \leqq \pi_2^{(k)}(u) \sup \Big\{\Big(\sum_{j=1}^k |(x^*, x_j)|^2\Big)^{1/2} \Big| x^*\in X^*, \|x^*\| \leqq 1\Big\},$$

for every sequence $x_1, \dots, x_k\in X$.

Obviously one has $\pi_2(u)=\sup_k \pi_2^{(k)}(u)$.

Theorem 1. *Let X, Y be Banach spaces and let $u\colon X\to Y$ be a linear operator of rank n. Then*

$$\pi_2^{(n)}(u) \leqq \pi_2(u) \leqq 2\pi_2^{(n)}(u).$$

Proof. We consider first the case $X=l_2^n$. Without loss of generality we may assume $\pi_2(u)=1$. Then there exist operators $v\colon l_2^n\to l_2^n$ with $\pi_2(v)=1$ and $w\colon l_2^n\to Y$ with $\|w\|=1$ such that $u=wv$. This follows from Pietsch's factorisation theorem for 2-absolutely summing operators (cf. [5]). We will construct an orthonormal basis (e_j) in l_2^n such that

$$\left(\sum_{j=1}^n \|ue_j\|^2\right)^{1/2} \geqq \tfrac{1}{2}.$$

The e_j's are chosen inductively so that for $j=1, \dots, n$

$$\|e_j\| = 1, \quad e_j \in E_j \quad \text{where} \quad E_1 = l_2^n, \quad E_k = [e_1, \dots, e_{k-1}]^\perp \quad \text{for} \quad k > 1,$$

$$\|wve_j\| = \|ve_j\| \, \|w_{|v(E_j)}\|.$$

Let $m \leqq n$ be the positive integer such that

$$\|w_{|v(E_{m+1})}\| < \sqrt{\tfrac{1}{2}} \leqq \|w_{|v(E_m)}\|.$$

It suffices to prove that $\sum_{j=1}^m \|ve_j\|^2 \geqq \frac{1}{2}$, because this yields

$$\pi_2^{(n)}(u) \geqq \left(\sum_{j=1}^n \|ue_j\|^2\right)^{1/2} \geqq \left(\sum_{j=1}^m \|wve_j\|^2\right)^{1/2}$$

$$= \left(\sum_{j=1}^m \|w_{|v(E_j)}\|^2 \|ve_j\|^2\right)^{1/2} \geqq \sqrt{\tfrac{1}{2}} \left(\sum_{j=1}^m \|ve_j\|^2\right)^{1/2} \geqq \tfrac{1}{2}.$$

Let $P\colon l_2^n \to l_2^n$ be the orthogonal projection onto E_{m+1}, let $Q = I - P$ and let $\alpha = \sum_{j=1}^m \|ve_j\|^2$.

Since for an operator acting in a Hilbert space its 2-absolutely summing norm equals to its Hilbert—Schmidt norm, we have

$$\alpha = \pi_2(vQ)^2, \quad \pi_2(vP)^2 = \pi_2(v)^2 - \pi_2(vQ)^2 = 1 - \alpha,$$

$$\pi_2(vQ + \beta vP) = [\pi_2(vQ)^2 + \beta^2 \pi_2(vP)^2]^{1/2}, \quad \text{for any real } \beta.$$

Thus, for each $b \in (0, 1]$ we get

$$\begin{aligned}
1 = \pi_2(wv) &\leqq \pi_2(bwvP) + \pi_2(wv - bwvP) \\
&\leqq b \, \|w_{|v(E_{m+1})}\| \pi_2(vP) + \|w\| \, \pi_2(vQ + (1-b)vP) \\
&< b \sqrt{\tfrac{1}{2}} \pi_2(vP) + [\pi_2(vQ)^2 + (1-b)^2 \pi_2(vP)^2]^{1/2} \\
&= b \sqrt{\tfrac{1}{2}(1-\alpha)} + [\alpha + (1-b)^2(1-\alpha)]^{1/2} \\
&= b \sqrt{\tfrac{1}{2}(1-\alpha)} + \left[(1-(1-\alpha)b)^2 + (1-\alpha)\alpha b^2\right]^{1/2} \\
&\leqq b \sqrt{\tfrac{1}{2}(1-\alpha)} + 1 - (1-\alpha)b + b^2.
\end{aligned}$$

For the last inequality observe that $(s^2+t^2)^{1/2} \leqq s + t^2/2s$, for $s > 0$.

It follows that for every $b \in (0, 1]$ we have

$$0 < b\sqrt{1-\alpha}\left[\sqrt{\tfrac{1}{2}} - \sqrt{1-\alpha}\right] + b^2.$$

This implies that $1-\alpha\leqq\frac{1}{2}$ and hence $\alpha\geqq\frac{1}{2}$. This proves the special case of Theorem 1.

Now let $u\colon X\to Y$ be an arbitrary operator of rank n. Let $x_1, \dots, x_m\in X$ satisfy

$$\sum_{j=1}^m |(x^*, x_j)|^2 \leqq \|x^*\|^2 \quad \text{for every} \quad x^*\in X^*.$$

We shall prove that

$$\left(\sum_{j=1}^m \|ux_j\|^2\right)^{1/2} \leqq 2\pi_2^{(n)}(u).$$

Let us define $U\colon l_2^m\to X$ by $Ue_j=x_j$ $(j=1, \dots, m)$, where (e_j) is the unit vector basis in l_2^m. Our assumption yields $\|U\|\leqq 1$. Let E be the orthogonal complement of the kernel of uU. Then $\dim E=\operatorname{rank} uU\leqq n$. We apply the special case of the theorem to the operator $uU_{|E}$. This yields

$$\begin{aligned}\left(\sum_{j=1}^m \|ux_j\|^2\right)^{1/2} &= \left(\sum_{j=1}^m \|uUe_j\|^2\right)^{1/2} \leqq \pi_2(uU) = \pi_2(uU_{|E}) \\ &\leqq 2\pi_2^{(n)}(uU_{|E}) \leqq 2\pi_2^{(n)}(u)\|U_{|E}\| \leqq 2\pi_2^{(n)}(u).\end{aligned}$$

This completes the proof of Theorem 1.

Remark. The computations in the proof can be made slightly simpler by setting $b=1$. The final constant becomes then $3\sqrt{2}$ (or 4, if $\sqrt{\frac{1}{2}}$ is replaced by $\frac{1}{2}$).

Let X be a Banach space. Let $\gamma_1, \gamma_2, \dots$ be a sequence of independent normalized Gaussian random variables on a probability space (Ω, μ). Following Maurey and Pisier, for each positive integer k we define the type 2 and cotype 2 constants of X (cf. e.g. [1]). These are the smallest positive numbers $\alpha_k(X)$ and $\beta_k(X)$ such that the following inequality

$$\beta_k(X)^{-1}\left(\sum_{j=1}^k \|x_j\|^2\right)^{1/2} \leqq \left(\int_\Omega \Big\|\sum_{j=1}^k \gamma_j x_j\Big\|^2 d\mu\right)^{1/2} \leqq \alpha_k(X)\left(\sum_{j=1}^k \|x_j\|^2\right)^{1/2}$$

holds for every sequence $x_1, \dots, x_k$ in X.

Theorem 2. *Let X be an n-dimensional normed space. Then for every integer $k\geqq n$ one has*

$$\alpha_n(X)\leqq\alpha_k(X)\leqq\sqrt{2\pi}\,\alpha_n(X),$$

$$\beta_n(X)\leqq\beta_k(X)\leqq 2\beta_n(X).$$

Proof. The inequalities for the cotype 2 constants of X are formal consequences of Theorem 1 and the following obvious formula valid for $k=1, 2, \dots$

$$\beta_k(X) = \sup\left\{\pi_2^{(k)}(u) \,\middle|\, u\colon l_2^k\to X,\ \int_\Omega \Big\|\sum_{j=1}^k \gamma_j ue_j\Big\|^2 d\mu = 1\right\}.$$

The latter formula is implicit in [3] (cf. Proposition 5 and Theorem 4).

The case of the type 2 constants is slightly more difficult. We need the dual form of Theorem 1 which can be stated as follows.

Given a positive integer k let B be a set of operators $v\colon l_2^k \to X$ which admit a factorisation

$$l_k^2 \xrightarrow[v_1]{} l_2^n \xrightarrow[\Delta]{} l_1^n \xrightarrow[v_2]{} X,$$

where Δ is a diagonal map and $\|v_1\|, \|\Delta\|, \|v_2\| \leqq 1$. Then every $w\colon l_2^k \to X$ with $\pi_2^*(w^*) \leqq \frac{1}{2}$ belongs to the convex hull $\tilde{B}$ of the set B.

Observe also that for each $j=1, 2, \ldots$ the formula

$$l(u) = \left(\int_\Omega \left\|\sum_{i=1}^j \gamma_i u e_i\right\|^2 d\mu\right)^{1/2}$$

defines an operator ideal norm on the space of linear operators $L(l_2^j, X)$ (cf. [2], [3]). In particular one has

$$l(v_2 \Delta v_1) \leqq l(v_2 \Delta)\|v_1\| \leqq l(v_2 \Delta).$$

Now, if $k \geqq n$, there is an operator $u\colon l_2^k \to X$ such that $\sum_{j=1}^k \|ue_j\|^2 = 1$ and $\alpha_k(X) = l(u)$. Clearly $\pi_2^*(u^*) \leqq 1$ and hence $\frac{1}{2} u \in \tilde{B}$. It follows that

$$\begin{aligned}\frac{1}{2} l(u) &\leqq \sup\{l(v_2 \Delta v_1) \mid v_2 \Delta v_1 \in B\}\\ &\leqq \sup\{l(v_2 \Delta) \mid \Delta\colon l_2^n \to l_1^n,\ v_2\colon l_1^n \to X,\ \|\Delta\| \leqq 1,\ \|v_2\| \leqq 1\}\\ &\leqq \alpha_n(X),\end{aligned}$$

because $\sum_{j=1}^n \|v_2 \Delta e_j\|^2 \leqq 1$. This completes the proof.

Remarks. 1°. Theorem 1 and 2 enable one to simplify some arguments and obtain sharper versions of several recent results. We can only mention Theorems 6.2, 6.3, 6.5 and 6.7 in [1], Section 10 of [2], Theorems 3 and 4 and Corollary 7 in [3]. Let us formulate a sample result of this kind. The definitions of the type p and cotype q constants of a space X, in symbols $K^{(p)}(X)$ and $K_{(q)}(X)$, and of the Banach—Mazur distance $d(X, l_2^{\dim X})$ can be found in each of the papers we refer to.

Corollary. *Let X be an n-dimensional normed space. Then*

$$d(X, l_2^n) \leqq 4K^{(p)}(X) K_{(q)}(X) n^{1/p-1/q}.$$

2° Easy examples show that in general $\pi_2^{(k)}(u) > \pi_2^{(n)}(u)$, if $k > n = \operatorname{rank} u$. On the other hand the method used in [1] Lemma 6.1 yields that $\pi_2^{(k)}(u) \leqq \pi_2^{(n^2)}(u)$ for $k = 1, 2, \ldots$.

3° There is an obvious extension of Theorem 2 to the case of the type 2 and cotype 2 constants of operators of rank n (cf. e.g. [2] Section 10).

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N. Tomczak-Jaegermann
Department of Mathematics
Warsaw University
PKiN, 00 901 Warszawa
Poland