A general high indices theorem with an application to a conjecture by Rényi

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Introduction

Let $\{a_n\}$ be a sequence of numbers and let k be a function in $\mathcal{S}(\mathbf{R})$. Assume that

(1) $$F(x) = \sum_1^\infty a_n \int_{\lambda_n - x}^\infty k(y)\,dy, \quad x \in \mathbf{R},$$

converges uniformly on every set $\{x \in \mathbf{R};\ x < x_0\}$ and defines a bounded function on $\mathbf{R}$. Let $\{\lambda_n\}$ be a given sequence of positive numbers, which are well separated, i.e.

(2) $$\lambda_{n+1} - \lambda_n \geqq c > 0.$$

If

(3) $$\lim_{x\to\infty} F(x) \quad \text{exists,}$$

what can then be said about the convergence of Σa_n?

For $k(y) = \exp(y - \exp(y))$ in (1), the well known high indices theorem by Hardy and Littlewood [3] (see also Ingham [4]) implies that Σa_n is convergent if (3) is true. This result was later generalized, by Levinson [7], to a wide class of kernels.

One restriction in Levinson's theorem is that the Fourier transform $\hat{k}$, of the kernel k, has an analytic continuation into the upper halfplane and is free from zeros there. Therefore, the question whether there is a high indices theorem or not for the series

(4) $$\sum a_n \frac{x^{p_n}}{1 + x^{p_n}}, \quad p_{n+1}/p_n \geqq \delta > 1, \quad x \in [0, 1),$$

cannot be decided by that result, because if (4) is transformed to the form (1), then

$$\hat{k}(u) = \Gamma(1 - iu)\zeta(-iu)(1 - 2^{1+iu}).$$

In 1959 Rényi [11] conjectured that there is a high indices theorem if $p_n=2^n$ in the case (4). This conjecture was proved by Halász [2] in 1967. In 1969 Korevaar [6] observed that there is no high indices theorem for the series (4) for a certain Hadamard sequences of exponents. But what is true for more general sequences $\{p_n\}$ and even for other kernels? We will give a general result in this direction and it will be seen that there is an interplay between the sequence $\{p_n\}$ or $\{\lambda_n\}$ and the zeros of the extension of $\hat{k}$.

Let us first give a sequence $\{p_n\}$, where there is no high indices theorem in the Rényi-case.

Example. Let $x\in(0, 1)$. Then

$$\sum_{-\infty}^{\infty} 2^n \frac{x^{2^n}}{1+x^{2^n}} = \lim_{N\to-\infty} \sum_N^{\infty} 2^n \left(\frac{x^{2^n}}{1-x^{2^n}} - 2\frac{x^{2^{n+1}}}{1-x^{2^{n+1}}}\right) \tag{5}$$

$$= \lim_{N\to-\infty} 2^N \frac{x^{2^N}}{1-x^{2^N}} = -\frac{1}{\log x}.$$

If x is replaced by $x^{\sqrt{2}}$ in (5), it follows that

$$\sum_{-\infty}^{\infty} (\sqrt{2})^{2n+1} \frac{x^{(\sqrt{2})^{2n+1}}}{1+x^{(\sqrt{2})^{2n+1}}} = -\frac{1}{\log x}. \tag{6}$$

A combination of (5) and (6) gives

$$\sum_{-\infty}^{\infty} (-\sqrt{2})^n \frac{x^{(\sqrt{2})^n}}{1+x^{(\sqrt{2})^n}} = 0.$$

Hence,

$$\sum_0^{\infty} (-\sqrt{2})^n \frac{x^{(\sqrt{2})^n}}{1+x^{(\sqrt{2})^n}} = \sum_1^{\infty} (-1)^{n+1} (\sqrt{2})^{-n} \frac{x^{(\sqrt{2})^{-n}}}{1+x^{(\sqrt{2})^{-n}}}. \tag{7}$$

The series to the right in (7) has a limit as x increases monotonically to 1, and therefore this also holds for the series to the left, but $2^{n/2}\not\to 0$, as $n\to\infty$.

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1. Preliminaries

We start to give some definitions. The notation of maximal density is due to Pólya (see [10], p. 559).

Definition. Let $\{\lambda_n\}$ be a sequence of positive numbers satisfying

$$\liminf_{n\to\infty}(\lambda_{n+1}-\lambda_n)=c>0. \tag{1.1}$$

Let

$$N(r)=\#\{\lambda_n\colon \lambda_n\leqq r\}.$$

Then

$$D=\lim_{\xi\uparrow 1}\limsup_{r\to\infty}\frac{N(r)-N(r\xi)}{r-r\xi} \tag{1.2}$$

is called the maximal density of the sequence $\{\lambda_n\}$.

A sequence fulfilling (1.1) and (1.2) is said to be of class $\Lambda(D, c)$.

Remark. $D\leqq 1/c$.

Using the notation k_α for the function defined by $k_\alpha(x)=k(x)\exp(\alpha x)$ $(x\in\mathbf{R})$, we will next define a special class of functions.

Definition. Suppose that k is a function such that $k_\alpha\in L^1(\mathbf{R})$, and that its Fourier transform $\hat{k}_\alpha$ has an extension, which is analytic in the open upper halfplane and continuous in the closed upper halfplane, and that $\hat{k}'_\alpha(u)$ exists, $u\in R$. Let $\beta>0$ and ξ be real, and suppose further that

$$\max_{|\xi|\leqq\beta}\left|\frac{\hat{k}_\alpha(u+\xi)}{\hat{k}_\alpha(u)}\right|\leqq\exp(\theta(u)) \tag{1.3}$$

$$\max_{|\xi|\leqq\beta}\left|\frac{\hat{k}_\alpha(w+\xi)}{\hat{k}_\alpha(w)}\right|\leqq C\exp(C|w|),\ \operatorname{Im} w\geqq 0 \quad (C \text{ is a constant}) \tag{1.4}$$

$$\left|\frac{\hat{k}'_\alpha(u)}{\hat{k}_\alpha(u)}\right|\leqq\exp(\theta(u)), \tag{1.5}$$

where $\theta(u)$ is a positive even function of u, monotonely increasing for $u>0$ and

$$\int_1^\infty\frac{\theta(u)}{u^2}\,du<\infty.$$

Then we say that $k\in\mathscr{L}_\alpha$.

Furthermore, we use the customary notations in distribution theory. (Cf. Rudin [12] or Schwartz [13]). Also, $|M|$ is used for the Lebesgue measure of the measurable set M and unspecified signs of integration will always denote integration over the whole real line. All sums are taken from 1 to ∞.

2. The main result

We start by stating a general high indices theorem. It can be formulated as follows:

Theorem 1. *Let* $\{\lambda_n\}$ *be a sequence of class* $\Lambda(D, c)$. *Suppose that there exists an* $\alpha \geqq 0$ *such that*

$$\exp(-\alpha x) \cdot \sum a_n \int_{\lambda_n - x}^{\infty} k(y)\, dy \tag{2.1}$$

converges uniformly on every set $\{x\colon x < x_0\}$, $x_0 < \infty$, *and defines a bounded function on* $\mathbf{R}$. *Suppose further that*

$$k \in \mathcal{L}_\alpha \cap \mathcal{S}, \tag{2.2}$$

$$\hat{k}(\cdot + iv) \in L^2(\mathbf{R}), \quad 0 < v < v_0, \quad \textit{for some} \quad v_0 > \alpha, \tag{2.3}$$

(2.4) *on every line* $\operatorname{Im} w = \sigma$, $0 \leqq \sigma < \alpha$, *there exists a closed interval* I *such that* $\hat{k}(u + i\sigma) \neq 0$ *for* $u \in I$, *where* $|I| = 2\pi D$ *if* $0 < \sigma < \alpha$ *and* $|I| = 2\pi/c$ *if* $\sigma = 0$,

$$\hat{k}(0) \neq 0. \tag{2.5}$$

Then

$$\sum a_n \int_{\lambda_n - x}^{\infty} k(y)\, dy = O(1), \quad x \in \mathbf{R}, \tag{2.6}$$

implies that

$$\sum\nolimits_{n \leqq x} a_n = O(1) \tag{2.7}$$

If, moreover,

$$\lim_{x \to \infty} \sum a_n \int_{\lambda_n - x}^{\infty} k(y)\, dy = 0 \tag{2.8}$$

then

$$\sum a_n = 0. \tag{2.9}$$

This theorem will follow by a combination of the Theorems 2 and 4.

Remark. If $\lambda_n = nc$, $c > 0$, $n = 1, 2, \ldots$, then instead of (2.4) it is enough that there is no line parallel to the real axis, where the extension of the Fourier transform $\hat{k}$ has zeros separated exactly by the distance $2\pi m/c$, $m = 1, 2, \ldots$. Furthermore, if there exists a line, where $\hat{k}$ has zeros separated by the distance $2\pi m/c$, $m = 1, 2, \ldots$, then the coefficients $\{a_n\}$ can be chosen in such a way that (2.8) is true but (2.9) is false.

3. Two Tauberian theorems

In this section we state and prove a somewhat different and, in a way, weaker result than Theorem 1. This is:

Theorem 2. *Suppose that the coefficients* $\{a_n\}$ *satisfy*

$$a_n = O(\exp(A\lambda_n)),$$

where A is a constant and $\{\lambda_n\}$ *is of class* $\Lambda(D, c)$. *Let the kernel* $k \in \mathcal{S}$ *and suppose that its Fourier transform* $\hat{k}$, *with* $\hat{k}(0) \neq 0$, *can be analytically continued into the strip* $0 < \operatorname{Im} w < v_0$, $v_0 > A$, *and that*

$$\hat{k}(\cdot + iv) \in L^2(\mathbf{R}), \quad 0 < v < v_0.$$

Moreover, suppose that on every line $\operatorname{Im} w = \sigma$, $0 \leqq \sigma \leqq A$, *there exists a closed interval I such that*

$$\hat{k}(u + i\sigma) \neq 0 \quad \text{for} \quad u \in I,$$

where $|I| = 2\pi D$ *if* $0 < \sigma \leqq A$ *and* $|I| = 2\pi/c$ *if* $\sigma = 0$. *Let*

$$F(x) = \sum a_n \int_{\lambda_n - x}^{\infty} k(y)\, dy, \quad x \in \mathbf{R}, \tag{3.1}$$

and suppose that

$$\lim_{x \to \infty} F(x) = 0. \tag{3.2}$$

Then

$$\sum a_n = 0. \tag{3.3}$$

Remark. Like in Theorem 1, boundedness of F implies boundedness of the sum $\sum_{n \leqq x} a_n$.

In order to prove Theorem 2, we need several auxiliary results. We start with an interpolation lemma, which in a weaker form can be found in Levinson [7]. We have:

Lemma 1. *Let* $\{\lambda_n\}$ *be a sequence of real numbers satisfying*

$$\lambda_{n+1} - \lambda_n \geqq c > 0, \quad n \in Z.$$

Then, for each integer n and each $\varepsilon > 0$ *there exists a function* $H_n \in \mathcal{S}$ *such that*

$$H_n(\lambda_n) = 1; \quad H_n(\lambda_k) = 0, \quad k \neq n, \tag{3.4}$$

and

$$\operatorname{supp}(\hat{H}_n) = \left[-\frac{\pi}{c} - \varepsilon, \frac{\pi}{c} + \varepsilon\right]. \tag{3.5}$$

If

$$\hat{G}_n(u) = \hat{H}_n(u)\, e^{iu\lambda_n}, \tag{3.6}$$

then

$$\|\hat{G}_n\|_N = \max_{0\le m\le N} \|D^m \hat{G}_n\|_\infty \leqq C_N < \infty \quad (\textit{independent of } n). \tag{3.7}$$

Proof. Keep n fixed and let $k\in\mathbf{Z}-\{0\}$. Define a sequence $\{x_k\}$ by

$$x_k = \begin{cases} kc, & \text{if} \quad |kc-\lambda_m+\lambda_n| > c/2 \quad \text{for all } m \\ kc, & \text{if} \quad kc-\lambda_m+\lambda_n = c/2 \quad \text{for some } m \\ \lambda_m-\lambda_n, & \text{if} \quad -c/2 \leqq kc-\lambda_m+\lambda_n < c/2 \quad \text{for some } m. \end{cases}$$

This determines the sequence $\{x_k\}$ uniquely. Moreover,

$$x_{k+1}-x_k \geqq c/2 \tag{3.8}$$

$$|x_k-kc| \leqq c/2. \tag{3.9}$$

Consider the function

$$T_n(z) = \prod_1^\infty (1-z/x_k)(1-z/x_{-k}) \tag{3.10}$$

and estimate

$$\left|\frac{T_n(z)}{\sin \pi z/c}\right| = \frac{c}{\pi|z|} \prod_1^\infty \left|\frac{1-z/x_k}{1-z/kc}\right| \left|\frac{1-z/x_{-k}}{1+z/kc}\right|. \tag{3.11}$$

A term in the infinite product (3.11) can be estimated by

$$\left|\frac{1-z/x_k}{1-z/kc}\right| \leqq 1+\frac{|z|\,|x_k-kc|}{|x_k|\,|z-kc|} \leqq 1+\frac{|z|}{|k|\,|z-kc|} \leqq \exp\left\{\frac{|z|}{|k|\,|z-kc|}\right\}, \tag{3.12}$$

where (3.8) and (3.9) have been used.

Let $\operatorname{Re} z \geqq 0$ and $Nc-c/2 \leqq |z| < Nc+c/2$, $N \geqq 5$. Then we find that

$$\begin{aligned} \sum_{k\ne 0,\pm N} \frac{1}{|k|\,|z-kc|} &\leqq \frac{1}{c}\sum_{-\infty}^{-N-1}\frac{1}{k^2}+\frac{1}{|z|}\sum_{-N+1}^{-1}\frac{1}{|k|} \\ &+\frac{3}{|z|}\sum_1^{[N/2]}\frac{1}{k}+\frac{2}{Nc}\sum_{[N/2]+1}^{N-1}\frac{1}{N-k-1/2}+\frac{1}{Nc}\sum_{N+1}^{2N}\frac{1}{k-N-1/2} \\ &+\frac{1}{c}\sum_{2N+1}^{\infty}\frac{1}{(k-N-1/2)^2} \leqq \frac{21}{|z|}+\frac{10\log|z|}{|z|}-\frac{10\log c}{|z|}. \end{aligned} \tag{3.13}$$

Inserting (3.12) and (3.13) in (3.11), we get

$$\left|\frac{T_n(z)}{\sin \pi z/c}\right| \leqq A_c|z|^9 \left|\frac{1-z/x_N}{1-z/Nc}\right| \left|\frac{1-z/x_{-N}}{1+z/Nc}\right|. \tag{3.14}$$

The inequality (3.14) is obviously true in the whole annulus $Nc-c/2 \leqq |z| \leqq Nc+c/2$, and therefore the inequality

$$|T_n(z)| \leqq A(1+|z|^9)\exp\left(\frac{\pi}{c}|\operatorname{Im} z|\right)$$

holds for all $z \in \mathbf{C}$. By a generalization of a well known theorem by Paley and Wiener (Rudin [12], theorem 7.23), we have

$$\operatorname{supp}(\hat{T}_n) \subset [-\pi/c, \pi/c].$$

Let φ be a function in $\mathcal{S}$ with $\operatorname{supp}\hat{\varphi} \subset [-\varepsilon, \varepsilon]$ and $\varphi(0)=1$. Putting $G_n = \varphi \cdot T_n$, we get, for each n, a function in $\mathcal{S}$ with Fourier transform

$$\hat{G}_n = \hat{T}_n * \hat{\varphi} \quad \text{and} \quad \operatorname{supp}(\hat{G}_n) \subset [-\pi/c-\varepsilon, \pi/c+\varepsilon].$$

These functions satisfy (3.7), and if H_n is defined by

$$H_n(z) = G_n(z-\lambda_n), \quad z \in \mathbf{C},$$

then the condition (3.4) follows from (3.10) and the condition (3.5) is easily seen to be valid.

The preceding interpolation result will be used to get global information from local behavior of a Dirichlet series on the axis of convergence. More precisely:

Lemma 2. *Let f be defined by the Dirichlet series*

$$f(w) = \sum \alpha_n e^{iw\lambda_n}, \ \operatorname{Im} w > 0.$$

where $\{\lambda_n\}$ is of class $A(D, c)$. Suppose that

$$f(\cdot + iv) \to \tilde{f} \quad in \quad \mathcal{D}'(I) \quad as \quad v \downarrow 0, \tag{3.15}$$

for some open interval I with $|I| > 2\pi/c$. Then there exist a finite constant N and a discrete measure $\mu \in \mathcal{S}'$ of the form

$$\mu = 2\pi \cdot \sum \alpha_n \tau_{-\lambda_n}\delta, \quad \alpha_n = O(\lambda_n^N),$$

such that

$$\hat{\mu} = \tilde{f} \quad in \ \mathcal{S}'.$$

(δ is the Dirac measure).

Proof. There is no loss in generality to assume that the interval I is symmetric around the origin and that $\lambda_{n+1} - \lambda_n \geqq c$. Then we can choose an $\varepsilon > 0$ such that the functions $\{H_k\}$, from Lemma 1, have $\operatorname{supp}(\hat{H}_k) \subset [-\pi/c-\varepsilon, \pi/c+\varepsilon] \subset I$.

Since $\tilde{f}$ is a bounded functional on $\mathcal{D}(I)$, we have

$$|\tilde{f}(\hat{H}_k)| \leqq C \|\hat{H}_k\|_N, \quad k = 1, 2, \ldots \tag{3.16}$$

for some finite N and some constant C.

Applying (3.15) to $\hat{H}_k$, we get

$$\tilde{f}(\hat{H}_k) = \lim_{v \downarrow 0} f(\cdot + iv)(\hat{H}_k) = \lim_{v \downarrow 0} \int f(u+iv)\hat{H}_k(u)\, du \tag{3.17}$$

$$= \lim_{v \downarrow 0} \sum \alpha_n e^{-v\lambda_n} \int \hat{H}_k(u) e^{iu\lambda_k} du = \lim_{v \downarrow 0} \sum a_n e^{-v\lambda_n} 2\pi H_k(\lambda_n) = 2\pi\alpha_k.$$

If (3.16) and (3.17) are combined with (3.6) and (3.7), we see that

$$|2\pi\alpha_k| \leqq C\|\hat{H}_k\|_N = O(\lambda_k^N)$$

and the lemma is proved.

Next we prove a distributional variant of a theorem by Paley and Wiener. It is

Lemma 3. *Let* $\theta\in\mathcal{S}'$ *and suppose that* $\operatorname{supp}(\theta)\subset(-\infty, 0)$. *Then the Fourier transform* $\hat{\theta}$ *has an analytic extension into the upper halfplane, i.e. there exists a function* ψ *analytic in* $u+iv, v>0$, *such that* $\lim_{v\downarrow 0}\psi(\cdot+iv)=\hat{\theta}$ in $\mathcal{S}'$.

Proof. Let φ be a C^∞-function such that $\varphi=1$ on $\operatorname{supp}\theta$ and $\operatorname{supp}\varphi\subset(-\infty, 0]$. If $v>0$, then

$$(e^{vx}\theta)^\wedge = (e^{vx}\varphi\theta)^\wedge = \hat{\theta}*\hat{\varphi}_v,$$

since $\varphi_v\in\mathcal{S}$, and

$$(e^{vx}\theta)^\wedge(u) = \hat{\theta}*\hat{\varphi}_v(u) = \hat{\theta}\big(\tau_u(\hat{\varphi}_v)^\vee\big) = \hat{\theta}\big(\hat{\varphi}(u+iv-\cdot)\big) = \hat{\theta}(u+iv).$$

To prove that $\hat{\theta}(u+iv)$ is analytic in $w=u+iv$, $v>0$, we use Morera's theorem. Let Γ be a regular closed curve in the upper halfplane. The map

$$\Gamma\ni w \to \hat{\varphi}(w-\cdot)\in\mathcal{S}$$

is easily seen to be continuous and therefore

$$H = \int_\Gamma \hat{\varphi}(w-\cdot)\,dw \tag{3.18}$$

is a well defined $\mathcal{S}$-valued integral (see for instance Rudin [12], chapter 3.) Hence, since evaluation is a tempered distribution, it follows that

$$H(t) = \int_\Gamma \hat{\varphi}(w-t)\,dw.$$

But $\hat{\varphi}(w)$ is analytic for $\operatorname{Im} w>0$ and we get $H=0$. Therefore, if $\hat{\theta}$ is applied to H, then by the representation (3.18)

$$0 = \hat{\theta}(H) = \int_\Gamma \hat{\theta}(\hat{\varphi}(w-\cdot))\,dw = \int \hat{\theta}(w)\,dw,$$

i.e. $\hat{\theta}$ is analytic in the upper halfplane.

It is readily seen that

$$\hat{\theta}(\cdot+iv) \to \hat{\theta} \quad \text{in } \mathcal{S}' \text{ as } \quad v\downarrow 0,$$

and the proof is completed.

We also need a theorem of Pólya (see Pólya [9]; Levinson [8], theorem XXIX). This result is stated in the following lemma.

Lemma 4. *Let*

$$f(w) = \sum \alpha_n e^{iw\lambda_n}$$

be a Dirichlet series with the axis of convergence $\operatorname{Im} w=\sigma$. *Suppose* $\{\lambda_n\}$ *is of class* $\Lambda(D, c)$. *Then every closed interval of length* $2\pi D$ *on the line* $\operatorname{Im} w=\sigma$ *contains at least one singular point of the function* f.

Next we treat a Tauberian result for a very special class of tempered distributions. We formulate the result as:

Theorem 3. *Let* $\{\lambda_n\}$ *be of class* $\Lambda(D, c)$ *and suppose that the coefficients* $\{a_n\}$ *satisfy*

$$a_n = O(\lambda_n^N) \quad \textit{for some finite constant } N.$$

Suppose further that $k\in\mathcal{S}$ *and that there exists a closed interval* I *of length* $2\pi/c$ *such that*

$$\hat{k}|_I \neq 0 \quad \textit{and} \quad \hat{k}(0) \neq 0.$$

If

(3.19) $$\sum a_n \int_{\lambda_n - x}^{\infty} k(y)\, dy = O(1), \quad x\in\mathbf{R},$$

then

(3.20) $$\sum\nolimits_{n\leqq x} a_n = O(1)$$

and if

(3.21) $$\sum a_n \int_{\lambda_n - x}^{\infty} k(y)\, dy = o(1), \quad x\to\infty,$$

then

(3.22) $$\sum\nolimits_{n\leqq x} a_n = o(1), \quad x\to\infty.$$

Proof. Let

(3.23) $$F(x) = \sum a_n \int_{\lambda_n - x}^{\infty} k(y)\, dy.$$

Then we have

$$F'(x) = \mu * k(-x),$$

where $\mu=\Sigma a_n \tau_{-\lambda_n}\delta$ is in $\mathcal{S}'$. Without loss of generality we can assume that $\lambda_{n+1}-\lambda_n\geqq c$ and that $\hat{k}(u)\neq 0$ for $u\in[-\beta,\beta]$, $\beta>\pi/c$. Thus there exists a function $\psi\in\mathcal{S}$ satisfying

$$\psi(u) = \frac{1}{\hat{k}(u)}, \quad u\in[-\beta,\beta].$$

Let $\{H_k\}$ be the functions given in Lemma 1 with $\operatorname{supp}\hat{H}_k\subset(-\beta,\beta)$. We get

$$((F')^{\vee} * \psi * \hat{H}_k)^{\wedge} = \hat{\psi}\cdot\hat{H}_k\cdot\hat{k}\,\hat{\mu} = (\mu * H_k)^{\wedge} \quad \text{in } \mathcal{S}',$$

and hence,

$$\mu * H_k = (F')^{\vee} * \psi * H_k = -\check{F} * \psi' * H_k.$$

It follows from (3.19) and (3.23) that F is bounded, thus

(3.24) $$|a_k| = |\mu * H_k(0)| = \left|\int F(y)(\psi' * H_k)(y)\, dy\right| \leqq \|F\|_{L^\infty}\|\psi'\|_{L^1}\|H_k\|_{L^1}$$

If (3.21) holds, we get a better estimate than (3.24) by

$$|a_k| \leqq \big(\sup_{y>\lambda_k/2} |F(y)|\big)\int_{y>\lambda_k/2} |\psi' * H_k(y)|\,dy + \big(\sup_{y<\lambda_k/2} |F(y)|\big)\int_{y<\lambda_k/2} |\psi' * H_k(y)|\,dy.$$

The second integral to the right can be estimated by

$$\int_{y<\lambda_k/2} |\psi' * H_k(y)|\,dy \leqq \int |\psi'(t)|\,dt \int_{y<\lambda_k/2} |H_k(y-t)|\,dy$$

$$= \int |\psi'(t)|\,dt \int_{y<-\lambda_k/2-t} |G_k(y)|\,dy \leqq \int_{t>-\lambda_k/4} |\psi'(t)|\,dt \int_{y<-\lambda_k/4} |G_k(y)|\,dy +$$

$$+\int_{t<-\lambda_k/4} |\psi'(t)|\,dt \int |G_k(y)|\,dy \leqq \|\psi'\|_{L^1} \int_{y<-\lambda_k/4} |G_k(y)|\,dy + \|G_k\|_{L^1} \int_{t<-\lambda_k/4} |\psi'(t)|\,dt,$$

where G_k is the function in Lemma 1.

Hence we see that

$$a_k = o(1), \quad k \to \infty. \tag{3.25}$$

In order to get (3.20) and (3.22) respectively, let $x_N = (\lambda_N + \lambda_{N+1})/2$ and suppose that $\hat{k}(0) = 1$. Then

$$\sum_1^N a_n - F(x_N) = \sum_1^N a_n \int_{-\infty}^{\lambda_n - x_N} k(y)\,dy - \sum_{N+1}^{\infty} a_n \int_{\lambda_n - x_N}^{\infty} k(y)\,dy. \tag{3.26}$$

Now (3.20) follows readily from (3.24) and (3.26). The conclusion (3.22) follows from (3.25) and (3.26), because

$$\Big|\sum_1^N a_n - F(x_N)\Big| \leqq \sum_1^{[N/2]} |a_n| \int_{-\infty}^{\lambda_n - x_N} |k(y)|\,dy$$

$$+ \sum_{[N/2]+1}^{N} |a_n| \int_{-\infty}^{\lambda_n - x_N} |k(y)|\,dy + \sum_{N+1}^{\infty} |a_n| \int_{\lambda_n - x_N}^{\infty} |k(y)|\,dy$$

$$\leqq C_1 \max |a_n| \sum_1^{[N/2]} \frac{1}{1+(\lambda_n - x_N)^2} + C_1 \max_{n \geqq [N/2]} |a_n| \sum_{[N/2]+1}^{N} \frac{1}{1-(\lambda_n - x_N)^2}$$

$$+ C_1 \max_{n \geqq N} |a_n| \sum_{N+1}^{\infty} \frac{1}{1+(\lambda_n - x_N)^2} \leqq C_1 \max |a_n| \sum_{[N/2]}^{N} \frac{1}{1+c^2 n^2} + C_2 \max_{n \geqq [N/2]} |a_n|,$$

where C_1 and C_2 are constants.

Proof of Theorem 2. Let G be the function defined by

$$G(x) = \begin{cases} 0, & x > 0 \\ 1, & x < 0. \end{cases}$$

Let $A < v < v_0$. We can rewrite (3.1) as

$$\begin{aligned} F(x) &= \sum a_n \int k(y) G(\lambda_n - x - y)\,dy \\ &= e^{vx} \sum a_n e^{-v\lambda_n} \int k_v(y) G_v(\lambda_n - x - y)\,dy \\ &= e^{vx} \sum a_n e^{-v\lambda_n} \int \hat{k}_v(u) \hat{G}_v(u) e^{iu(\lambda_n - x)}\,du, \end{aligned} \tag{3.27}$$

where the last equality is Parseval's formula.

By Fubini's theorem, the order of summation and integration can be changed in (3.27) to get

$$F(x) = \int \hat{k}(u+iv)\hat{G}(u+iv)\left(\sum a_n e^{i(u+iv)\lambda_n}\right)e^{-i(u+iv)x}\,du. \tag{3.28}$$

If we define a function h by the Dirichlet series

$$h(w) = \sum a_n e^{iw\lambda_n},\ \operatorname{Im} w > A,$$

then

$$\hat{g}(\cdot+iv) = \hat{k}(\cdot+iv)h(\cdot+iv) \in L^2(\mathbf{R}). \tag{3.29}$$

Thus, there exists a function g such that

$$\left(g(y)e^{vy}\right)^{\wedge}(u) = \hat{g}(u+iv). \tag{3.30}$$

Using (3.30) in (3.28), it gives

$$F(x) = \int \left(g(y)e^{vy}\right)^{\wedge}(u)\hat{G}_v(u)e^{-i(u+iv)x}\,du = \int_{-x}^{\infty} g(y)\,dy.$$

Since $F\in\mathcal{S}'$, we have that $g\in\mathcal{S}'$ and g has a Fourier transform.

Let φ_1 be a C^∞-function with $\operatorname{supp}\varphi_1\subset(-\infty, 0)$ and $\varphi_1(x)=1$ if $x<-1$. Take $\varphi_2=1-\varphi_1$. From Lemma 3 we get

$$(\varphi_1 g)^{\wedge}(w) \text{ is analytic for } \operatorname{Im} w > 0. \tag{3.31}$$

Moreover, by a combination of (3.29) and (3.30), we see that

$$\int |(g(t)e^{vt}|^2\,dt < \infty \quad \text{for} \quad A < v < v_0.$$

Hence,

$$\int |(\varphi_2 g)(t)e^{vt}|^2\,dt < \infty \quad \text{for} \quad v < v_0,$$

and this implies that

$$(\varphi_2 g)^{\wedge}(w) \text{ is analytic for } \operatorname{Im} w < v_0. \tag{3.32}$$

Putting (3.31) and (3.32) together, we get

$$\hat{g}(w) = ((\varphi_1+\varphi_2)g)^{\wedge}(w) = (\varphi_1 g)^{\wedge}(w)+(\varphi_2 g)^{\wedge}(w)$$

is analytic in the strip $0<\operatorname{Im} w<v_0$, i.e.

$$\hat{g}(w) = \hat{k}(w)h(w) \text{ is analytic for } 0 < \operatorname{Im} w < v_0. \tag{3.33}$$

Suppose that the axis of convergence for the Dirichlet series $h(w)$ is $\operatorname{Im} w=\sigma>0$. Then, on the line $\operatorname{Im} w=\sigma$ the function $h(w)$ has at least one singular point in each closed interval of length $2\pi D$. The formula (3.33) shows that $h(w)$ has a meromorphic extension into the halfplane $\operatorname{Im} w>0$, and therefore these singularities must be poles. Moreover, the product $\hat{k}(w)h(w)$ is analytic for $0<\operatorname{Im} w<v_0$, but this contradicts the hypothese on $\hat{k}(w)$. Thus, the series defining h converges not only for $\operatorname{Im} w>A$, but

$$h(w) = \sum a_n e^{iw\lambda_n},\ \operatorname{Im} w > 0. \tag{3.34}$$

The assumptions on the Fourier transform $\hat{k}$ imply that there exist a closed interval I, with $|I|>2\pi/c$, and an $\varepsilon>0$ such that

(3.35) $$\hat{k}(u+iv) \neq 0, \quad \text{when} \quad u\in I \quad \text{and} \quad 0 \leqq v < \varepsilon$$

and

(3.36) $$D^n\hat{k}(\cdot + iv) \to D^n\hat{k} \quad \text{in uniform norm on } I, \quad v\downarrow 0, \quad n = 0, 1, \ldots.$$

Since

(3.37) $$\hat{g}(\cdot + iv) \to \hat{g} \quad \text{in } \mathscr{S}', \quad v\downarrow 0,$$

the combination of (3.35), (3.36) and (3.37) (see Rudin [11], Theorem 6.18) gives

(3.38) $$h(\cdot + iv) \to \tilde{h} \quad \text{in } \mathscr{D}'(\operatorname{Int} I), \text{ as } \quad v\downarrow 0.$$

From (3.34) and (3.38) it follows, by Lemma 2, that there exists a finite constant N such that

(3.39) $$a_n = O(\lambda_n^N).$$

Recalling (3.1) and (3.2), we see that

$$\sum a_n \int_{\lambda_n - x}^{\infty} k(y)\,dy = o(1), \quad x \to \infty,$$

where the coefficients $\{a_n\}$ satisfy (3.39). Now we can use Theorem 3 to get the conclusion (3.3) and Theorem 2 is proved.

4. Estimation of the coefficients

The second step in proving Theorem 1 will be accomplished by the next theorem.

Theorem 4. *Let the sequence $\{\lambda_n\}$ be of class $A(D, c)$ and let $k\in\mathscr{L}_\alpha$. Suppose that there exists an $\alpha\geqq 0$ such that*

(4.1) $$\exp(-\alpha x)\cdot \sum a_n \int_{\lambda_n - x}^{\infty} k(y)\,dy$$

converges uniformly on every set $\{x\colon x<x_0\}$, $x_0<\infty$, and defines a bounded function on $\mathbf{R}$. Then

$$a_n = O(\exp(\alpha\lambda_n)).$$

In proving this theorem we first state a lemma, which is almost identical to a result of Levinson. For a proof see Johansson [5] or Levinson [7].

Lemma 5. *There exists a function ψ, analytic in the upper halfplane, with*

(4.2) $$|\psi(u+iv)| \leqq C\exp(Cv), \quad (C \text{ is a constant})$$

(4.3) $$|\psi(u)| \quad \text{is even}$$

and

(4.4) $$|\psi(u)| \leqq C_N(1+|u|^N)^{-1}\exp\{-\theta(u)\}, \quad N=1,2,\dots,$$

where $\{C_N\}$ *are finite constants and* $\theta(u)$ *is a positive function of* u, *monotonely increasing for* $u>0$, *and with*

$$\int_1^\infty \frac{\theta(u)}{u^2}\,du < \infty.$$

Moreover,

(4.5) $$\hat{\psi}(0)=1.$$

The next lemma is crucial to the proof of Theorem 4. The proof of this result follows by a slight modification of a method used by Levinson [7].

Lemma 6. *Suppose that the hypotheses in Theorem 4 are satisfied. Then there exists a sequence* $\{B_n\}_1^\infty$ *of functions, defined on* $\mathbf{R}\times[2,\infty)$, *such that*

(4.6) $$\int |B_n(x,A)|\,dx \leqq C_1$$

(4.7) $$B_n(x,A)=0 \quad \text{for} \quad x \geqq C_2 A$$

(4.8) $$\int B_n(x,A)e^{\alpha(\lambda_m-x)}\int_{\lambda_m-x}^\infty k(y)\,dy\,dx = 0, \quad m\neq n,$$

and

(4.9) $$\lim_{A\to\infty}\int B_n(x,A)e^{\alpha(\lambda_n-x)}\int_{\lambda_n-x}^\infty k(y)\,dy\,dx = \hat{k}_\alpha(0),$$

where C_1 *and* C_2 *are constants.*

Proof. Define

(4.10) $$\hat{B}_n(u,A) = -\frac{i(u+i\alpha)}{\hat{k}_\alpha(u)}(\hat{k}_\alpha\cdot\psi^A)*\hat{H}_n(u), \quad n=1,2,\dots,$$

where the functions $\{\hat{H}_n\}$ are given in Lemma 1 with $\operatorname{supp}\hat{H}_n\subset(-\beta,\beta)$, $\beta>\pi/c$. The function ψ^A is defined by $\psi^A(u)=A\psi(Au)$, where ψ is the function in Lemma 5 except that we instead of $\theta(u)$ here use $2\theta(u)$.

By (1.3) and (3.7), we get

$$|\hat{B}_n(u,A)| \leqq C_3|u|e^{\theta(u)}\int_{u-\beta}^{u+\beta}|\psi^A(t)|\,dt$$
$$= C_3|u|e^{\theta(u)}\int_{A(u-\beta)}^{A(u+\beta)}|\psi(t)|\,dt.$$

If $|u|>2\beta$, then it follows from (4.3) and (4.4) that

(4.11) $$|\hat{B}_n(u,A)| \leqq C_3|u|e^{\theta(u)}\int_{|u|}^\infty|\psi(t)|\,dt \leqq \frac{C_4}{1+u^2}.$$

For $|u|\leqq 2\beta$ the inequality is obvious. In a similar manner, by using (1.3), (1.5), (3.7), (4.3) and (4.4), we can make an estimate of

$$\frac{d}{du}\{e^{iu\lambda_n}\hat{B}_n(u, A)\} = -\frac{i(u+i\alpha)}{\hat{k}_\alpha(u)}\int \hat{k}_\alpha(t)\psi^A(t)e^{it\lambda_n}\hat{G}_n'(u-t)\,dt$$

$$-\frac{i}{\hat{k}_\alpha(u)}\left(1-(u+i\alpha)\frac{\hat{k}_\alpha'(u)}{\hat{k}_\alpha(u)}\right)\int \hat{k}_\alpha(t)\psi^A(t)e^{it\lambda_n}\hat{G}_n(u-t)\,dt,$$

where $\hat{G}_n(t)=\hat{H}_n(t)\exp(it\lambda_n)$, to get

$$\left|\frac{d}{du}\{e^{iu\lambda_n}\hat{B}_n(u, A)\}\right| \leqq \frac{C_5}{1+u^2}.$$

Thus, by Carlson's inequality,

$$\int |B_n(x, A)|\,dx \leqq C_6.$$

We have that

$$\hat{B}_n(w, A) = -\frac{i(w+i\alpha)}{\hat{k}_\alpha(w)}\int \hat{k}_\alpha(w-t)\psi^A(w-t)\hat{H}_n(t)\,dt$$

is analytic for $\operatorname{Im} w>0$ and continuous for $\operatorname{Im} w\geqq 0$. Using (1.4), (3.7) and (4.2), we find that

$$|\hat{B}_n(w, A)| \leqq C_7|w|Ae^{C_8Av}e^{C_9|w|} \leqq C_7e^{C_{10}A|w|}. \tag{4.12}$$

By a Phragmén—Lindelöf argument, (4.11) and (4.12) imply that

$$|\hat{B}_n(w, A)e^{i2C_{10}Aw}| \leqq \frac{C_{11}}{1+|w|^2},\quad \operatorname{Im} w \geqq 0.$$

Hence, applying Paley—Wiener's theorem, we get

$$B_n(x, A) = 0 \quad \text{for} \quad x \geqq 2C_{10}A.$$

Parseval's formula gives

$$\frac{1}{2\pi}\int \hat{B}_n(u, A)\hat{k}_\alpha(u)e^{iut}\,du = \int B_n(x, A)k_\alpha(t-x)\,dx = \int B_n(x, A)e^{\alpha(t-x)}k(t-x)\,dx$$

and this can be rewritten as

$$\frac{1}{2\pi}\int \hat{B}_n(u, A)\hat{k}_\alpha(u)e^{i(u+i\alpha)t}\,du = \int B_n(x, A)e^{-\alpha x}k(t-x)\,dx. \tag{4.13}$$

Integrating (4.13) with respect to t, we get

$$\frac{1}{2\pi}\int \hat{B}_n(u, A)\hat{k}_\alpha(u)\frac{e^{i(u+i\alpha)T}-e^{i(u+i\alpha)s}}{i(u+i\alpha)}\,du \tag{4.14}$$

$$= \int B_n(x, A)e^{-\alpha x}\int_s^T k(t-x)\,dt\,dx.$$

We have

$$\lim_{T\to\infty}\int \frac{\hat{B}_n(u, A)\hat{k}_\alpha(u)}{i(u+i\alpha)}\, e^{i(u+i\alpha)T}\, du = 0,$$

which is obvious if $\alpha>0$ and follows from Riemann—Lebesgue's lemma for $\alpha=0$. Thus, if $T\to\infty$ in (4.14),

$$\int B_n(x, A)e^{\alpha(s-x)}\int_{s-x}^{\infty} k(y)\,dy\,dx = -\frac{1}{2\pi}\int \frac{\hat{B}_n(u, A)\hat{k}_\alpha(u)}{i(u+i\alpha)}\, e^{ius}\, du.$$

Together with (4.10) this implies that

$$\int B_n(x, A)e^{\alpha(s-x)}\int_{s-x}^{\infty} k(y)\,dy\,dx = H_n(s)(\hat{k}_\alpha\cdot\psi^A)\hat{}\,(-s).$$

Hence,

$$\int B_n(x, A)e^{\alpha(\lambda_m-x)}\int_{\lambda_m-x}^{\infty} k(y)\,dy\,dx = 0 \quad \text{if} \quad m\neq n$$

and

$$\lim_{A\to\infty}\int B_n(x, A)e^{\alpha(\lambda_n-x)}\int_{\lambda_n-x}^{\infty} k(y)\,dy\,dx = \lim_{A\to\infty}(\hat{k}_\alpha\cdot\psi^A)\hat{}\,(-\lambda_n)$$
$$= \lim_{A\to\infty}\int k_\alpha(t)\psi((t-\lambda_n)/A)\,dt = \hat{k}_\alpha(0),$$

where the last equality is given by the theorem of dominated convergence and (4.5).

Proof of Theorem 4. From (4.1) and (4.6) it follows that

$$\int B_n(x, A)e^{-\alpha x}\sum a_m\int_{\lambda_m-x}^{\infty} k(y)\,dy\,dx = O(1). \tag{4.15}$$

Because of (4.7) and since the convergence is uniform, the order of integration and summation can be changed in (4.15).

Thus,

$$\sum a_m e^{-\alpha\lambda_m}\int B_n(x, A)e^{\alpha(\lambda_m-x)}\int_{\lambda_m-x}^{\infty} k(y)\,dy\,dx = O(1). \tag{4.16}$$

By the properties (4.8) and (4.9), we find that (4.16) implies that

$$a_n e^{-\alpha\lambda_n}\hat{k}_\alpha(0) = O(1)$$

and this proves the theorem.

5. Best possible results and some applications

It can be seen by an example that the hypothesis (1.3) is essential in Theorem 1 (see Levinson [8] or Johansson [5]).

The question whether the separation (2) of the sequence $\{\lambda_n\}$ is necessary, will be answered by the next theorem.

Theorem 5. *Let $k\in L^1(R)$ and let the sequence $\{\lambda_n\}$ satisfy*

$$0<\lambda_1<\lambda_2<\ldots<\lambda_n\to\infty,\quad n\to\infty$$

and

$$\liminf_{n\to\infty}(\lambda_{n+1}-\lambda_n)=0.$$

Then there exist coefficients $\{a_n\}$ such that

$$F(x)=\sum a_n\int_{\lambda_n-x}^{\infty}k(y)\,dy$$

converges uniformly on every set $\{x: x<x_0\}$, $x_0<\infty$, and

$$\lim_{x\to\infty}F(x)=0$$

but

$$a_n\nrightarrow 0,\quad n\to\infty,$$

Proof. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two disjoint subsequences of $\{\lambda_n\}$ satisfying

$$\alpha_1<\beta_1<\alpha_2<\beta_2<\ldots$$

and

$$\sum(\beta_n-\alpha_n)<\infty.$$

Define

$$F(x)=\sum a_n\int_{\lambda_n-x}^{\infty}k(y)\,dy,$$

where

$$a_k=\begin{cases}1 & \text{if}\quad \lambda_k\in\{\alpha_n\}\\ -1 & \text{if}\quad \lambda_k\in\{\beta_n\}\\ 0 & \text{elsewhere.}\end{cases}$$

Thus we have

$$F(x)=\sum\Big(\int_{\alpha_n-x}^{\infty}-\int_{\beta_n-x}^{\infty}\Big)k(y)\,dy=\sum\int_{\alpha_n-x}^{\beta_n-x}k(y)\,dy.$$

Let $\varepsilon>0$ and choose N such that $\sum_{N+1}^{\infty}(\beta_n-\alpha_n)<\varepsilon$. We now see that

$$\limsup_{x\to\infty}|F(x)|\leqq\limsup_{x\to\infty}\sum_1^N\int_{\alpha_n-x}^{\beta_n-x}|k(y)|\,dy$$

$$+\limsup_{x\to\infty}\sum_{N+1}^{\infty}\int_{\alpha_n}^{\beta_n}|k(y-x)|\,dy\leqq\max_{|M|=\varepsilon}\int_M|k(y)|\,dy$$

i.e.

$$\lim_{x\to\infty}F(x)=0.$$

We shall now see that the kernel in (4), studied by Rényi, is just a special case of a class of functions, where Theorem 1 may be applicable. These functions are of the form

$$f(x)=\Big(\log\frac{1}{x}\Big)^{\beta}\sum_1^{\infty}\alpha_m x^m,\quad x\in(0,1),\ f'\in L(0,1),\tag{5.1}$$

with $\alpha_m=O(m^{\gamma})$ for some finite γ.

Suppose that

(5.2) $$\lim_{x\uparrow 1} \sum a_n f(x^{p_n}) = 0, \quad p_{n+1}/p_n \geqq \theta > 1.$$

If (5.2) is transformed to the form (1), i.e.

$$\lim_{t\to\infty} \sum a_n \int_{\lambda_n - t}^{\infty} k(y)\, dy = 0,$$

by letting $x = \exp(-\exp(-t))$ and $\lambda_n = \log p_n$, then

$$\hat{k}(u) = -iu\Gamma(\beta - iu)\chi(\beta - iu).$$

The function χ is, for $\operatorname{Re} w > \gamma + 1$, defined by the Dirichlet series

$$\chi(w) = \sum_1^\infty \frac{\alpha_m}{m^w}$$

and has an analytic extension into $\operatorname{Re} w > \beta$ (if $\beta < \gamma + 1$).

It is easily seen that for each kernel k there exists an $\alpha \geqq 0$ such that $k \in \mathscr{L}_\alpha$. For this α it also holds that

$$\left(\log \frac{1}{x}\right)^\alpha \sum a_n f(x^{p_n})$$

converges uniformly on every set $\{x\colon x < x_0\}$, $x_0 < 1$, and defines a bounded function on [0, 1], i.e. (2.1) is satisfied. Thus, in the case of the series (5.2), the existence of a high indices theorem depends solely on the location of the zeros of the function χ as Theorem 1 shows.

In the Rényi case we have

$$\hat{k}(u) = \Gamma(1 - iu)\zeta(-iu)(1 - 2^{1+iu}),$$

which is in $\mathscr{S}$ and $\lambda_n = n \log 2$. From the knowledge of the ζ-function, we know that there exists a closed interval I with $|I| = 2\pi/\log 2$, where $\chi(-iu + \sigma) = \zeta(\sigma - iu)(1 - 2^{1-\sigma+iu}) \neq 0$ for $u \in I$ and $\sigma \geqq 0$ (see for instance Widder [14]). Since all hypotheses in Theorem 1 are satisfied the desired conclusion (2.9) follows and Rényi's conjecture is proved.

A direct consequence of Theorem 4 is the following: let $\gamma \geqq 0$ and let $\{p_n\}$ be a Hadamard sequence ($p_1 > 0$). Then

(5.3) $$\sum a_n e^{-p_n t} = O(t^{-\gamma}), \quad t \to +0$$

implies that

(5.4) $$a_n = O(p_n^\gamma).$$

This problem has been studied by Gaier [1], who gives an estimate less precise than ours. The estimate (5.4) is in fact best possible, as we can readily see if (5.3) is applied to the real line by letting $t = \exp(-x)$. Let $a_n = \exp(\gamma \log p_n)$, $\gamma > 0$.

Then

$$\exp(-\gamma x)\cdot\sum a_n \exp(-\exp(\log p_n - x))$$
$$= \sum \exp(-\gamma(x-\log p_n)-\exp(-(x-\log p_n))) = O(1),$$

where the last equality is obvious, since $k(x)=\exp(-\gamma x-\exp(-x))$ is in $\mathcal{S}$.

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