On the asymptotic distribution of the eigenvalues of pseudodifferential operators in $\mathbf{R}^n$

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Dedicated to Lars Gårding on his 60th birthday

1. Introduction

In a recent paper [2] we have developed the Weyl calculus of pseudo-differential operators in $\mathbf{R}^n$ for quite general symbols. As an application we shall now extend and improve the estimates given by Tulovskii and Šubin [5] for the error term in the asymptotic formula for the number $N(\lambda)$ of eigenvalues $\leqq \lambda$ of certain pseudo-differential operators $P = p^w(x, D)$ in $\mathbf{R}^n$,

$$N(\lambda) \sim (2\pi)^{-n} \iint_{p(x,\xi)<\lambda} dx\, d\xi \tag{1.1}$$

The first results of this type were obtained by H. Weyl for second order operators, and R. Courant later isolated the maximum-minimum principle for eigenvalues from the proof. A new approach which also gives asymptotic formulas for the eigenfunctions by means of Tauberian arguments was introduced by Carleman, and these methods were developed by Gårding [1] to a general proof of (1.1) for higher order elliptic operators in bounded sets in $\mathbf{R}^n$. The methods used in [5] are more closely related to the earlier methods of Weyl, however. They start from the observation that $N(\lambda)$ is the trace of the orthogonal projection E on the space spanned by the eigenvectors corresponding to eigenvalues $\leqq \lambda$. Thus E is self adjoint and

$$E^2 - E = 0, \quad E(P-\lambda)E \leqq 0, \quad (I-E)(P-\lambda)(I-E) \geqq 0. \tag{1.2}$$

Now the calculus of pseudo-differential operators allows one to satisfy these conditions approximately; one just takes E with symbol $\chi_\lambda(p)$ where χ_λ is a smooth approximation to the Heaviside function $H(\lambda - .)$. Arguments from [5] which we recall in Section 2 lead from approximate solutions of (1.2) to estimates of $N(\lambda)$.

The main new point here is a careful discussion in Section 3 of the symbol classes containing $\chi_\lambda(p)$. The error estimates in (1.1) which follow in Section 4 are better than those of [5] since we have more general pseudo-differential operators available so that we can use a better smooth approximation χ_λ to the Heaviside function. The generality is needed even if p is a "classical" symbol. It would not suffice to use the Beals—Fefferman calculus. (Robert [3] has recently given error estimates for (1.1) similar to those of [5] for such operators.) However, as indicated by an example discussed in Section 4, it is not likely that the improved results given here are optimal.

I would like to thank Victor Guillemin who called my attention to [5] when I lectured at MIT on the Weyl calculus.

2. Approximate spectral projections

Let P be a self adjoint operator in a Hilbert space H which has a discrete spectrum and is bounded from below. Denote by $N(\lambda)$ the number of eigenvalues $\leqq\lambda$. The following lemma is implicit in [5, Section 6] (see also [4, Section 28]).

Lemma 2.1. *Let E be a self adjoint operator of trace class such that PE is bounded. If*

$$((P-\lambda)Eu, Eu)\leqq A(u,u),\quad u\in H, \tag{2.1}$$

where $\lambda\in\mathbf{R}$ and $A\geqq 0$, then

$$N(\lambda+4A)\geqq \operatorname{Tr}E-2\|E-E^2\|_{\mathrm{Tr}}. \tag{2.2}$$

If instead of (2.1) *we have in the domain of P*

$$((P-\lambda)(u-Eu), u-Eu)\geqq -B(u,u), \tag{2.3}$$

where $\lambda\in\mathbf{R}$ and $B\geqq 0$, then

$$N(\lambda-4B-0)\leqq \operatorname{Tr}E+2\|E-E^2\|_{\mathrm{Tr}}. \tag{2.4}$$

Proof. Let W_0 and W_1 be the orthogonal complementary spaces spanned by eigenvectors of E corresponding to eigenvalues $<1/2$ and $\geqq 1/2$ respectively. Since

$$\|E-E^2\|_{\mathrm{Tr}}=\sum|\lambda_j-\lambda_j^2|$$

where λ_j are the eigenvalues of E, we have

$$|\dim W_1-\operatorname{Tr}E|\leqq 2\|E-E^2\|_{\mathrm{Tr}}. \tag{2.5}$$

In fact,

$$|1-\lambda_j|\leqq 2|\lambda_j-\lambda_j^2|\quad\text{if}\quad \lambda_j\geqq 1/2,\qquad |\lambda_j|\leqq 2|\lambda_j-\lambda_j^2|\quad\text{if}\quad \lambda_j<1/2$$

so (2.5) follows by summation. When $u \in W_1$ we have $(u, u) \leqq 4(Eu, Eu)$ so

$$\big((P-\lambda-4A)Eu, Eu\big) \leqq 0, \quad u \in W_1.$$

But $EW_1 = W_1$ so the maximum minimum principle shows that P has at least $\dim W_1$ eigenvalues $\leqq 4A+\lambda$. Since $(u, u) \leqq 4(u-Eu, u-Eu)$, $u \in W_0$, we obtain in the same way

$$\big((P-\lambda+4B)v, v\big) \geqq 0, \quad v \in W_0,$$

provided that v is in the domain of P. Hence $P-\lambda+4B$ has at most $\dim W_1$ negative eigenvalues, which proves the lemma.

3. Hypoelliptic symbols

Let g be a σ temperate metric in $\mathbf{R}^{2n}$ (see [2, Def. 4.1]) and set as in [2, (4.6)],

$$h(x, \xi)^2 = \sup\left(\frac{g_{x,\xi}}{g^{\sigma}_{x,\xi}}\right).$$

We shall always assume that $h \leqq 1$ ("the uncertainly principle") and later on we shall also have to require that for some $\delta > 0$ and C

$$h(x, \xi) \leqq C(1+|x|+|\xi|)^{-\delta}. \tag{3.1}$$

Let p be a positive g continuous function such that p is a symbol of weight p,

$$p \in S(p, g). \tag{3.2}$$

(See [2, section 2]) For special choices of the metric this is essentially a well known sufficient condition for hypoellipticity, and it always implies that p has a parametrix:

Lemma 3.1. *For any positive integer N one can find $q \in S(1/p, g)$ so that*

$$q^w p^w - 1 = r_N^w, \quad r_N \in S(h^N, g). \tag{3.3}$$

Proof. For $q_0 = 1/p$ we have $q_0 \in S(1/p, g)$ by [1, Lemma 2.4], so the calculus [1, Theorem 4.2] gives

$$1 - q_0^w p^w = r^w, \quad r \in S(h, g).$$

It follows that

$$q^w p^w = 1 - (r^w)^N \quad \text{if} \quad q^w = \big(1 + r^w + \ldots + (r^w)^{N-1}\big) q_0^w,$$

which proves (3.3).

A standard argument can be used to construct q so that (3.3) is valid for any N, but we have no need for this in what follows. In the preceding argument the important point was that $1/p \in S(1/p, g)$. The proof of [1, Lemma 2.4] easily gives also that $p^a \in S(p^a, g)$ for any real number a and not only for $a=-1$. (See also the more general Proposition 3.5 below.) Using this fact with $a=-1/2$ we shall now prove

Lemma 3.2. *For any N we can find $q \in S(\sqrt{p}, g)$ so that*

$$q^w(q^w)^* - p^w = r_N^w, \quad r_N \in S(h^N p, g). \tag{3.4}$$

Proof. Put $q_0 = p^{-1/2} \in S(p^{-1/2}, g)$. Then the calculus gives

$$1 - q_0^w p^w q_0^w = r^w$$

where $r \in S(h, g)$ is real. If we denote the sum of the first N terms in the power series expansion of $(1-x)^{1/2}$ at 0 by $T_N(x)$, then $1-x-T_N(x)^2$ is a polynomial divisible by x^N. Hence $1-r^w-T_N(r^w)^2 \in S(h^N, g)$, so

$$T_N(r^w)^2 - q_0^w p^w q_0^w \in S(h^N, g).$$

The product of $T_N(r^w)$ to the left by a parametrix for q_0, constructed according to Lemma 3.1, will now satisfy (3.4) since $T_N(r^w)$ is self adjoint.

The following simple technical lemma should have been included in [2]. Just as Lemmas 3.1 and 3.2 it does not require (3.1) but only that $h \leqq 1$.

Lemma 3.3. *If χ_j is a bounded sequence in $S(1, g)$ with limit χ, then $\chi_j^w u \to \chi^w u$ in L^2 for every $u \in L^2$.*

Proof. We may assume that $\chi=0$ and in view of the uniform L^2 bound [2, Theorem 5.3], we may take $u \in \mathcal{S}$. Then we have $\chi_j^w u \to 0$ in $\mathcal{S}'$, and since $\chi_j^w u$ is bounded in $\mathcal{S}$ by [2, Theorem 5.2] we obtain $\chi_j^w u \to 0$ in $\mathcal{S}$.

Theorem 3.4. *If (3.1) is fulfilled and $p \in S(p, g)$ then $p^w(x, D)$ defines a self adjoint operator P in L^2. It is bounded from below, and if $p \to \infty$ at ∞ then the spectrum is discrete.*

Proof. The domain of P consists of all $u \in L^2$ with $Pu = p^w(x, D)u \in L^2$. That P is closed follows from the continuity of p^w in $\mathcal{S}'$. To prove that P is self adjoint it is sufficient to show that P is the closure of its restriction to $\mathcal{S}$. Choose a sequence $\chi_j \in C_0^\infty(\mathbf{R}^{2n})$ which is bounded in $S(1, g)$ and converges to 1. We may for example take the partial sums of a partition of unity corresponding to the metric g. If $u \in L^2$ we have $\chi_j^w u \in \mathcal{S}$ and $\chi_j^w u \to u$ in L^2 by Lemma 3.3. According to Lemma 3.1 we can choose $q \in S(1/p, g)$ so that $1 - q^w p^w = r^w$, $r \in S(1/p, g)$, for $1/p$ is bounded by a power of $1/h$ since p is σ temperate and (3.1) is assumed. If $u \in L^2$, $p^w(x, D)u \in L^2$ we write

$$p^w \chi_j^w u = p^w \chi_j^w q^w p^w u + p^w \chi_j^w r^w u.$$

The symbols of $p^w \chi_j^w r^w$ and $p^w \chi_j^w q^w$ belong to a bounded set in $S(1, g)$ and converge to the symbols of $p^w r^w$ and $p^w q^w$ respectively, so with L^2 convergence we obtain from Lemma 3.3

$$p^w \chi_j^w u \to p^w q^w (p^w u) + p^w r^w u = p^w u.$$

To prove that p is bounded from below we choose q according to Lemma 3.2 so that $q^w\ (q^w)^* - p^w = r^w$, $r \in S(1, g)$. Then r^w is bounded in L^2 and

$$(p^w u, u) \geqq -(r^w u, u), \quad u \in \mathcal{S},$$

which proves the assertion. Now assume that $p \to \infty$ at ∞ and choose q according to Lemma 3.1 with $N=1$, say $q=1/p$. Then q^w and r^w are both compact. Hence

$$u = q^w p^w u - r^w u$$

belongs to a compact set in L^2 when $\|u\|$ and $\|Pu\|$ are bounded, which proves that the spectrum is discrete and completes the proof of Theorem 3.4.

So far we have only considered the symbol $\chi(p)$ when $\chi(t) = t^\alpha$, but now we shall consider more general choices. Let $a(t)^2 dt^2$ be a slowly varying metric on $\mathbf{R}^+$, that is,

$$a(t)\,|s-t| < c \Rightarrow \frac{1}{C} \leqq \frac{a(t)}{a(s)} \leqq C. \tag{3.5}$$

A sufficient condition for this is that

$$|a'(t)| \leqq C' a(t)^2 \tag{3.6}$$

for then we have

$$\left|\frac{1}{a(t)} - \frac{1}{a(s)}\right| \leqq C' |t-s|$$

which implies (3.5) with $C=2$ and $c=1/2C'$. Let m be $a^2 dt^2$- continuous, say

$$|m'(t)| \leqq C'' m(t) a(t) \tag{3.7}$$

and let $\chi \in S(m, a^2 dt^2)$, thus

$$|\chi^{(k)}(t)| \leqq C_k m(t) a(t)^k. \tag{3.8}$$

For what metrics G can we then conclude that $\chi(p) \in S(m(p), G)$? For first order derivatives this requires that

$$|\chi'(p)\langle dp, t\rangle| \leqq m(p)\, G(t)^{1/2}$$

when (3.8) is fulfilled, so we need to know that

$$a(p)^2\, |dp|^2 \leqq G.$$

The second differential of $\chi(p)$ is

$$\chi''(p)\langle dp, t_1\rangle\langle dp, t_2\rangle + \chi'(p)\, p''(t_1, t_2).$$

There is no new problem in estimating the first term, but for the second one we just have the bound

$$Cm(p)\,a(p)\,pg(t_1)^{1/2}g(t_2)^{1/2}$$

so we also need to know that $a(p)pg \leqq G$. Thus we are led to define

$$G = a(p)^2\,|dp|^2 + \big(1+pa(p)\big)\,g \tag{3.9}$$

where a term g has been included to make $G \geqq g$.

Proposition 3.5. *If $p \in S(p, g)$ and $a(t)^2dt^2$ is a slowly varying metric on $\mathbf{R}^+$, then the metric G defined by* (3.9) *is slowly varying. If m is a^2dt^2 continuous and $\chi \in S(m, a)$ then $m(p)$ is G continuous and $\chi(p) \in S\big(m(p), G\big)$.*

Proof. To simplify notation we shall write X instead of (x, ξ) and so on, for the symplectic structure plays no role yet. Let Y be in a small G ball with center at X, that is,

$$a\big(p(X)\big)^2\langle Y-X, dp(X)\rangle^2 + \big(1+p(X)a\big(p(X)\big)\big)g_X(Y-X) < c \tag{3.10}$$

where c is small. Since $g \leqq G$ we obtain a bound for g_Y/g_X and $p(Y)/p(X)$ if c is small enough. For $0<s<1$ it follows from Taylor's formula that

$$|\langle Y-X, dp(X+s(Y-X)) - dp(X)\rangle| \leqq Cp(X)g_X(Y-X)$$

so (3.10) gives

$$a\big(p(X)\big)|\langle Y-X, dp\,(X+s(Y-X))\rangle| \leqq c^{1/2}+Cc.$$

Hence, by Taylor's formula,

$$a\big(p(X)\big)\,|p(Y)-p(X)| \leqq c^{1/2}+Cc.$$

If we choose c small enough then (3.5) gives a bound for $a\big(p(Y)\big)/a\big(p(X)\big)$, and if m is a^2dt^2 continuous we also obtain a bound for $m\big(p(Y)\big)/m\big(p(X)\big)$. By Taylor's formula we obtain as above, using (3.10),

$$\begin{aligned} a\big(p(X)\big)^2|\langle t, dp(X)-dp(Y)\rangle|^2 &\leqq Ca\big(p(X)\big)^2p(X)^2g_X(Y-X)g_X(t) \\ &\leqq C\big(1+a\big(p(X)\big)\,p(X)\big)\,g_X(t), \end{aligned}$$

and this completes the proof that the metric varies slowly.

To show that $\chi(p) \in S\big(m(p), G\big)$ we observe that the k-th differential of $\chi(p)$ is a linear combination of terms of the form

$$\chi^{(j)}(p)\,p^{(k_1)}\dots p^{(k_j)}$$

with all $k_\nu \geqq 1$. We can estimate $\chi^{(j)}(p)$ by $m(p)a(p)^j$ so it suffices to prove that for $i \geqq 1$ we have

$$|a(p)\,p^{(i)}(t_1, \dots, t_i)| \leqq C_i \textstyle\prod_1^i G(t_\nu)^{1/2}.$$

For $i=1$ this is obvious because of the first term in (3.9), and for $i>1$ we note that the condition $p\in S(p, g)$ implies

$$a(p)|p^{(i)}(t_1, \ldots, t_i)| \leqq a(p)C_i p \prod_1^i g(t_\nu)^{1/2}$$
$$= C_i pa(p)(1+pa(p))^{-i/2} \prod_1^i ((1+pa(p))g(t_\nu))^{1/2}.$$

This completes the proof, for the factor in front of the product is bounded.

Next we shall examine when the uncertainty principle is valid.

Lemma 3.6. *Let g be a positive definite quadratic form in a symplectic vector space W and let*

$$G(t) = \sigma(t,f)^2+g(t), \quad t\in W$$

where f is a fixed element in W. If $g\leqq h^2 g^\sigma$ *it follows then that*

$$G \leqq 2(g(f)+h^2)G^\sigma.$$

Proof. We must show that $G^\sigma(w)\leqq 1$ implies $G(w)\leqq 2(g(f)+h^2)$. Now $G^\sigma(w)\leqq 1$ means by definition that

$$\sigma(t, w)^2 \leqq \sigma(t,f)^2+g(t).$$

The form $(\sigma(t, f), t)\to\sigma(t, w)$ can then be extended to $\mathbf{R}\oplus W$ with norm 1, so

$$\sigma(t, w) = a\sigma(t,f)+\sigma(t, b) \quad \text{where} \quad b\in W, \quad a\in\mathbf{R}, \quad a^2+g^\sigma(b) \leqq 1.$$

Thus we have $w=af+b$, hence

$$G(w) \leqq \sigma(b,f)^2+g(af+b) \leqq g^\sigma(b)g(f)+2(a^2g(f)+g(b))$$
$$\leqq g^\sigma(b)(g(f)+2h^2)+2a^2g(f) \leqq 2(g(f)+h^2).$$

Let us now return to the metric (3.9). It is of the form discussed in Lemma 3.6 with $f=a(p)H_p$, where H_p is the Hamilton field of p, and g replaced by $(1+pa(p))g$. Assuming that for the given metric

$$g \leqq h^2 g^\sigma$$

we obtain

$$G \leqq 2((1+pa(p))a(p)^2g(H_p)+h^2(1+pa(p))^2)G^\sigma.$$

We may assume that $\langle dp, t\rangle^2\leqq p^2g(t)$. Then

$$g^\sigma(H_p) = \sup\frac{\sigma(H_p, t)^2}{g(t)} = \sup\frac{\langle dp, t\rangle^2}{g(t)} \leqq p^2$$

so $g(H_p)\leqq h^2p^2$ and

$$G \leqq 2(1+pa(p))^3h^2G^\sigma. \tag{3.11}$$

Let γ be any positive number such that

(3.12) $$h \leqq Cp^{-\gamma}$$

and let $0<\delta<2\gamma/3$. If

(3.13) $$1+ta(t) \leqq C(1+t)^{\delta}$$

it follows from (3.11) that G/G^{σ} is bounded by a negative power of p. From now on we also assume that for some C and N

(3.14) $$1+|x|+|\xi| \leqq Cp(x, \xi)^{N}.$$

Note that (3.1) follows from (3.14) and (3.12). Then it follows that G has all the properties of the metric g required at the beginning of the section:

Proposition 3.7. *Assume that* (3.5) *and* (3.12)—(3.14) *are fulfilled and that* $\delta<2\gamma/3$. *Then the metric* G *is* σ *temperate and*

$$\frac{G_{x,\xi}}{G^{\sigma}_{x,\xi}} \leqq C(1+|x|+|\xi|)^{-c}$$

for some $C, c>0$. *If* m *is* $a(t)^2dt^2$ *continuous and*

(3.15) $$\frac{m(s)}{m(t)} \leqq C(1+s+t)^{N}$$

then $m(p)$ *is* σ, G *temperate.*

Proof. There is a fixed bound for $G^{\sigma}_{y,\eta}/G^{\sigma}_{x,\xi}$ if $G_{x,\xi}(y-x, \eta-\xi)<c$, and if $G_{x,\xi}(y-x, \eta-\xi)\geqq c$ then $G^{\sigma}_{x,\xi}(y-x, \eta-\xi)>cH(x, \xi)^{-2}$ which bounds a positive power of $1+|x|+|\xi|$. Since

$$g_{x,\xi} \leqq G_{x,\xi} \leqq C\big(1+p(x, \xi)a(p(x, \xi))\big)^2 g_{x,\xi} \leqq C'\big(1+p(x, \xi)\big)^{2\delta} g_{x,\xi}$$

we have

$$g^{\sigma}_{x,\xi} \geqq G^{\sigma}_{x,\xi} \geqq g^{\sigma}_{x,\xi}\big(1+p(x, \xi)\big)^{-2\delta}C'^{-1},$$

$$C'^{-1}\frac{G^{\sigma}_{y,\eta}}{G^{\sigma}_{x,\xi}} \leqq \big(1+p(x, \xi)\big)^{2\delta}\frac{g^{\sigma}_{y,\eta}}{g^{\sigma}_{x,\xi}} \leqq C\big(1+p(x, \xi)\big)^{2\delta}\big(1+g^{\sigma}_{x,\xi}(x-y, \xi-\eta)\big)^{N}$$

$$\leqq C_1\big(1+p(C_1, \xi)\big)^{2\delta(N+1)}\big(1+G^{\sigma}_{x,\xi}(x-y, \xi-\eta)\big)^{N} \leqq C_2\big(1+G^{\sigma}_{x,\xi}(x-y, \xi-\eta)\big)^{N'}.$$

Hence G is σ temperate and the last assertion follows in the same way.

As explained in the introduction we are interested in functions $\chi_\lambda\in S(1, a^2dt^2)$ which approximate the Heaviside function $H(\lambda-.)$ well when λ is a large positive number. Choose $\chi\in C^\infty$ so that $\chi(t)=1$ for $t<0$ and $\chi(t)=0$ for $t>1$, and set

$$\chi_{\lambda,\varepsilon}(t) = \chi\left(\frac{t-\lambda}{\varepsilon}\right).$$

The derivative of order j has the bound $C_j\varepsilon^{-j}$ when $\lambda<t<\lambda+\varepsilon$ and vanishes elsewhere. Thus we want to choose $a(t)$ roughly equal to $1/\varepsilon$ in $(\lambda, \lambda+\varepsilon)$ and so that (3.6) is fulfilled, that is, $1/a(t)$ is Lipschitz continuous. To make a as small as possible we therefore define

$$a_{\lambda,\varepsilon}(t) = \left(\varepsilon^2+(t-\lambda)^2\right)^{-1/2}. \tag{3.16}$$

The condition (3.13) then becomes

$$t^2\left(\varepsilon^2+(t-\lambda)^2\right)^{-1} \leqq C^2(1+t)^{2\delta}, \quad t \geqq 0.$$

This condition is of course fulfilled if $t<2|t-\lambda|$, and if $|t-\lambda|\leqq t/2$ then $t/2\leqq \lambda\leqq 3t/2$ so the condition is fulfilled if

$$\varepsilon = \lambda^{1-\delta}. \tag{3.17}$$

Summing up, (3.17) implies (3.6) and (3.13). With this choice of ε we shall write χ_λ and a_λ instead of $\chi_{\lambda,\varepsilon}$ and $a_{\lambda,\varepsilon}$. The corresponding metric defined by (3.9) will be denoted by G_λ, and we have $e_\lambda\in S(1, G_\lambda)$ if $e_\lambda=\chi_\lambda(p)$. The calculus of pseudo-differential operators shows that the symbol of $e_\lambda^w(1-e_\lambda^w)$ is in $S(1, G_\lambda)$ and that it can be estimated by any negative power of $(1+|x|+|\xi|)$ except when $\lambda<p<\lambda+\varepsilon$. This estimate is uniform with respect to λ, for in Propositions 3.5 and 3.7 the constants implied by the conclusions can be estimated in terms of those occurring in the hypotheses. We shall now estimate the trace norm (see [5, Section 4] and [2, Section 7]).

Lemma 3.8. *If $q\in C_0^\infty(\mathbf{R}^{2n})$ and $N\geqq n+1$ then with B denoting the unit ball*

$$\|q^w\|_{\mathrm{Tr}} \leqq C_N\Big(\|q\|_{L^1}+\iint \big(\sup_{(y,\eta)\in B,\, |\alpha|=N} |D^\alpha q(x+y, \xi+\eta)|\big)\, dx\, d\xi\Big), \tag{3.18}$$

Proof. It is clear that q^w is of trace class if $q\in\mathscr{S}$ (see e.g. [2, Lemma 7.2]). If L is a real linear form in $\mathbf{R}^{2n}$ and $q_L=qe^{iL}$ then the trace norm of q_L^w is equal to that of q^w. In fact, by the unitary equivalence theorem [2, Theorem 4.3], we may assume that $L(x, \xi)$ depends on x only, and then $q_L^w=e^{iL/2}\,q^w e^{iL/2}$ which proves the assertion. Hence we obtain

$$\|(fq)^w\|_{\mathrm{Tr}} \leqq \left\|\frac{\hat f}{(2\pi)^{2n}}\right\|_{L^1} \|q^w\|_{\mathrm{Tr}}, \quad f\in C_0^\infty,$$

if we express f in terms of $\hat f$ by Fourier's inversion formula. Choosing $q=1$ in B we obtain by Bernstein's theorem

$$\|f^w\|_{\mathrm{Tr}} \leqq C \textstyle\sum_{|\alpha|\leqq n+1} \sup|D^\alpha f|, \quad f\in C_0^{n+1}(B). \tag{3.19}$$

Fix $\varphi\in C_0^\infty(B)$ with $\iint \varphi\, dx\, d\xi=1$ and set, now with any $q\in C_0^\infty(\mathbf{R}^{2n})$,

$$q_{x,\xi}(y, \eta) = \varphi(y-x, \eta-\xi)\, q(y, \eta).$$

If we show that

(3.20) $$\|q^w_{x,\xi}\|_{\mathrm{Tr}} \leqq C\big(\|q_{x,\xi}\|_{L^1} + \sup_{(y,\eta)\in B, |\alpha|=N} |D^\alpha q(x+y, \xi+\eta)|\big),$$

then an integration with respect to (x, ξ) gives (3.18). It is of course sufficient to prove (3.20) when $x=\xi=0$. Write

$$q(y, \eta) = T(y, \eta)+R(y, \eta)$$

where $T(y, \eta)$ is the Taylor polynomial of degree $N-1$. Then

$$\sup_{(y,\eta)\in B, |\alpha|\leqq N} |D^\alpha R(y, \eta)| \leqq C \sup_{(y,\eta)\in B, |\alpha|=N} |D^\alpha q(y, \eta)|.$$

Hence the L^1 norm of $\varphi T = q_{0,0}-\varphi R$ can be estimated in terms of the right hand side of (3.20), so this is also possible for the coefficients of T since all norms in the finite dimensional vector space of polynomials of degree $<N$ are equivalent. Thus we have

$$\sup_{y,\eta} \sup_{|\alpha|\leqq N} |D^\alpha q_{0,0}(y, \eta)| \leqq C\big(\|q_{0,0}\|_{L^1} + \sup_{(y,\eta)\in B, |\alpha|=N} |D^\alpha q(y, \eta)|\big)$$

so (3.20) follows from (3.19).

Theorem 3.9. *Let g be slowly varying and let m be g continuous, $g\leqq h^2 g^\sigma$ where $h\leqq 1$. Then we have for every integer $k\geqq 0$*

$$\|q^w\|_{\mathrm{Tr}} \leqq C_k(\|q\|_{L^1}+\|h^k m\|_{L^1}\|q\|), \quad q\in S(m, g),$$

where $\|q\|$ is a seminorm of q in the symbol class $S(m, g)$ which only depends on k and the constants in the slow variation and g continuity assumed.

Proof. Let φ_i be the partition of unity constructed in [2, Section 2], let g_i be the metric at the center of the support of φ_i and let $m=m_i$, $h=h_i$ there. It suffices to show that with $q_i=\varphi_i q$ we have when $N>n$

(3.21) $$\|q_i^w\|_{\mathrm{Tr}} \leqq C_N(\|q_i\|_{L^1}+h_i^{N/2}\,|\det g_i|^{-1/2} \sup |q_i|^{g_i}_N).$$

In doing so we may assume that φ_i is centered at 0 and that

$$g_i = \textstyle\sum_1^n \lambda_j(x_j^2+\xi_j^2), \quad \lambda_j \leqq h_i.$$

The measure of the set of points at distance $\leqq 1$ from the support of q_i is bounded by a constant times $|\det g_i|^{-1/2}$ and

$$\textstyle\sum_{|\alpha|=N} \sup |D^\alpha q_i| \leqq C h_i^{N/2} \sup |q_i|^{g_i}_N$$

so (3.21) follows from (3.18).

We can apply Theorem 3.9 to $e^w_\lambda(1-e^w_\lambda)$, with $m=1$ and g replaced by G_λ. When k is chosen so large that H^k_λ is integrable, if $H^2_\lambda = \sup G_\lambda/G^\sigma_\lambda$, and we recall that the symbol is $\leqq CH_\lambda^k$ except when $\lambda<p<\lambda+\varepsilon$, we obtain

$$\|e_\lambda^w(1-e_\lambda^w)\|_{\mathrm{Tr}} \leqq C_1 \iint_{\lambda<p<\lambda+\varepsilon} dx\,d\xi + C_2. \tag{3.22}$$

To complete the preparations for the application of Lemma 2.1 in Section 4 we shall finally prove

Lemma 3.10. *There exists a constant C such that*

$$e_\lambda^w(p^w-\lambda)e_\lambda^w \leqq C\varepsilon, \quad (1-e_\lambda^w)(p^w-\lambda)(1-e_\lambda^w) \geqq -C\varepsilon. \tag{3.23}$$

Proof. Let $f(t)=|t|$ when $|t|>1$, $f\in C^\infty$ and $f>0$ everywhere, and set

$$f_\lambda(t) = \varepsilon f\left(\frac{t-\lambda}{\varepsilon}\right), \quad \varepsilon = \lambda^{1-\delta}.$$

Then $f_\lambda(t)=-(t-\lambda)$ when $t<\lambda-\varepsilon$ and $f(t)=t-\lambda$ when $t>\lambda+\varepsilon$, and it is clear that $a_\lambda f_\lambda$ has fixed upper and lower bounds. To prove that f_λ satisfies (3.15) we observe that

$$\frac{a_\lambda(t)}{a_\lambda(s)} \leqq 1+\frac{|t-s|}{\varepsilon} \leqq C(1+t+s)^{1+N}$$

unless

$$(t+s)^N < \varepsilon^{-1} = \lambda^{\delta-1}.$$

If N is large this implies that $t+s<\lambda/2$ and then $|t-\lambda|$ and $|s-\lambda|$ lie between $\lambda/2$ and λ so $a_\lambda(t)/a_\lambda(s)$ is bounded. Thus a_λ satisfies (3.15) so f_λ does, and $F_\lambda=f_\lambda(p)$ is G_λ temperate. It is clear that

$$F_\lambda\in S(F_\lambda, G_\lambda)$$

with a uniform bound for each seminorm.

We shall now prove that

$$\|e_\lambda^w(p^w-\lambda+F_\lambda^w)e_\lambda^w\| \leqq C\varepsilon, \quad \|(1-e_\lambda^w)(p^w-\lambda-F_\lambda^w)(1-e_\lambda^w)\| \leqq C\varepsilon. \tag{3.24}$$

To prove the first estimate we observe that $p-\lambda+F_\lambda$ is uniformly bounded in $S\big(F_\lambda(p/\lambda)^\nu, G_\lambda\big)$ for any $\nu>0$ since it is 0 for $p<\lambda-\varepsilon$. (That p is G_λ continuous is obvious since $g\leqq G_\lambda$, and p is σ, G_λ temperate since (3.15) is valid for $m(t)=t$.) All terms in the composition series for

$$e_\lambda^w(p^w-\lambda+F_\lambda^w)e_\lambda^w \tag{3.24}$$

vanish except when $|p-\lambda|<\varepsilon$. Thus the symbol of (3.24) is bounded in $S(F_\lambda, G_\lambda)$, and when $|p-\lambda|\geqq\varepsilon$ it is bounded in $S\big(F_\lambda(p/\lambda)^\nu H_\lambda^N, G_\lambda\big)$ for any ν, N. Now recall that $H_\lambda\leqq Cp^{-\Gamma}$ for some $\Gamma>0$ depending only on the choice of δ. If $N\Gamma>\nu+1$ we conclude that the symbol of (3.24) is bounded in $S(\varepsilon, G_\lambda)$ which proves the first estimate (3.24). The second one follows in the same way if we observe that $p-\lambda-F_\lambda=0$ when $p>\lambda+\varepsilon$ and that $1-e_\lambda$ is bounded in $S\big((p/\lambda)^\nu, G_\lambda\big)$ for any ν.

What remains now is to establish a lower bound for F_λ^w. As in the proof of Theorem 3.4 it follows from Lemma 3.2 that one can find R_λ bounded in $S(F_\lambda H_\lambda^N, G_\lambda)$ for any desired N so that

$$F_\lambda^w \geqq R_\lambda^w. \tag{3.25}$$

However, $F_\lambda H_\lambda^N$ need not be small near 0 so we have to use some supplementary arguments to show that the lower bound of F_λ^w is $\geqq -C\varepsilon$.

Choose a decreasing function φ on $\mathbf{R}$ so that $\varphi(t)=1$ for $t<1$, $\varphi(t)=0$ for $t>2$ and $\varphi(t)^2+\psi(t)^2=1$ for another C^∞ function ψ. Then

$$p_\mu = p\varphi\left(\frac{p}{\mu}\right)$$

is uniformly bounded in $S(\mu, g)$ so for some $C>1$

$$\|p_\mu\| \leqq C\mu, \quad \mu > 1.$$

When $C\mu=\lambda$ we obtain

$$\lambda - p_\mu^w \geqq 0.$$

Write $\Phi_\lambda=\varphi(3p/\mu)$ which has support where $p<\mu$, thus $p_\mu=p$. Then

$$\Phi_\lambda^w(\lambda - p_\mu^w)\Phi_\lambda^w \geqq 0$$

and we shall estimate the difference between the operator on the left hand side and the operator with symbol $\Phi_\lambda^2(\lambda-p)$ which is the leading term in the composition series while the first order terms cancel. The symbol of the difference is bounded in $S((\lambda+p)h^2, g)$ and when $p>\mu$ it is bounded in $S((\lambda+p)h^N, g)$ for any N since all terms in the composition series vanish. In that case

$$h \leqq Cp^{-\gamma} \leqq C'\lambda^{-\gamma}$$

so if $1-(N-1)\gamma<0$ we obtain that the symbol is bounded in $S(\lambda^{1-\gamma}, g)$ when $p>\mu$. When $\mu/3<p<\mu$ the symbol is bounded in $S(\lambda^{1-2\gamma}, g)$ since $\lambda h^2<C\lambda^{1-2\gamma}$ then. Finally, when $p<\mu/3$ it is convenient to note that $\tilde{\Phi}_\lambda=1-\Phi_\lambda\in S((p/\lambda)^\nu, g)$ for any ν. The error is therefore bounded in the symbol space with weight

$$\lambda h^2\left(\frac{p}{\lambda}\right)^\nu < \lambda p^{-2\gamma}\left(\frac{p}{\lambda}\right)^\nu < \lambda^{1-2\gamma}$$

if we choose $\nu=2\gamma$. Summing up, the symbol of

$$\Phi_\lambda^w(\lambda-p_\mu^w)\Phi_\lambda^w-(\Phi_\lambda^2(\lambda-p))^w$$

is bounded in $S(\lambda^{1-\gamma}, g)$ so the lower bound of $(\Phi_\lambda^2(\lambda-p))^w$ is $\geqq -C\varepsilon$.

With $\Psi_\lambda=\psi(3p/\mu)$ we obtain from (3.25)

$$\Psi_\lambda^w F_\lambda^w \Psi_\lambda^w \geqq \Psi_\lambda^w R_\lambda^w \Psi_\lambda^w. \tag{3.26}$$

Here Ψ_λ is bounded in $S((p/\lambda)^\nu, G_\lambda)$ and R is bounded in $S(F_\lambda H_\lambda^N, G_\lambda)$ for some large N which we can choose as we like, so the symbol of the right hand side of (3.26) is bounded in $S(m, G_\lambda)$ with

$$m = F_\lambda H_\lambda^N \left(\frac{p}{\lambda}\right)^\nu \leqq C(p+\lambda)p^{-N\Gamma}\left(\frac{p}{\lambda}\right)^\nu < C'\lambda^{1-\nu}$$

if N is chosen so large that $N\Gamma > \nu+1$. Hence the norm of the operator on the right hand side of (3.26) is $O(\varepsilon)$. The symbol of

$$\Psi_\lambda^w F_\lambda^w \Psi_\lambda^w - (F_\lambda \Psi_\lambda^2)^w$$

is bounded in $S(m, G_\lambda)$ if $m = F_\lambda H_\lambda^2$, and outside $\operatorname{supp} d\Psi_\lambda$ it is bounded in the symbol space with $m = (p/\lambda)^\nu F_\lambda H_\lambda^N$ for any N, so the arguments above are applicable there. In $\operatorname{supp} d\Psi_\lambda$ we have $\mu/3 \leqq p \leqq 2\mu/3$ so H_λ is equivalent to h and F_λ to λ there. Hence $F_\lambda H_\lambda^2 < \lambda^{1-2\gamma}$ there so the symbol in question is actually bounded in $S(\lambda^{1-2\gamma}, G_\lambda)$. Hence the lower bound of $(\Psi_\lambda^2 F_\lambda)^w$ is $\geqq -C\varepsilon$. The same is true of $(\Phi_\lambda^2 F_\lambda)^w$ and adding we conclude that the lower bound of F_λ^w is $\geqq -C\varepsilon$, which completes the proof.

4. The eigenvalue estimate

Let $p \in S(p, g)$ and assume as in Section 3 that

$$h(x, \xi)^2 = \sup \frac{g_{x,\xi}}{g_{x,\xi}^\sigma} \leqq Cp(x, \xi)^{-2\gamma}, \tag{4.1}$$

$$1+|x|+|\xi| \leqq Cp(x, \xi)^N. \tag{4.2}$$

Denote the number of eigenvalues $\leqq \lambda$ of $p^w(x, D)$ by $N(\lambda)$, and let

$$W(\lambda) = (2\pi)^{-n} \iint_{p(x,\xi) \leqq \lambda} dx\, d\xi$$

be the expected approximation. We shall prove

Theorem 4.1. *If* $0<\delta<2\gamma/3$ *then one can find* C_δ *so that for large enough* λ

$$|N(\lambda) - W(\lambda)| \leqq C_\delta(W(\lambda+\lambda^{1-\delta}) - W(\lambda-\lambda^{1-\delta})). \tag{4.3}$$

Before the proof it is useful to make some preliminary observations on the measure of the set where $|p/\lambda - 1| < \lambda^{-\delta}$ which occurs in the right hand side of (4.3). With appropriate symplectic coordinates y, η centered at a point where $p=\lambda$, the metric form is bounded by $h(|y|^2+|\eta|^2)$ and $|p/\lambda - 1| < \lambda^{-\delta}$ if for a suitable $c>0$

$$h^{1/2}|y_1| + h(|y|^2+|\eta|^2) < c\lambda^{-\delta}.$$

The volume of this set (which is a symplectic invariant) is proportional to

$$(h\lambda^{\delta})^{1-n}\lambda^{-3\delta/2}h^{-1} > C'\lambda^{(\gamma-\delta)(n-1)+\gamma-3\delta/2}$$

so it tends to ∞ with λ.

Proof of Theorem 4.1. As explained in Section 3 we choose an approximation χ_λ to $H(\lambda-.)$, where H is the Heaviside function, so that $\chi_\lambda=1$ in $(-\infty, \lambda)$ and $\chi_\lambda=0$ in $(\lambda+\lambda^{1-\delta}, \infty)$. If $e_\lambda=\chi_\lambda(p)$ then

$$\operatorname{Tr} e_\lambda^w = (2\pi)^{-n}\iint e_\lambda(x, \xi)\,dx\,d\xi$$

lies between $W(\lambda)$ and $W(\lambda+\lambda^{1-\delta})$. By (3.22) we have

$$\|e_\lambda^w(1-e_\lambda^w)\|_{\mathrm{Tr}} \leqq C_1\big(W(\lambda+\lambda^{1-\delta})-W(\lambda-\lambda^{1-\delta})\big)+C_2,$$

and we observed after the statement of the theorem that the second term on the right is much smaller than the first one for large λ so it can be dropped. By Lemma 3.10 we have

$$e_\lambda^w(p^w-\lambda)e_\lambda^w \leqq C\lambda^{1-\delta}, \quad (1-e_\lambda^w)(p^w-\lambda)(1-e_\lambda^w) > -C\lambda^{1-\delta},$$

so Lemma 2.1 gives

$$N(\lambda+4C\lambda^{1-\delta}) \geqq W(\lambda)-C_3\big(W(\lambda+\lambda^{1-\delta})-W(\lambda-\lambda^{1-\delta})\big),$$
$$N(\lambda-4C\lambda^{1-\delta}) \leqq W(\lambda)+C_3\big(W(\lambda+\lambda^{1-\delta})-W(\lambda-\lambda^{1-\delta})\big).$$

If we introduce $\mu=\lambda\pm 4C\lambda^{1-\delta}$ as new variable, we conclude that

$$|N(\mu)-W(\mu)| \leqq C_1\big(W(\mu+C_2\mu^{1-\delta})-W(\mu-C_2\mu^{1-\delta})\big)$$

for some new constants C_1, C_2. If we replace δ by a smaller number, the constant C_2 may be omitted and the theorem is proved.

We shall now compare Theorem 4.1 with the results proved in [5]. In our notation the hypotheses made in [5] are first of all that $p\in S(p, g)$ where for some $\varrho'>0$

$$g = p^{-2\varrho'}(|dx|^2+|d\xi|^2).$$

In addition (4.2) is assumed and a hypothesis is made which implies

$$W(\lambda+\mu)-W(\lambda) \leqq CW(\lambda)\mu\lambda^{a-1}, \quad 0<\mu<\lambda^{1-a},$$

where $0\leqq a<\varrho'$. Then the conclusion in [5] is that for $\delta<\varrho'$

$$|N(\lambda)-W(\lambda)| \leqq C_\delta W(\lambda)\lambda^{a-\delta}. \tag{4.4}$$

If we apply Theorem 4.1 in this situation we can take $\gamma=2\varrho'$ and $\delta<4\varrho'/3$. With $\mu=\lambda^{1-\delta}$ we have $\mu\ll\lambda^{1-a}$ if $\delta>a$. If just $a<4\varrho'/3$ we can choose δ with $a<\delta<4\varrho'/3$ and obtain from (4.3) that (4.4) is valid. Thus we have replaced the hypothesis $a<\varrho'$ by $a<4\varrho'/3$ and improved the error estimate (4.4) so that it holds for any $\delta<4\varrho'/3$ instead of any $\delta<\varrho'$.

Let us consider as an example the harmonic oscillator corresponding to

$$p(x, \xi) = \sum_1^n (x_j^2+\xi_j^2)+1$$

(we have added 1 to make Theorem 4.1 applicable). Then $\varrho'=1/2$ and $a=0$ so our error estimate means that

$$\Big|N(\lambda)-(2\pi)^{-n}\iint_{p<\lambda} dx\, d\xi\Big| \leqq C_\delta \lambda^{n-\delta} \quad \text{when} \quad \delta<2/3. \tag{4.5}$$

Now the eigenvalues of p^w are

$$1+\sum_1^n (2\alpha_j+1) = n+1+2\sum_1^n \alpha_j$$

where α_j are non-negative integers. Thus $N(\lambda)$ jumps just at the integers congruent to $n+1 \bmod 2$, and when $\lambda=n+1+2\mu$, μ an integer, then

$$N(\lambda) = \#\{(\alpha_1, \dots, \alpha_n);\ \alpha_j \geqq 0,\ \ \sum \alpha_j \leqq \mu\} = \binom{\mu+n}{n}$$
$$= \frac{(\mu+n)\dots(\mu+1)}{n!} = \frac{\mu^n}{n!}+O(\mu^{n-1}).$$

On the other hand

$$(2\pi)^{-n}\iint_{p<\lambda+1} dx\, d\xi = \lambda^n (2\pi)^{-n}\iint_{|x|^2+|\xi|^2<1} dx\, d\xi = \frac{\lambda^n}{2^n n!}$$

so (4.5) is actually valid when $\delta=1$ but for no larger value of δ, because of the jumps in $N(\lambda)$.

If p has an asymptotic expansion in homogeneous terms

$$p \sim p_m(x, \xi)+p_{m-1}(x, \xi)+\dots$$

and $p_m \neq 0$ except at 0, then we can take $\gamma=2/m$ and obtain from (4.3)

$$\Big|N(\lambda)-(2\pi)^{-n}\lambda^{2n/m}\Big(\iint_{p_m(x,\xi)<1} dx\, d\xi-\lambda^{-1/m}\langle p_{m-1}, \delta(p_m-1)\rangle\Big)\Big| \leqq C_\delta \lambda^{2n/m-\delta} \tag{4.6}$$

for every $\delta<4/3m$. Here $\delta(p_m-1)=dS/|p_m'|$ where dS is the Euclidean element of area on the surface $p_m=1$, so the second term in the parenthesis vanishes if p is a polynomial in (x, ξ). The example above raises the question if (4.6) is always valid for $\delta=2/m$. This is made plausible by the analogy with pseudo-differential operators on a compact manifold where the methods used in this paper should give

$$\Big|N(\lambda)-(2\pi)^{-n}\lambda^{n/m}\iint_{p_m(x,\xi)<1} dx\, d\xi\Big| \leqq C_\delta \lambda^{n/m-\delta}$$

for every $\delta<2/3m$ whereas Fourier integral operator methods give this result for $\delta=1/m$. However that may be, we have still thought it of some interest to examine the scope of the methods introduced in [5] when combined with the technical refinements of the calculus given in [2].

Theorem 4.1 may be given a more general form containing the "quasi-claassical asymptotics" of Šubin [4, Appendix 2]. These concern the operator $p^w(\varepsilon x, \varepsilon D)$ or the unitarily equivalent operator $p^w(x, \varepsilon^2 D)$ where $0<\varepsilon<1$. Set $p_\varepsilon(x,\xi)=p(\varepsilon x, \varepsilon\xi)$. Then

$$p_\varepsilon \in S(p_\varepsilon, {}^\varepsilon g)$$

uniformly for $0<\varepsilon<1$ if

$${}^\varepsilon g_{x,\xi}(y,\eta) = g_{\varepsilon x, \varepsilon\xi}(\varepsilon y, \varepsilon\eta).$$

This is uniformly σ-temperate (see [2, Section 7]), and by [2, (7.7)′]

$$\frac{{}^\varepsilon g_{x,\xi}}{{}^\varepsilon g^\sigma_{x,\xi}} \leqq h(\varepsilon x, \varepsilon\xi)^2 \varepsilon^4 \quad \text{if} \quad h(x,\xi)^2 = \sup \frac{g_{x,\xi}}{g^\sigma_{x,\xi}}.$$

From (4.1) we therefore obtain

$$\frac{{}^\varepsilon g_{x,\xi}}{{}^\varepsilon g^\sigma_{x,\xi}} \leqq \left(p_\varepsilon(x,\xi)\varepsilon^{-2/\gamma}\right)^{-2\gamma}$$

so (4.1) is also satisfied by the new metric and the symbol $p_\varepsilon(x,\xi)\varepsilon^{-2/\gamma}$. For large N we have

$$Cp_\varepsilon(x,\xi)\varepsilon^{-2/\gamma} \geqq (1+\varepsilon|x|+\varepsilon|\xi|)^{1/N}\varepsilon^{-2/\gamma} \geqq (1+|x|+|\xi|)^{1/N}$$

so (4.2) is also uniformly satisfied. It follows that the conclusion of Theorem 4.1 is valid uniformly in ε.

Let $N_\varepsilon(\lambda)$ be the number of eigenvalues $\leqq\lambda$ of $p^w(\varepsilon x, \varepsilon D)$. Then $p^w(\varepsilon x, \varepsilon D)\varepsilon^{-2/\gamma}$ has $N_\varepsilon(\varepsilon^{2/\gamma}\lambda)$ eigenvalues $\leqq\lambda$, and

$$(2\pi)^{-n}\iint_{p(\varepsilon x, \varepsilon\xi)\varepsilon^{-2/\gamma}<\lambda} dx\, d\xi = \varepsilon^{-2n}W(\varepsilon^{2/\gamma}\lambda).$$

When $0<\delta<2\gamma/3$ we obtain from Theorem 4.1 for large λ, $0<\varepsilon<1$,

$$|N_\varepsilon(\varepsilon^{2/\gamma}\lambda)-\varepsilon^{-2n}W(\varepsilon^{2/\gamma}\lambda)| \leqq C_\delta\varepsilon^{-2n}\left(W(\varepsilon^{2/\gamma}(\lambda+\lambda^{1-\delta}))-W(\varepsilon^{2/\gamma}(\lambda-\lambda^{1-\delta}))\right).$$

Changing notations we obtain

$$(4.3)' \qquad |N_\varepsilon(\lambda)-\varepsilon^{-2n}W(\lambda)| \leqq C_\delta\varepsilon^{-2n}\left(W(\lambda(1+\varepsilon^{2\delta/\gamma}\lambda^{-\delta}))-W(\lambda(1-\varepsilon^{2\delta/\gamma}\lambda^{-\delta}))\right)$$

provided that $\lambda\varepsilon^{-2/\gamma}$ is large enough.

For a fixed λ which is not a critical value of p the parenthesis on the right hand side of (4.3)′ is $O(\varepsilon^{2\delta/\gamma})$ so it follows that as $\varepsilon\to 0$

$$(4.3)'' \qquad |N_\varepsilon(\lambda)-\varepsilon^{-2n}W(\lambda)| \leqq C_\theta\varepsilon^{\theta-2n}$$

for every $\theta<4/3$. When $\theta<1$ this is Proposition 2.2 of Šubin [3, p. 241].

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