On the regularity of difference schemes

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1. Introduction

In this paper general elliptic difference schemes in Lipschitz regions with Dirichlet boundary conditions are studied. It is shown that the inverse of the difference operator is a uniformly bounded mapping from the analogue of the Sobolev space $H^{\Theta-m}(\Omega)$ onto the analogue of $H_0^{\Theta+m}(\Omega)$ for $|\Theta|<1/2$ ($2m$: order of the differential operator). This property is important for the convergence proof of multi-grid iterations applied to difference schemes, since it is possible to obtain optimal error estimates that are similar to the estimates known from Galerkin approximations. The result is also useful for proving ℓ_∞ stability of difference operators.

Let $\mathbf{Z}$ be the set of all integers, while $\mathbf{Z}_+$ contains all non-negative integers. Following norms will be used for multi-indices $\alpha=(\alpha_1, \dots, \alpha_d)\in\mathbf{Z}_+^d$ and $\nu=(\nu_1, \dots, \nu_d)\in\mathbf{Z}^d$:

$$|\alpha| = \alpha_1+\dots+\alpha_d, \quad \|\nu\| = (\nu_1^2+\dots+\nu_d^2)^{1/2} \quad (\alpha\in\mathbf{Z}_+^d,\ \nu\in\mathbf{Z}^d).$$

We define the differential operator

$$D^\alpha = i^{-|\alpha|}(\partial/\partial x_1)^{\alpha_1}\dots(\partial/\partial x_d)^{\alpha_d} \quad (\alpha\in\mathbf{Z}_+^d).$$

Let Ω be a domain in $\mathbf{R}^d$ and consider the boundary value problem

$$(1.1) \qquad Lu = f, \quad u\in H_0^m(\Omega),$$

where L is the differential operator

$$(Lu)(x) = \sum_{|\alpha|,|\beta|\leqq m} D^\alpha a_{\alpha\beta}(x) D^\beta u(x) \quad (x\in\Omega)$$

of order $2m$. For the notation of the Sobolev spaces $H^s(\Omega)$ and $H_0^s(\Omega)$ compare, e.g., [11]. The boundary values are given by $u\in H_0^m(\Omega)$: $(\partial/\partial n)^\nu u(x)=0$ ($0\leqq\nu<m$, $x\in\partial\Omega:=$boundary of Ω, $\partial/\partial n$: normal derivative).

Introduce the regular grid $G_h\subset R^d$ with size h and the grid $\Omega_h\subset G_h$ of Ω by

$$G_h = \{x = \nu h:\ \nu\in\mathbf{Z}^d\}, \quad \Omega_h = G_h\cap\Omega \quad (h\in(0, h_0]).$$

The boundary value problem is discretized by some difference scheme

(1.2) $$L_h u_h = f_h \quad \text{at} \quad x \in \Omega_h, \quad u_h = 0 \quad \text{at} \quad x \in G_h \setminus \Omega_h.$$

For the formulation of ellipticity, consistency and convergence we refer, e.g. to Thomée [17], Thomée and Westergren [18], Bramble and Thomée [2] and Stummel [16].

Here we are interested in the *'regularity'* of L_h. Under suitable conditions the inverse L^{-1} of the differential operator maps $H^0(\Omega) = L^2(\Omega)$ onto $H^{2m}(\Omega) \cap H_0^m(\Omega)$. Let $\mathscr{H}^s(\Omega_h)$ and $\mathscr{H}_0^s(\Omega_h)$ be discrete analogues of $H^s(\Omega)$ and $H_0^s(\Omega)$. The counterpart of the property mentioned above is

$$L_h^{-1}: \mathscr{H}^0(\Omega_h) \to \mathscr{H}^{2m}(\Omega_h) \cap \mathscr{H}_0^m(\Omega_h) \text{ bounded independently of } h.$$

This is proved even for nonlinear problems in the case where Ω is a rectangle or parallelepiped (cf., e.g., D'jakonov [4], Guilinger [6], Lapin [10]). Dryja [5] showed the same result for a convex grid. (Note that in general Ω_h is not convex even if Ω is).

It is well-known that $L^{-1}: L^2(\Omega) \to H^{2m}(\Omega)$ is not valid in the case of more general regions. Nevertheless, $L^{-1}: H^{\Theta-m}(\Omega) \to H_0^{\Theta+m}(\Omega)$ ($|\theta| < 1/2$) is proved by Nečas [13] for Lipschitz regions Ω. In this paper we show the analogous result $L_h^{-1}: \mathscr{H}^{\Theta-m}(\Omega_h) \to \mathscr{H}_0^{\Theta+m}(\Omega_h)$ for the difference scheme (1.2) in a Lipschitz region Ω. An important application of this result is the convergence proof of multi-grid iterations for difference equations as mentioned in Section 4.2.

It is to be noted that the regularity of L_h is different from the *interior regularity* studied, e.g., by Thomée [19], Thomée and Westergren [18], Shreve [15].

In Section 2.1 we recall the result of Nečas [13] for the operator L of (1.1). The difference scheme is introduced in Section 2.2. The discrete analogues of the Sobolev spaces $H^s(\mathbf{R}^d)$ and $H_0^s(\Omega)$ and their norms are explained in Section 2.3. The main theorem about the regularity of L_h is contained in Section 2.4. In this theorem L_h is assumed to have smooth coefficients. In practice difference schemes are often used with quite different discretizations at points near the boundary. In Section 2.5 we discuss a discretization of this type. It is shown that regularity can be proved for this scheme, too.

The proof of Theorem 2.2 is given in Section 3. Section 3.1 contains preparing lemmata. A convolution operator discussed in Section 3.2 is used in Section 3.3 for the construction of an operator R_ϑ. By means of this operator the proof is completed in Section 3.4.

Section 4 contains applications of our results. An optimal error estimate is proved in Section 4.1. In the following subsection we explain the application to the convergence proof for multi-grid methods. ℓ_∞ stability of difference operators is discussed in Section 4.3.

2. Regularity of the Difference Operator

2.1. Regularity of the Differential Operator

Before considering the discrete problem (1.2) we recall the properties of the differential operator L. The *ellipticity* of L is expressed by

$$\text{(2.1)}\quad \operatorname{Re}\sum\nolimits_{|\alpha|=|\beta|=m} a_{\alpha\beta}(x)\xi^{\alpha+\beta} \geqq \varepsilon\|\xi\|^{2m} \quad \text{for all} \quad x\in\Omega, \quad \xi\in\mathbf{R}^d \quad (\varepsilon>0),$$

where $\xi^\alpha=\xi_1^{\alpha_1}\dots\xi_d^{\alpha_d}$ and $\|\xi\|^2=\xi_1^2+\dots+\xi_d^2$. Here and in the following ε and C denote generic constants.

Let $C^n(\bar\Omega)$ $(n\in\mathbf{Z}_+)$ be the set of functions u with continuous and uniformly bounded derivatives $D^\alpha u$ $(|\alpha|\leqq n)$ on the closure of Ω, while the n^{th} derivatives of the functions of $C^{n+\varkappa}(\bar\Omega)$ $(n\in\mathbf{Z}_+,\ 0<\varkappa<1)$ are uniformly Hölder continuous with exponent $\varkappa$.

$\Omega\subset\mathbf{R}^d$ is called a *Lipschitz region* if there is $\varepsilon>0$ such that for all spheres $S_\varepsilon(x_0)$ with midpoint $x_0\in\partial\Omega$ and radius ε the following property holds: There are local coordinates $(y_1,\dots,y_d)=(y',y_d)=U\cdot(x-x_0)$ (U: matrix with $\det(U)=1$) and a Lipschitz continuous function α: $\mathbf{R}^{d-1}\to\mathbf{R}$ such that

$$S_\varepsilon(x_0)\cap\partial\Omega=\{x_0+U^{-1}y\colon\ y=(y',\alpha(y'))\}\cap S_\varepsilon(x_0).$$

The Lipschitz constant of α must be independent of $x_0\in\partial\Omega$.

The discrete analogue of the following theorem is desired.

Theorem 2.1. (Nečas [13]). *Let* $\Theta\in(-1/2,1/2)$. *Assume* (2.1), $a_{\alpha\beta}\in L^\infty(\Omega)$, $a_{\alpha\beta}$ *real if* $|\alpha|=|\beta|=m$,

$$a_{\alpha\beta}\in C^\varkappa(\bar\Omega)\quad \textit{if}\quad \begin{Bmatrix}|\alpha|=m \ \textit{and}\ \Theta>0\\ |\alpha|=|\beta|=m\\ |\beta|=m\ \textit{and}\ \Theta<0\end{Bmatrix},\quad \textit{where}\quad \varkappa>|\Theta|>0\quad \textit{or}\quad \varkappa\geqq\Theta=0.$$

Furthermore, Ω is assumed a Lipschitz region. Then $(L+\lambda I)^{-1}\colon H^{\Theta-m}(\Omega)\to H_0^{\Theta+m}(\Omega)$ *is bounded for suitable* $\lambda\in\mathbf{R}$. *If* $a_{\alpha\beta}$ $(|\alpha|=|\beta|=m)$ *is complex, the same result holds as long as* $|\Theta|$ *is sufficiently small.*

For the regularity of L in the case of smooth coefficients and a smooth boundary $\partial\Omega$ compare, e.g., Lions and Magenes [11].

2.2. Difference Scheme

Let $h\in I_0:=(0,h_0]$ be a fixed grid size and define the grids G_h and Ω_h as in Section 1. Grid functions of G_h are $u=(u_\nu)_{\nu\in\mathbf{Z}^d}$, where $u_\nu=u(\nu h)$. In the following the subscript h of u_h in (1.2) is omitted. Grid functions u of Ω_h are identified with $\tilde u$ defined on G_h by $\tilde u_\nu=u_\nu$ if $\nu h\in Q_h$, $\tilde u_\nu=0$ otherwise.

The *translation* operator T_h^α $(\alpha \in \mathbf{Z}^d)$ is defined by

$$(T_h^\alpha u)(x) = u(x+\alpha h) \quad (x \in G_h) \quad \text{or} \quad (T_h^\alpha u)_\nu = u_{\nu+\alpha}.$$

The discrete analogues of $\partial/\partial x_j$ and $i^{|\alpha|}D^\alpha$ are:

$$\partial_{h,j} = h^{-1}(I - T_h^{-e_j}), \quad e_j = j^{\text{th}} \text{ unit vector } \in \mathbf{Z}^d \quad (1 \leqq j \leqq d),$$
$$\partial_h^\alpha = \partial_{h,1}^{\alpha_1} \dots \partial_{h,d}^{\alpha_d} \qquad (\alpha \in \mathbf{Z}_+^d).$$

A general difference scheme discretizing L has the representation

$$L_h = h^{-2m} \sum_{\gamma \in \mathbf{Z}^d} b_\gamma(\cdot, h) T_h^\gamma, \tag{2.2'}$$

where $(b_\gamma(\cdot, h)v)(x) = b_\gamma(x, h)v(x)$ $(x \in G_h)$ and $b_\gamma \equiv 0$ except of a finite number of subscripts. The scheme (2.2′) can be rewritten in the form

$$L_h = \sum_{|\alpha|, |\beta| \leqq m} \sum_{\gamma, \delta \in \mathbf{Z}^d} \partial_h^\alpha T_h^\gamma c_{\alpha\beta\gamma\delta}(\cdot, h) T_h^\delta \partial_h^\beta, \tag{2.2}$$

where again $c_{\alpha\beta\gamma\delta}$ vanishes for almost all subscripts. The relationship of (2.2′) and (2.2) is discussed in [18]. For the formulation of consistency by means of $a_{\alpha\beta}$ and $c_{\alpha\beta\gamma\delta}$ compare [18, 19], too.

L_h is called *elliptic* if (2.3) holds (cf. [18, Lemma 2.3]):

$$\operatorname{Re} p(x, \xi) \geqq \varepsilon \left[\sum_{j=1}^d \sin^2(\xi_j/2)\right]^m \quad \text{for all} \quad x \in \mathbf{R}^d, \ \xi \in Q = [-\pi, \pi]^d \subset \mathbf{R}^d, \tag{2.3}$$

where

$$p(x, \xi) = \sum_{|\alpha|=|\beta|=m} \sum_{\gamma, \delta} c_{\alpha\beta\gamma\delta}(x, 0) e^{-i\xi\cdot(\gamma+\delta)} \prod_{j=1}^d [1 - e^{i\xi_j}]^{\alpha_j + \beta_j}.$$

Example 2.1. Let $d=1$, $m=1$, $Lu(x) = -[a(x)u'(x)]' + c(x)u(x)$, i.e. $a_{00}(x) = c(x)$, $a_{11}(x) = a(x)$, $a_{\alpha\beta} = 0$ otherwise. The usual discretization is

$$(L_h u)(x) = -h^{-2}\{a(x+h/2)[u(x+h) - u(x)]$$
$$- a(x-h/2)[u(x) - u(x-h)]\} + c(x)u(x).$$

Hence, (2.2) holds with $c_{0000}(x, h) = c(x)$, $c_{1101}(x, h) = -a(x+h/2)$, $c_{\alpha\beta\gamma\delta} = 0$ otherwise. Since

$$p(x, \xi) = -a(x)e^{-i\xi}(1 - e^{i\xi})^2 = 4a(x)\sin^2(\xi/2),$$

(2.3) is valid with $\varepsilon = \inf\{a(x) \colon x \in G_h\} > 0$. The generalization to $L = -\nabla \cdot a(x)\nabla + c(x)$ is obvious.

Although we consider the discrete problem (1.2) with homogeneous boundary values, the results of this paper hold for the *inhomogeneous* problem

$$(L_h v)_\nu = g_\nu \quad \text{at} \quad \nu h \in \Omega_h, \quad v_\nu = w_\nu \quad \text{at} \quad \nu h \in G_h \setminus \Omega_h,$$

too. $w \in \mathscr{H}^{\Theta+m}$ and $g \in \mathscr{H}_0^{\Theta-m}$ yield $f := g - L_h w \in \mathscr{H}_0^{\Theta-m}$. Let u be the solution of (1.2) with f defined above. Theorem 2.2 will show $u \in \mathscr{H}_0^{\Theta+m}$. Hence

$$|v|_{m+\Theta} \leqq C \cdot (|w|_{m+\Theta} + |g|_{\Theta-m,0})$$

holds (for the notation compare the following section).

2.3. *Discrete Sobolev Spaces*

Throughout the paper we are only interested in the discrete spaces $\mathscr{H}^s = \mathscr{H}^s(G_h)$ and $\mathscr{H}_0^s = \mathscr{H}_0^s(\Omega_h)$. The discrete analogue of $L^2(\mathbf{R}^d)$ is $\mathscr{H}^0$ consisting of all complex-valued grid function with finite $|\cdot|_0$ norm:

$$|u|_0 = h^{d/2} \left[\sum_{\nu \in \mathbf{Z}^d} |u_\nu|^2 \right]^{1/2}.$$

$\mathscr{H}^0$ is a Hilbert space with the scalar product

$$(u, v) = h^d \sum_{\nu \in \mathbf{Z}^d} u_\nu \bar{v}_\nu.$$

Usually, this Hilbert space is denoted by ℓ_2. The discrete Fourier transform of u is the periodic function

$$\hat{u}(\xi) = \sum_{\nu \in \mathbf{Z}^d} u_\nu e^{i\nu\xi} \quad (\xi \in Q = [-\pi, \pi]^d \subset \mathbf{R}^d),$$

where $\nu\xi = \nu_1 \xi_1 + \ldots + \nu_d \xi_d$. Note that

$$|u|_0 = [h/(2\pi)]^{d/2} \|\hat{u}\|_{L^2(Q)}.$$

Let $\mathscr{H}^s$ $(s \in \mathbf{R})$ be all grid functions with $|u|_s < \infty$, where

$$|u|_s = [h/(2\pi)]^{d/2} \left\| \left[1 + h^{-2} \sum_{j=1}^d \sin^2(\xi_j/2) \right]^{s/2} \hat{u}(\xi) \right\|_{L^2(Q)}. \tag{2.4}$$

This is a natural definition since for $s = n = 1$ it coincides with the usual definition

$$|u|_n^* = \left[|u|_0^2 + \sum_{|\alpha|=n} |\partial_h^\alpha u|_0^2 \right]^{1/2} \quad (n \in \mathbf{Z}_+). \tag{2.4*a}$$

One easily verifies that $|\cdot|_n$ is equivalent to $|\cdot|_n^*$ for $n \in \mathbf{Z}_+$. A generalization to non-integers $s = n + t > 0$ is given by

$$\begin{aligned} |u|_s^* &= \left[|u|_n^{*2} + \sum_{|\alpha|=n} |||\partial_h^\alpha u|||_t^{*2} \right]^{1/2} \quad (s = n+t,\ n \in \mathbf{Z}_+,\ 0 < t < 1), \\ |||v|||_t^* &= h^{-t} \left[\sum_{\mu \in \mathbf{Z}^d,\, 0 < \|\mu\| \leqq \varepsilon/h} |(I - T_h^\mu) v|_0^2 / \|\mu\|^{d+2t} \right]^{1/2} \quad (\varepsilon > 0). \end{aligned} \tag{2.4*b}$$

The equivalence of $|\cdot|_s$ and $|\cdot|_s^*$ follows from the representation

$$(2\pi)^d |||v|||_t^{*2} = h^{d-2t} \int_Q |\hat{v}(\xi)|^2 \left[\sum_{0 < \|\mu\| \leqq \varepsilon/h} \sin^2(\mu\xi/2) / \|\mu\|^{d+2t} \right] d\xi.$$

For negative s, $|\cdot|_s$ is equivalent to

$$|u|_s^* = \sup \{ |(u, v)| / |v|_{-s}^* : 0 \neq v \in \mathscr{H}^{-s} \} \quad (s < 0).$$

The counterpart of $H_0^s(\Omega)$ $(s \geqq 0)$ is

$$\mathscr{H}_0^s = \{u \in \mathscr{H}^s\colon\ u_\nu = 0 \text{ if } \nu h \in G_h \setminus \Omega_h\} \subset \mathscr{H}^s \quad (s \geqq 0)$$

endowed with the norm of $\mathscr{H}^s$:

$$|\cdot|_{s,0} = |\cdot|_s \quad (s \geqq 0).$$

This choice corresponds to the fact that the extension of $u \in H_0^s(\Omega)$ by zero outside is a continuous mapping into $H^s(\mathbf{R}^d)$ except of $s-1/2 \in \mathbf{Z}$ (cf. [11, p. 60]). According to the embedding $\mathscr{H}_0^s \subset \mathscr{H}^s$, the dual space is $\mathscr{H}_0^{-s} \supset \mathscr{H}^{-s}$ with

$$|u|_{-s,0} = \sup\{|(u, v)|/|v|_s\colon\ 0 \neq v \in \mathscr{H}_0^s\} \quad (s \geqq 0).$$

Note that $|u|_{-s,0}=0$ implies $u_\nu=0$ only at $\nu h \in \Omega_h$. By this definition the operator $L_h\colon \mathscr{H}^m \to \mathscr{H}^{-m}$ of (2.2) can be considered as an operator from $\mathscr{H}_0^m$ into $\mathscr{H}_0^{-m}$, too; although $u \in \mathscr{H}_0^m$ does not imply support $(L_h u) \subset \Omega_h$. To avoid difficulties we sometimes use $L_{h,0}$ instead of L_h, where

$$(L_{h,0}u)_\nu = (L_h u)_\nu \quad \text{at} \quad \nu h \in \Omega_h, \quad (L_{h,0}u)_\nu = 0 \quad \text{at} \quad \nu h \in G_h \setminus \Omega_h.$$

As usual we define the operator norms

$$\|A\|_{\mathscr{H}^r \to \mathscr{H}^s} = \sup\{|Au|_s/|u|_r\colon\ 0 \neq u \in \mathscr{H}^r\}, \quad \|A\|_{\mathscr{H}_0^r \to \mathscr{H}_0^s} = \sup\{|Au|_{s,0}/|u|_{r,0}\colon\ 0 \neq u \in \mathscr{H}_0^r\}$$

of $A\colon \mathscr{H}^r \to \mathscr{H}^s$ or $A\colon \mathscr{H}_0^r \to \mathscr{H}_0^s$, respectively.

Instead of Lipschitz regions Ω we consider grids Ω_h with the following property.

Property C. Ω_h has *'property C_h'* if there are numbers $0 \leqq \varepsilon_0 < \infty$, $\varepsilon_1 > 0$, $\varepsilon_2 > 0$ and a function $n \in C^m(\mathbf{R}^d)$ mapping into the set $\{x \in \mathbf{R}^d\colon \|x\|=1\}$ of unit vectors such that $x \in G_h \setminus \Omega_h$ implies $C(x+h\varepsilon_0 n(x), n(x), \varepsilon_1, \varepsilon_2) \cap \Omega_h = \emptyset$ for the cone

$$C(z, n, \varepsilon_1, \varepsilon_2) = \{z+y \in \mathbf{R}^d\colon\ (y, n) \in [0, \varepsilon_1],\ \|y-(y, n)n\| \leqq \varepsilon_2 \cdot (y, n)\}$$

with axis n and vertex at z. A set of grids $\{\Omega_h\}_{h \in I_0}$ with $I_0=(0, h_0]$ has 'property C' if all Ω_h's have 'property C_h' with the same constants ε_0, ε_1, ε_2 and functions $n(x)$ such that $|D^\alpha n(x)|$ $(x \in \mathbf{R}^d,\ |\alpha| \leqq m)$ is uniformly bounded.

The following note shows that 'property C' is a natural analogue of a Lipschitz region Ω.

Note 2.1. If Ω is a Lipschitz region, $\{\Omega_h\}_{h \in I_0}$ $(I_0=(0, h_0],\ \Omega_h=G_h \cap \Omega)$ has 'property C'.

The following lemma is the discrete counterpart of the interpolation property $H_0^s(\Omega)=[H_0^m(\Omega), H^0(\Omega)]_{s/m}$ $(s-1/2 \notin \mathbf{Z}$, cf. [11, p. 64]).

Lemma 2.1. *Let Ω have 'property C'. Define*

$$L_h = \left[I - \sum_{j=1}^d T_h^{e_j}(\partial_{h,j})^2\right]^m$$

and denote the restriction of L_h on $\mathscr{H}_0^m$ by $L_{h,0}\colon \mathscr{H}_0^m \to \mathscr{H}_0^{-m}$. Then the norms $|u|_{s,0}$

and $|L_{h,0}^{s/(2m)}u|_{0,0}$ $(|s|\leqq m)$ *are equivalent (uniformly with respect to* $h\in I_0=(0,h_0]$ *and* $|s|\leqq m)$, *i.e.*

$$\frac{1}{C}|u|_{s,0}\leqq |L_{h,0}^{s/(2m)}u|_{0,0}\leqq C|u|_{s,0}\quad (-m\leqq s\leqq m,\ u\in\mathscr{H}_0^s).$$

The proofs of this and the next lemma are given in Section 3.5.

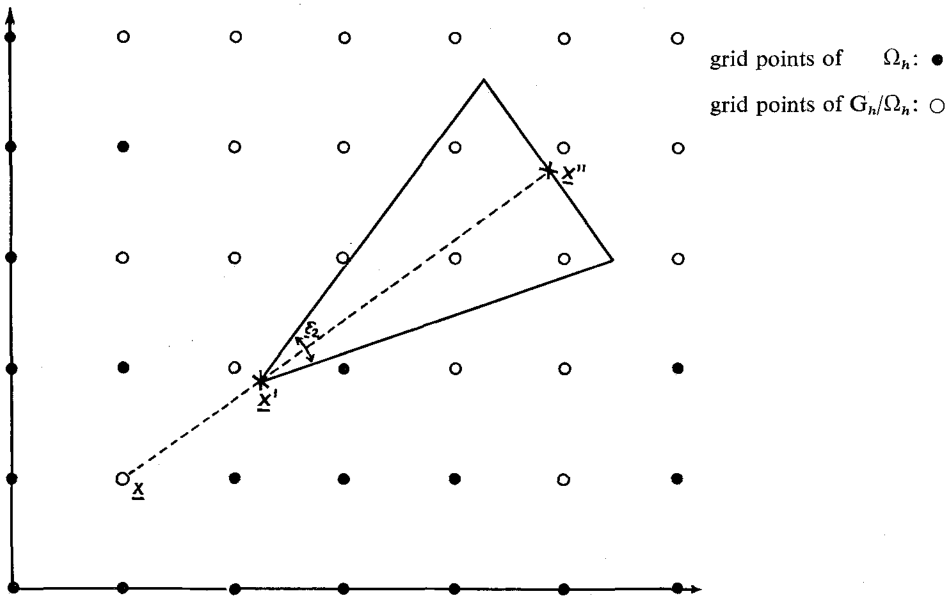

Fig. 1. Cone $C(x', n(x), \varepsilon_1, \varepsilon_2)$ with $x'=x+h\varepsilon_0 n(x)$, $x''=x'+\varepsilon_1 n(x)$

Lemma 2.2. i) *Assume*

$$\{x\in\Omega_h\colon \text{distance}(x, G_h\setminus\Omega_h)\geqq Ch\}\subset\Omega_h'\subset\{x\in G_h\colon \text{distance}(x,\Omega_h)\leqq Ch\}$$

and let $\{\Omega_h\}_{h\in I_0}$ *have 'property C'. Then* $\{\Omega_h'\}_{h\in I_0}$ *has 'property C', too.*

ii) *Let* $\{\Omega_h\}_{h\in I_0}$ *have 'property C' and assume* $\Gamma_h\subset\{x\in\Omega_h\colon \text{distance}(x, G_h\setminus\Omega_h)\leqq Ch\}$. *Define the restriction* γ *by* $(\gamma u)_\nu=u_\nu$ *if* $\nu h\in\Gamma_h$, $(\gamma u)_\nu=0$ *otherwise. Then the following estimate is valid with* C' *independent of* u, s, t, *and* h:

$$|u|_{s,0}\leqq C' h^{t-s}|u|_{t,0}\quad \textit{for}\quad s,t\in[-2m,2m],\quad h\in I_0.$$

2.4. Theorem on the Regularity of a Difference Operator

A difference scheme L_h (more precisely: $L_{h,0}$; cf. Section 2.3) is called stable with respect to $\mathscr{H}_0^0$ if the inverse mapping $L_{h,0}^{-1}\colon \mathscr{H}_0^0\to\mathscr{H}_0^0$ is bounded independently of h:

$$\|L_{h,0}^{-1}\|_{\mathscr{H}_0^0\to\mathscr{H}_0^0}\leqq C\quad\text{for all}\quad h\in I_0.$$

The main result of this paper is the following counterpart of Theorem 2.1. It will be proved in Section 3.

Theorem 2.2. *Let* $\Theta \in (-1/2, 1/2)$. *Assume the difference operator* L_h *of* (2.2) *to be elliptic* [cf. (2.3)] *and let* $\{\Omega_h\}_{h \in I_0}$ *have 'property C'. The coefficients of* L_h *must satisfy:*

$$|c_{\alpha\beta\gamma\delta}(x, h)| \leqq C \quad \text{for all} \quad x \in \mathbf{R}^d, \quad h \in I_0,$$

$$c_{\alpha\beta\gamma\delta} \in C^\varkappa(\mathbf{R}^d \times I_0) \quad \text{if} \quad \begin{Bmatrix} |\alpha| = m \ \text{and} \ \Theta > 0 \\ |\alpha| = |\beta| = m \\ |\beta| = m \ \text{and} \ \Theta < 0 \end{Bmatrix}, \quad \text{where} \quad \varkappa > |\Theta| > 0 \quad \text{or} \quad \varkappa \geqq \Theta = 0.$$

Finally, assume $L_{h,0}\colon \mathscr{H}_0^m \to \mathscr{H}_0^{-m}$ *to be stable with respect to* $\mathscr{H}_0^0$. *Then the estimate*

$$\|L_{h,0}^{-1}\|_{\mathscr{H}_0^{\Theta-m} \to \mathscr{H}_0^{\Theta+m}} \leqq C' \quad (h \in I_0, \ C' = C'(\theta)) \tag{2.5}$$

holds if the function $p(x, \xi)$ *of* (2.3) *is real-valued. If* $p(x, \xi)$ *is complex-valued,* (2.5) *holds for* $|\Theta| \in [0, \Theta_0)$ *with* $\Theta_0 \leqq 1/2$ *sufficiently small.*

2.5. Case of Irregular Discretizations near the Boundary

In the previous section we assumed that the scheme (2.2) has smooth coefficients for all $x \in \Omega_h$. Usually, the discretization is regular at interior points, while the difference equations at the points near the boundary depend on certain distances from the boundary.

In the following we give a criterion for $\mathscr{H}_0^{\Theta+m}$-solutions of irregular schemes. As application two examples are discussed.

Let L_h be the scheme (2.2) and consider the disturbed scheme

$$\tilde{L}_{h,0} = L_{h,0} + l_{h,0}.$$

In the following $L_h^{-1}\colon \mathscr{H}_0^0 \to \mathscr{H}_0^0$ is written instead of $L_{h,0}^{-1}$. By

$$\tilde{L}_h^{-1} = L_h^{-1}(I + l_h L_h^{-1})^{-1}$$

the inverse $\tilde{L}_h^{-1}$ satisfies (2.5) if $(I - l_h L_h^{-1})^{-1}\colon \mathscr{H}_0^{\Theta-m} \to \mathscr{H}_0^{\Theta-m}$ is bounded and if L_h fulfils (2.5).

Criterion 2.1. Assume

$$|(l_h u, v)| \leqq \varkappa |||u|||_m |||v|||_m, \quad \varkappa < 1, \quad \frac{1}{C}|u|_m \leqq |||u|||_m \leqq C|u|_m \quad (u, v \in \mathscr{H}_0^m),$$

where $|||u|||_m := [\operatorname{Re}(L_h u, u)]^{1/2}$ is required to be a norm on $\mathscr{H}_0^m$. Moreover,

$$\|l_h\|_{\mathscr{H}_0^{m+s} \to \mathscr{H}_0^{s-m}} \leqq C$$

must hold for some $s>0$. Let L_h satisfy the assumptions of Theorem 2.2 for some $\Theta>0$. Then

(2.5*) $$\|\tilde{L}_h^{-1}\|_{\mathscr{H}_0^{\Theta-m}\to\mathscr{H}_0^{\Theta+m}} \leqq C'$$

holds for sufficiently small $\Theta\in[0, \Theta_0)$, where Θ_0 does not depend on $h\in I_0$.

A similar criterion can be formulated for $\Theta\in(-\Theta_0, 0]$.

Proof. Let $|||\cdot|||_{-m}$ be the dual norm of $|||\cdot|||_m$. From

$$|||L_h^{-1}u|||_m = \mathrm{Re}(u, L_h^{-1}u)/|||L_h^{-1}u|||_m \leqq |||u|||_{-m}$$

it follows that

$$\|L_h^{-1}\|_{\mathscr{H}_0^{-m}\to\mathscr{H}_0^{m}} \leqq 1$$

if $\mathscr{H}_0^{\pm m}$ are endowed with $|||\cdot|||_{\pm m}$. The assumption on l_h yields

(2.6a) $$\|l_h\|_{\mathscr{H}_0^{m}\to\mathscr{H}_0^{-m}} \leqq \varkappa;$$

hence

(2.6b) $$\|l_h L_h^{-1}\|_{\mathscr{H}_0^{-m}\to\mathscr{H}_0^{-m}} \leqq \varkappa < 1.$$

Let Λ_h be the positive definite matrix $[(L_{h,0}+L_{h,0}^*)/2]^{1/(2m)}$. The equivalence of $|\cdot|_m$ and $|||\cdot|||_m$ implies the equivalence of $|u|_t$ and $|||u|||_t:=|\Lambda_h^t u|_0$ for all $t\in[0, m]$ by virtue of Lemma 2.1 and the interpolation theorem (cf. [9, Lemma 4]). We may assume $s<1/2$ (or $s<\Theta_0$, resp.) for $s>0$ appearing in Criterion 2.1. Otherwise, use again interpolation with (2.6a). Hence (2.5) yields

(2.6c) $$\|l_h L_h^{-1}\|_{\mathscr{H}_0^{s-m}\to\mathscr{H}_0^{s-m}} \leqq C.$$

By equivalence of $|\cdot|_t$ and $|||\cdot|||_t$ for $t=m-s$, the inequalities (2.6b,c) prove

(2.6d) $$\|\Lambda_h^{-t} l_h L_h^{-1}\Lambda_h^t\|_{\mathscr{H}_0^{0}\to\mathscr{H}_0^{0}} \leqq C(t)$$

at $t=m$ and $t=m-s$ with $C(m)=\varkappa$, $C(m-s)=C$. Interpolation yields (2.6d) for all $t\in[m-s, m]$ with $C(t)=\varkappa\cdot[C/\varkappa]^{(m-t)/s}$ (cf. Lemma 4 of [9]). Because of $\varkappa<1$ there exists $\Theta_0\in(0, s]$ such that $C(m-\Theta)<1$ for all $\Theta\in[0, \Theta_0)$. Hence

$$\|(I+l_h L_h^{-1})^{-1}\|_{\mathscr{H}_0^{\Theta-m}\to\mathscr{H}_0^{\Theta-m}} \leqq C'/[1-C(m-\Theta)] < \infty \quad (0\leqq\Theta\leqq\Theta_0)$$

and (2.5) yield the desired result. ∎

Example 2.2. Consider the discretization of $-\Delta u=f$ in $\Omega\subset\mathbf{R}^2$ and $u=0$ on $\partial\Omega$ by the usual five-point formula at interior points. Near the boundary interpolation is used (cf. Collatz [3, p. 344f]). $\Omega_h'\subset\Omega_h$ is the set of all 'interior' grid points, i.e., $x\pm he_j\in\bar{\Omega}$ holds for $x\in\Omega_h', j=1, 2$. We recall that $e_j\in\mathbf{Z}^d$ is the j^{th} unit vector. $\Gamma_h=\Omega_h\setminus\Omega_h'$ consists of the grid points near the boundary. For all $x\in\Gamma_h$ there are a direction $\pm e_j$ and a number $\varkappa\in[0, 1)$ such that $x\mp he_j\in\Omega'_h$, $x\pm\varkappa he_j\in\partial\Omega$, and $x\pm he_j\notin\bar{\Omega}$. At those points the five-point formula is replaced with interpolation:

$$u(x)=\varkappa\cdot u(x\mp he_j)/(1+\varkappa).$$

By these equations all grid points of Γ_h can be eliminated from the system of difference equations. It results a scheme $\tilde{L}_h$ that differs from the five-point formula at $x\in\Gamma'_h$, where $\Gamma'_h\subset\Omega'_h$ consists of the points $x\mp he_j$ involved by interpolation. Defining the spaces $\mathscr{H}_0^s$ by means of Ω'_h instead of Ω_h we shall prove in Section 3.6:

Note 2.2. Let Ω be a bounded Lipschitz region. The scheme of Example 2.2 satisfies (2.5*) for $0\leqq\Theta\leqq\Theta_0$ (Θ_0 sufficiently small).

Example 2.3 (Shortley—Weller scheme). Discretize the Poisson equation of Example 2.2 by the five-point formula at interior grid points $x\in\Omega'_h$ and by the Shortley—Weller scheme at points $x\in\Gamma_h$ near the boundary (cf. [14], [12, p. 203]). It is based on the discretization of $-(\partial/\partial x_j)^2u(x)$ $(x\in\Gamma_h)$ by

$$h^{-2}\left\{\frac{2}{\varkappa_1\varkappa_2}u(x)-\frac{2}{\varkappa_1(\varkappa_1+\varkappa_2)}u(x-\varkappa_1 he_j)-\frac{2}{\varkappa_2(\varkappa_1+\varkappa_2)}u(x+\varkappa_2 he_j)\right\},$$

where $\varkappa_j\in(0, 1]$ and either $x+(-1)^i\varkappa_i he_j\in\partial\Omega$ or $\varkappa_i=1$.

Note 2.3. Let Ω be a bounded Lipschitz region. The scheme of Example 2.3 satisfies (2.5*) for $0\leqq\Theta\leqq\Theta_0$ (Θ_0 sufficiently small). The proof is also given in Section 3.6.

3. Proofs

3.1. *Preparing Lemmata*

Theorem 2.2 is proved in the Sections 3.2 to 3.4. The crux of the proof is the construction of an operator R_ϑ: $\mathscr{H}^{t+\vartheta}\to\mathscr{H}^t$ such that support$(R_\vartheta u)\subset\Omega_h$ holds whenever support$(u)\subset\Omega_h$. It can be shown that the form $(L_h R_{2\vartheta}u, v)$ is $\mathscr{H}_0^{m+\vartheta}$-coercive (cf. Theorem 3.1). Then Theorem 2.2 is an immediate result. The properties of R_ϑ are proved in Section 3.3. R_ϑ is constructed by means of operators $R_{\vartheta s}^{\chi}$ introducted in Section 3.2.

This section contains five lemmata recalling standard techniques for treating variable coefficients.

In the sequel we shall use the symbol $\eta_t(u)$ $(u\in\mathscr{H}^t,\ t\geqq0)$ as an abbreviation of the following inequality: For all $\varepsilon>0$ there exists $C(\varepsilon)<\infty$ such that the term $\eta_t(u)$ can be estimated by

$$|\eta_t(u)|^2\leqq\varepsilon|u|_t^2+C(\varepsilon)|u|_0^2.$$

For $t=0$ the estimate degenerates into $|\eta_0(u)|\leqq C|u|_0$. It is well-known that

$$(3.1)\qquad |u|_s=\eta_t(u),\quad |u|_s|u|_t=\eta_t^2(u)\quad\text{if}\quad 0\leqq s<t.$$

The following lemmata are needed.

Lemma 3.1. $|gu|_t\leqq C\|g\|_{C^\varkappa(G_h)}|u|_t$ *holds for* $t\in\mathbf{R}$, $u\in\mathscr{H}^t$, $g\in C^\varkappa(\mathbf{R}^d)$ *with* $C=C(t)$ *and* $\varkappa>|t|\notin\mathbf{Z}$ *or* $\varkappa\geqq|t|\in\mathbf{Z}$. $\|g\|_{C^\varkappa(G_h)}$ *is the maximal value of* $|\partial_h^\alpha g(x)|$ $(|\alpha|\leqq\varkappa,\ x\in G_h)$ *and the corresponding Hölder constants.*

Proof. The estimate is valid for $t=0$ with $C=1$. Assume that the estimate holds for $0\leqq t-1\in\mathbf{Z}$ and note

$$\partial_h^\alpha(gu)=g\,\partial_h^\alpha u+\textstyle\sum_{|\beta|\leqq|\alpha|-1}g_\beta\,\partial_h^\beta u\quad(g_\beta\in C^{\varkappa-|\alpha|+|\beta|}(\mathbf{R}^d)).$$

Hence, $|\partial_h^\alpha(gu)|_0\leqq C\|g\|_{C^t(G_h)}|u|_t$ holds for $|\alpha|=t$, and (2.4*a) proves the inequality. If $0<t\notin\mathbf{Z}$ the result follows from (2.4*b) by the same argument. For negative t use

$$|(gu,v)|=|(u,\bar g v)|\leqq|u|_t|\bar g v|_{-t}\leqq C\|g\|_{C^\varkappa(G_h)}|u|_t|v|_{-t}.\qquad\blacksquare$$

Lemma 3.2. *Let* $g_k\in C^\varkappa(\mathbf{R}^d)$ $(k\in\mathbf{Z})$ *be a family of functions with the following properties:*

1) *For all* $x^*\in\mathbf{R}^d$ *and* $K>0$ *there is* $N(K)<\infty$ *such that at most* $N(K)$ *functions* g_k *do not vanish on the sphere* $S_K(x^*)=\{x:\ \|x-x^*\|\leqq K\}$.

2) *The diameters of the supports of* g_k *are uniformly bounded by* $\varrho<\infty$.

3) $\|g_k\|_{C^\varkappa(G_h)}\leqq C$ *for all* $k\in\mathbf{Z}$.

Then

$$\textstyle\sum_{k\in\mathbf{Z}}|g_k u|_t^2\leqq C|u|_t^2$$

holds for $u\in\mathscr{H}^t$, $0\leqq t<\varkappa$ *(or* $0\leqq t\leqq\varkappa\in\mathbf{Z}$*).*

Proof. There is a finite number of subsets $I_l\subset\mathbf{Z}$ $(l=1,\dots,L)$ such that $\bigcup_{l=1}^L I_l=\mathbf{Z}$ and that the supports of g_k $(k\in I_l)$ have a distance greater than $2\cdot\max(th,\varepsilon)$ [$\varepsilon>0$ from (2.4*b)]. Then (2.4*a, b) shows $\sum_{k\in I_l}|g_k u|_t^{*2}=|(\sum_{k\in I_l}g_k)u|_t^{*2}$. Hence $L<\infty$ and Lemma 3.1 yield the desired inequality. $\blacksquare$

Let $e_k\in C^\infty(\mathbf{R}^d)$ $(k\in\mathbf{Z})$ be a *partition of unity:*

$$\textstyle\sum_k e_k^2(x)=1\quad\text{for all}\quad x\in\mathbf{R}^d,$$

$$\|e_k\|_{C^t(G_h)}\leqq C(t)\quad\text{for all}\quad k\in\mathbf{Z}\quad\text{and all}\quad t\geqq 0.$$

Let U_k be the support of e_k and fix $x_k\in U_k$. It is required that

$$\varrho:=\sup\{\|x-x_k\|:\ x\in U_k,\ k\in\mathbf{Z}\}<\infty$$

and that all spheres $S_K(x^*)=\{x:\ \|x-x^*\|\leqq K\}$ $(K>0,\ x^*\in\mathbf{R}^d)$ have non-empty intersections with only $N=N(K)<\infty$ supports U_k. The magnitude ϱ will be chosen sufficiently small. We recall the following property of the partition $\{e_k\}$.

Lemma 3.3. $\frac{1}{C}|u|_t^2 \leqq \sum_{k\in\mathbf{Z}} |e_k u|_t^2 \leqq C|u|_t^2$ $(C=C(t, \{e_k\}))$ *for all* $u\in\mathscr{H}^t$, $t\geqq 0$.

Proof. The second inequality follows by Lemma 3.2 stated above. The first inequality holds with $C=1$ for $t=0$. Assume its validity for $0\leqq t-1\in\mathbf{Z}$. Let $|\alpha|=t$. Note that

$$e_k\,\partial_h^\alpha u-\partial_h^\alpha e_k u = \sum_{|\beta|\leqq t-1} g_{k\beta}\,\partial_h^\beta u \quad \text{for some} \quad g_{k\beta}\in C^\infty(\mathbf{R}^d);$$

hence,

$$|e_k\,\partial_h^\alpha u|_0^2 \leqq C\big[|e_k u|_t^2+\sum_{|\beta|\leqq t-1}|g_{k\beta}\,\partial_h^\beta u|_0^2\big].$$

Summation over $k\in\mathbf{Z}$ results in

$$|\partial_h^\alpha u|_0^2 = \sum_k |e_k\,\partial_h^\alpha u|_0^2 \leqq C'\big[\sum_k |e_k u|_t^2+\sum_{s=0}^{t-1}|u|_s^2\big]$$

by virtue of Lemma 3.2. Using (2.4*a) we obtain

$$|u|_t^2 \leqq C''\big[\sum_k |e_k u|_t^2+\sum_{s=0}^{t-1}|u|_s^2\big].$$

(3.1) shows

$$\sum_{s=0}^{t-1}|u|_s^2 \leqq [1/(2C'')]\,|u|_t^2+C'''\,|u|_0^2.$$

Together with

$$|u|_0^2 = \sum_k |e_k u|_0^2 \leqq \sum_k |e_k u|_t^2$$

the first inequality of Lemma 3.3 follows for $0\leqq t\in\mathbf{Z}$. For non-integers $t>0$ use the norm $|\cdot|_t^*$ and $|||v|||_\tau^{*2}-\sum_k |||e_k v|||_\tau^{*2}=\eta_\tau^2(v)$ [cf. (2.4*b); $0<\tau<1$]. ■

Lemma 3.4. *Let* $\sigma\in C^\varkappa(\mathbf{R}^d)$, *and* e_k, x_k, ϱ *as defined above. Then*

$$\big\|\sigma(x)-\sum_k e_k^2(x)\sigma(x_k)\big\|_{C^t(\mathbf{R}^d)} \leqq \varepsilon(\varrho)$$

holds for $0\leqq t<\varkappa$ *or* $0\leqq t\leqq\varkappa\in\mathbf{Z}$, *where* $\varepsilon(\varrho)\searrow 0$ *as* $\varrho\to 0$.

The proof is obvious. The proof of Lemma 3.3 shows the following result, too.

Lemma 3.5. $|e_k\partial_h^\alpha u-\partial_h^\alpha e_k u|_t\leqq C|u|_{t+|\alpha|-1}$ *for* $t\geqq 1$, $u\in\mathscr{H}^t$, $C=C(t,\alpha)$.

3.2. Operators $R_{\vartheta s}^\chi$

Now we start constructing operators $R_{\vartheta s}$ and $R_{\vartheta s}^\chi$. By means of the function χ the value $(R_{\vartheta s}^\chi u)_\nu$ depends on only a finite number of components $u_{\nu+\mu s}$. Let $\varepsilon_1>0$ be the number appearing in the definition of 'property C' and choose a real function $\chi(t)$ such that

(3.2) $\qquad \chi(t)\in C^\infty(\mathbf{R}),\quad \chi(t)=1 \quad \text{for} \quad t\leqq \varepsilon_1/2,\quad \chi(t)=0 \quad \text{for} \quad t\geqq\varepsilon_1.$

Let $s \in \mathbf{Z}^d$ and $\vartheta \in [0, 1)$ and define the operators $R_{\vartheta s}$ and $R^\chi_{\vartheta s}$ by

$$(R_{\vartheta s} u)_\nu = h^{-\vartheta} \sum_{\mu=0}^{\infty} e^{-\mu h} \binom{\vartheta}{\mu} (-1)^\mu u_{\nu+\mu s},$$

$$(R^\chi_{\vartheta s} u)_\nu = h^{-\vartheta} \sum_{\mu=0}^{\infty} e^{-\mu h} \binom{\vartheta}{\mu} (-1)^\mu \chi(\mu h \|s\|) u_{\nu+s\mu},$$

where

$$\binom{\vartheta}{0} = 1, \quad \binom{\vartheta}{\mu} (-1)^\mu = -\vartheta(1-\vartheta) \dots (\mu-1-\vartheta)/\mu!.$$

The following note describes the Fourier transform of $R_{\vartheta s} u$ and proves some useful estimates.

Note 3.1 (Properties of $R_{\vartheta s}$, $R^\chi_{\vartheta s}$). Let $t \in \mathbf{R}$. (3.3a—d) are valid with $C = C(t, s, \chi)$:

(3.3a) $$\widehat{(R_{\vartheta s} u)}(\xi) = [(1-e^{-h-i\xi s})/h]^\vartheta \hat{u}(\xi),$$

(3.3b) $$|(R_{\vartheta s} - R^\chi_{\vartheta s}) u|_t \leqq C |u|_t \qquad (u \in \mathscr{H}^t),$$

(3.3c) $$|R^\chi_{\vartheta s} u|_t \leqq C |u|_{t+\vartheta} \qquad (u \in \mathscr{H}^{t+\vartheta}),$$

(3.3d) $$|e_k R^\chi_{\vartheta s} u - R^\chi_{\vartheta s}(e_k u)|_t \leqq C |g_k u|_t \quad (u \in \mathscr{H}^t, \ k \in \mathbf{Z}),$$

where $g_k \in C^\infty(\mathbf{R}^d)$ must satisfy $g_k(x) = 1$ if the distance of x from $U_k = \text{support}(e_k)$ is less than ε_1 [cf. (3.2)].

Proof. 1) (3.3a) follows from $\sum_{\mu=0}^{\infty} \binom{\vartheta}{\mu} (-z)^\mu = (1-z)^\vartheta$ and

$$\widehat{(T_h^\alpha u)}(\xi) = e^{-i\xi\alpha} \hat{u}(\xi).$$

2) Discrete Fourier transformation of $(R_{\vartheta s} - R^\chi_{\vartheta s}) u$ yields

$$\left[\sum_{\mu=1}^{\infty} (1-\chi(\mu h \|s\|)) \binom{\vartheta}{\mu} (-1)^\mu e^{-\mu(h+i\xi s)}\right] h^{-\vartheta} \hat{u}(\xi).$$

By $|1-\chi(\mu h \|s\|)| = |\chi(0) - \chi(\mu h \|s\|)| \leqq \mu h C$,

$$\left|\binom{\vartheta}{\mu} \mu\right| \leqq \binom{\vartheta-1}{\mu-1} (-1)^{\mu-1},$$

and $(1-e^{-h})^{-1} \leqq (1+h)/h$ the sum in brackets is bounded by

$$Ch \sum_{\mu=1}^{\infty} \binom{\vartheta-1}{\mu-1} (-1)^{\mu-1} e^{-\mu h} = Che^{-h}(1-e^{-h})^{\vartheta-1} \leqq C' h^\vartheta.$$

Hence, (3.3b) follows from the definition of $|\cdot|_t$.

3) By (3.3b) it suffices to prove $|R_{\vartheta s} u|_t \leqq C |u|_{t+\vartheta}$ instead of (3.3c). This estimate is a conclusion of (3.3a) and

$$[1 + \sum_{j=1}^{d} h^{-2} \sin^2(\xi_j/2)]^{-\vartheta/2} [(1-e^{-h-i\xi s})/h]^\vartheta \leqq C.$$

4) Note that the left-hand side of (3.3d) depends only on u_ν with distance $(\nu h, U_k) \leqq \varepsilon_1$. Therefore, u may be replaced by $g_k u$. Thus, it is sufficient to show (3.3d) with $C|u|_t$ on the right-hand side. $e_k R^\chi_{\vartheta s} u - R^\chi_{\vartheta s}(e_k u)$ has the representation $\sum_{\mu=1}^\infty Y_\mu u$, where

$$(Y_\mu u)_\nu = h^{-\vartheta} c_\mu e^{-\mu h}[e_k(\nu h) - e_k(\nu h + \mu s h)] u_{\nu+\mu s}, \quad c_\mu = \binom{\vartheta}{\mu}(-1)^\mu \chi(\mu h \|s\|).$$

Lemma 3.1 implies $|Y_\mu u|_t \leqq C h^{1-\vartheta} \mu c_\mu e^{-\mu h} |u|_t$. As in the proof of (3.3b),

$$\left|\sum_{\mu=1}^\infty Y_\mu u\right|_t \leqq C h^{1-\vartheta} \sum_{\mu=1}^\infty \binom{\vartheta-1}{\mu-1} (-1)^{\mu-1} e^{-\mu h} |u|_t \leqq C|u|_t$$

yields the desired estimate. ■

3.2. Operator R_ϑ

$(R^\chi_{\vartheta s} u)_\nu$ depends only on $u_{\nu+\mu s}$ with $0 \leqq \mu \leqq \varepsilon_1/(\|s\| h)$. In (3.5) we shall define R_ϑ as a combination of those $R^\chi_{\vartheta s}$ so that all appearing grid points $(\nu+\mu s)h$ are contained in a certain cone C. Therefore, the coefficients σ_s of R_ϑ must vanish if $(\nu+\mu s)h \notin C$. Let $S \subset \mathbf{Z}^d$ be a finite subset with $1 \leqq \|s\| \leqq C_R$ for $s \in S$. In the sequel we need functions $\sigma_s(x)$ for all $s \in S$ with following properties:

(3.4a) $\sigma_s \in C^m(\mathbf{R}^d)$, $\sigma_s(x) \geqq 0$, $\sum_{s\in S} \sigma_s(x) = 1$ for all $x \in \mathbf{R}^d$,

for all $x \in G_h$ there is a subset $S_0 = S_0(x) \subset S$ such that

(3.4b) $\sigma_s(x) \geqq C_R^{-d} > 0$ for $s \in S_0$ and such that $\xi = 0$ is the only common zero of $\sin(\xi s/2)$ $(s \in S_0)$ in $Q = [-\pi, \pi]^d \subset \mathbf{R}^d$.

A third condition on the support of $\sigma_s(x)$ is formulated in the following note.

Note 3.2. i) Let $C(x, n, \varepsilon_1, \varepsilon_2)$ be the cone mentioned in the definition of 'property C'. Choose $\sigma_s(x) \in C^m(\mathbf{R}^d)$ $(s \in S)$ according to (3.4a) so that

$$\sigma_s(x) = 0 \quad \text{if} \quad sh \notin C(h\varepsilon_0 n(x), n(x), \infty, \varepsilon_2),$$
$$\sigma_s(x) \geqq C_R^{-d} \quad \text{if} \quad sh \in C(h\varepsilon_0 n(x), n(x), \infty, \varepsilon_2/2).$$

If $1/C_R$ and h are small enough, (3.4b) is valid.

ii) (3.4b) implies

$$\text{(3.4b')} \qquad \sum_{s\in S} \sigma_s(x) \sin^\vartheta(|s\xi|/2) \geqq \varepsilon\left[\sum_{j=1}^d \sin^2(\xi_j/2)\right]^{\vartheta/2}$$

with $\varepsilon = \varepsilon(C_R) > 0$ for all $\xi \in Q$, $\vartheta \in [0, 1)$.

Proof. i) Choose C_R so that $d+1$ indices $\{s_0, s_1, \ldots, s_d\} \subset \mathbf{Z}^d \setminus \{0\}$ with $s_j = s_0 + e_j$ $(1 \leqq j \leqq d$; e_j: j^{th} unit vector) belong to $S_0 := S \cap C(h\varepsilon_0 n(x), n(x), \infty, \varepsilon_2/2)$. Assume $\xi \in Q$ a zero of $\sin(\xi s_j/2)$ $(0 \leqq j \leqq d)$. Then $\xi s_0 \equiv \xi s_j \pmod{2\pi}$ holds. Hence, $\xi_j = \xi e_j = \xi(s_j - s_0) \equiv 0 \pmod{2\pi}$ and $\xi \in Q$ prove $\xi = 0$.

ii) Since $\xi=0$ is the only zero of the left-hand side of (3.4b'), l.h.s.$\geqq\varepsilon'|\xi|^{\vartheta}\geqq$r.h.s follows.

By means of $\sigma_s(x)$ we define the operators

$$R_\vartheta = \sum_{s\in S}\sigma_s(\cdot)R^\chi_{\vartheta s}, \quad R_\vartheta(x_k) = \sum_{s\in S}\sigma_s(x_k)R^\chi_{\vartheta s} \tag{3.5}$$

for $k\in\mathbf{Z}$, $0\leqq\vartheta<1$, where $x_k\in U_k$ is defined above. The symbol $\sigma_s(\cdot)$ means $(\sigma_s(\cdot)u)(x)=\sigma_s(x)u(x)$. $R_\vartheta(x_k)$ is the operator R_ϑ 'frozen' at x_k. Note that $R_\vartheta(x_k)$ is a convolution operator, whereas R_ϑ is not. The properties of convolution operators can be analysed by means of Fourier transformations.

Note 3.3 (Properties of R_ϑ, $R_\vartheta(x_k)$). Let $\vartheta\in[0, 1)$ and $c_{\alpha\beta\gamma\delta}$ as in Theorem 2.2 ($\Theta=-\vartheta/2$) and assume 'property C'. If σ_s is chosen according to Note 3.2 the properties (3.6a—e) hold:

$$\text{support}(u)\subset\Omega_h \text{ implies support}(R_\vartheta u)\subset\Omega_h, \tag{3.6a}$$

$$|R_\vartheta u|_t\leqq C|u|_{t+\vartheta}, \quad |R_\vartheta(x_k)u|_t\leqq C|u|_{t+\vartheta} \quad (u\in\mathscr{H}^{t+\vartheta},\ |t|\leqq m), \tag{3.6b}$$

$$\operatorname{Re}(R_\vartheta u, u)\geqq\varepsilon|u|^2_{\vartheta/2}-C|u|^2_0 \quad (\varepsilon>0) \quad \text{for all} \quad u\in\mathscr{H}^{\vartheta/2}, \tag{3.6c}$$

$$\begin{aligned}&\Big|\big(T^\gamma_h\partial^\alpha_h c_{\alpha\beta\gamma\delta}(\cdot, 0)T^\delta_h\partial^\beta_h R_\vartheta u, u\big)\\ &-\sum_{k\in Z}\big(T^\gamma_h\partial^\alpha_h c_{\alpha\beta\gamma\delta}(x_k, 0)T^\delta_h\partial^\beta_h R_\vartheta(x_k)e_k u, e_k u\big)\Big|\\ &\leqq\varepsilon(\varrho)|u|^2_{m+\vartheta/2}+\eta^2_{m+\vartheta/2}(u) \quad (|\alpha|=|\beta|=m,\ \varepsilon(\varrho)\searrow 0 \text{ as } \varrho\to 0),\end{aligned} \tag{3.6d}$$

$$\operatorname{Re}\big(L_k R_\vartheta(x_k)u, u\big)\geqq\varepsilon|u|^2_{m+\vartheta/2}-C|u|^2_0 \quad (\varepsilon>0,\ u\in\mathscr{H}^{m+\vartheta/2}), \tag{3.6e}$$

where

$$L_k = \sum_{|\alpha|=|\beta|=m}\sum_{\gamma,\delta}T^\gamma_h\partial^\alpha_h c_{\alpha\beta\gamma\delta}(x_k, 0)T^\delta_h\partial^\beta_h. \tag{3.7}$$

(3.6a), (3.6c), and (3.6e) are the characteristic properties of R_ϑ. By virtue of (3.6d) estimates involving R_ϑ can be replaced with those involving $R_\vartheta(x_k)$.

Proof. 1) Let $x=\nu h\notin\Omega_h$. Because of (3.2) and the choice of σ_s (cf. Note 3.2) $(R_\vartheta u)_\nu$ depends only on $u_{\nu+\mu}$ with $\mu h\in C(h\varepsilon_0 n(x), n(x), \varepsilon_1, \varepsilon_2)$. By definition of 'property C' this cone belongs to $\mathbf{R}^d\setminus\Omega$. Therefore, $x\in\Omega_h$ implies $u_{\nu+\mu}=0$ and $(R_\vartheta u)_\nu=0$.

2) (3.6b) follows by applying Lemma 3.1 and (3.3c).

3) Lemmata 3.1 and 3.4, and (3.6b) yield

$$\Big|(R_\vartheta u, u)-\sum_k\big(e^2_k R_\vartheta(x_k)u, u\big)\Big|\leqq\varepsilon(\varrho)|u|^2_{\vartheta/2}.$$

Choose g_k appearing in (3.3d) so that Lemma 3.2 applies:

$$\begin{aligned}&\Big|\sum_k\big(e_k R_\vartheta(x_k)u-R_\vartheta(x_k)e_k u, e_k u\big)\Big|\\ &\leqq\sum_k\big|\big(e_k R_\vartheta(x_k)-R_\vartheta(x_k)e_k\big)u\big|_0|e_k u|_0\\ &\leqq C\big[\sum_k|g_k u|^2_0\big]^{1/2}\big[\sum_k|e_k u|^2_0\big]^{1/2}\leqq C'|u|^2_0=\eta^2_{\vartheta/2}(u).\end{aligned}$$

Assume that

(3.6c′) $$\operatorname{Re}\big(R_\vartheta(x_k)v, v\big) \geqq \varepsilon' |v|^2_{\vartheta/2} - C' |v|^2_0 \quad (\varepsilon' > 0)$$

holds with ε' and C' independent of $k \in \mathbf{Z}$. Substituting $v = e_k u$ and summing over $k \in \mathbf{Z}$ one obtains

$$\operatorname{Re} \sum_k \big(R_\vartheta(x_k)e_k u, e_k u\big) \geqq \varepsilon'' |u|^2_{\vartheta/2} - C' |u|^2_0 \quad (\varepsilon'' > 0)$$

by means of Lemma 3.3. Choose ϱ so that $\varepsilon(\varrho) < \varepsilon''$. Then the three foregoing inequalities result in (3.6c) with $0 < \varepsilon < \varepsilon'' - \varepsilon(\varrho)$.

It remains to show (3.6c′). Thanks to (3.3b) it suffices to prove (3.6c′) with $R_k := \sum_{s \in S} \sigma_s(x_k)\ R_{\vartheta s}$ instead of $R_\vartheta(x_k)$. Note that $|\arg(1-z)| < \pi/2$ ($|z| < 1$) implies

$$\operatorname{Re}[(1-e^{-h-i\xi s})/h]^\vartheta \geqq \cos(\vartheta\pi/2)\,|(1-e^{-h-i\xi s})/h|^\vartheta$$
$$\geqq \varepsilon[1 + h^{-\vartheta} \sin^\vartheta(|\xi s|/2] \quad \text{with} \quad \varepsilon = \varepsilon(\vartheta) > 0.$$

Hence, (3.4b′) proves

$$\operatorname{Re} \hat{R}_k(\xi) \geqq \varepsilon' \big[1 + h^{-2} \textstyle\sum_{j=1}^d \sin^2(\xi_j/2)\big]^{\vartheta/2},$$

where $\hat{R}_k(\xi) = \sum_{s \in S} \sigma_s(x_k)[(1-e^{-h-i\xi s})/h]^\vartheta$. (3.6c′) follows from

$$(R_k v, v) = [h/(2\pi)]^d \int_Q \hat{R}_k(\xi)\,|\hat{v}(\xi)|^2\, d\xi$$

[cf. (3.3a)].

4) Lemmata 3.4 and 3.1, and (3.6b) $(t = -\vartheta/2)$ show

$$\Big|\big(T_h^\gamma \partial_h^\alpha c_{\alpha\beta\gamma\delta}(\cdot, 0) T_h^\delta \partial_h^\beta R_\vartheta u, u\big) - \sum_k \big(e_k c_{\alpha\beta\gamma\delta}(x_k, 0) T_h^\delta \partial_h^\beta R_\vartheta u, e_k (T_h^\gamma \partial_h^\alpha)^* e_k u\big)\Big|$$
$$\leqq \varepsilon(\varrho)\,|u|^2_{m+\vartheta/2}.$$

Applying Lemma 3.5 to $g_k R_\vartheta u$ and $g_k u$ with g_k as in (3.3d), we obtain that each term of the last sum differs from

$$\big(c_{\alpha\beta\gamma\delta}(x_k, 0) T_h^\delta \partial_h^\beta e_k R_\vartheta u, (T_h^\gamma \partial_h^\alpha)^* e_k u\big)$$

by

$$C[|e_k R_\vartheta u|_{m-\vartheta/2} |g_k u|_{m-1+\vartheta/2} + |g_k R_\vartheta u|_{m-1-\vartheta/2} |g_k u|_{m+\vartheta/2}].$$

Using $|e_k R_\vartheta u|_{m-\vartheta/2} \leqq C |g_k u|_{m+\vartheta/2}$ [cf. (3.4b)] and $|g_k R_\vartheta u|_{m-1-\vartheta/2} \leqq C |\tilde{g}_k u|_{m-1+\vartheta/2}$ with $\tilde{g}_k$ similarly defined as g_k, this bound becomes

$$\eta^2_{m+\vartheta/2}(g_k u) + \eta^2_{m+\vartheta/2}(\tilde{g}_k u)$$

[cf. (3.1)]. Summation over k and application of Lemma 3.2 yield

$$\sum_k \Big|\big(e_k c_{\alpha\beta\gamma\delta}(x_k, 0) T_h^\delta \partial_h^\beta R_\vartheta u, e_k (T_h^\gamma \partial_h^\alpha)^* u\big)$$
$$-\big(c_{\alpha\beta\gamma\delta}(x_k, 0) T_h^\delta \partial_h^\beta e_k R_\vartheta u, (T_h^\gamma \partial_h^\alpha)^* e_k u\big)\Big| = \eta^2_{m+\vartheta/2}(u).$$

Thus, (3.6d) is proved.

5) Let R_k as in 4).

$$\left|(L_k R_\vartheta(x_k)u, u) - (L_k R_k u, u)\right| \leqq C\,|u|_m^2 = \eta_{m+\vartheta/2}^2(u)$$

can be concluded from (3.3b) and (3.1) if $\vartheta>0$. In the case of $\vartheta=0$ the difference vanishes because of $R_\vartheta(x_k)=R_k=I$. Hence, it suffices to prove $\mathrm{Re}\,(L_k R_k u, u) \geqq \varepsilon\,|u|_{m+\vartheta/2}^2$. This estimate follows for $0 \leqq \vartheta < 1$ as in the proof of (3.6c′), if $p(x_k, \xi)$ is real. Otherwise, (2.3) implies $\left|\arg\left(p(x, \xi)\right)\right| \leqq (1-\varepsilon)\pi/2$ $(\varepsilon>0)$ for all $x, \xi \in \mathbf{R}^d$. Hence, $\left|\arg\left(p(x_k, \xi)\hat{R}_k(\xi)\right)\right| < \pi/2$ holds for $0 \leqq \vartheta < 2\Theta_0$ with sufficiently small $\Theta_0 \in (0, 1/2]$. ■

3.3. Proof of Theorem 2.2

The proof of Theorem 2.2 is prepared by two lemmata. The first one allows to estimate $\|A^{-1}\|$ by means of $\|(A-\lambda I)^{-1}\|$. The second one is the trivial remark that $(A-\lambda I)^{-1}$ is bounded for coercive forms (Au, v).

Lemma 3.6. *Let A be an (unbounded) operator with dense domain in $\mathscr{H}_0^0$ and assume $\|A^{-1}\|_{\mathscr{H}_0^0 \to \mathscr{H}_0^0} \leqq C_1$ (stability). Then*

$$\|(A+\lambda I)^{-1}\|_{\mathscr{H}_0^{-s} \to \mathscr{H}_0^r} \leqq C_2 \quad (s, r \geqq 0) \quad \text{for some} \quad \lambda \in \mathbf{R}$$

implies

$$\|A^{-1}\|_{\mathscr{H}_0^{-s} \to \mathscr{H}_0^r} \leqq C_3.$$

Proof. Set $A_\lambda = A - \lambda I$. $A^{-1} = A_\lambda^{-1} - \lambda A^{-1} A_\lambda^{-1}$ shows $\|A^{-1}\|_{\mathscr{H}_0^{-s} \to \mathscr{H}_0^0} \leqq C' := C_2 + |\lambda| C_1 C_2$ by virtue of $|\cdot|_0 \leqq |\cdot|_r$. Hence, $A^{-1} = A_\lambda^{-1} - \lambda A_\lambda^{-1} A^{-1}$ proves

$$\|A^{-1}\|_{\mathscr{H}_0^{-s} \to \mathscr{H}_0^r} \leqq C_2 + |\lambda|\, C_2 C' =: C_3. \qquad ■$$

Lemma 3.7. $\mathrm{Re}\,(Au, u) \geqq \varepsilon\,|u|_s^2 - \lambda\,|u|_0^2$ $(\varepsilon>0,\ \lambda \in \mathbf{R},\ s>0)$ *for all $u \in \mathscr{H}_0^s$ implies* $\|(A+\lambda I)^{-1}\|_{\mathscr{H}_0^{-s} \to \mathscr{H}_0^s} \leqq 1/\varepsilon$.

As announced in Section 3.1 we show $\mathscr{H}_0^{m+\Theta}$-coerciveness of $L_h R_{2\Theta}$. In the proof we apply the partition of unity and use the fact that $L_k R_{2\Theta}(x_k)$ is coercive [cf. (3.6e)].

Theorem 3.1. *Assume the conditions of Theorem 2.2 for $\Theta = -\vartheta/2 \in (-1/2, 0]$ except of the stability. Then there is $\varepsilon>0$ with*

$$\mathrm{Re}\,(L_h R_\vartheta u, u) \geqq \varepsilon\,|u|_{m+\vartheta/2}^2 - C\,|u|_0^2 \quad \text{for all} \quad u \in \mathscr{H}_0^{m+\vartheta/2}.$$

Proof. Set $R := R_\vartheta$ and $R_k := R_\vartheta(x_k)$ $(k \in \mathbf{Z})$. In order to show that the principle part

$$L_h^P = \sum\nolimits_{|\alpha|=|\beta|=m} \sum\nolimits_{\gamma,\delta \in \mathbf{Z}^d} T_h^\gamma\, \partial_h^\alpha c_{\alpha\beta\gamma\delta}(\cdot, 0)\, T_h^\delta\, \partial_h^\beta$$

of L_h satisfies

(3.8a) $$|(L_h R u, u) - (L_h^P R u, u)| = \eta_{m+\vartheta/2}^2(u) \quad \text{for all} \quad u \in \mathscr{H}^{m+\vartheta/2}$$

three cases are to be discussed. If $|\alpha|<m$ and $|\beta|=m$

$$\begin{aligned}&|(T_h^\gamma\partial_h^\alpha c_{\alpha\beta\gamma\delta}(\cdot,h)T_h^\delta\partial_h^\beta Ru,u)|\\ &\leqq|c_{\alpha\beta\gamma\delta}(\cdot,h)T_h^\delta\partial_h^\beta Ru|_{-\vartheta/2}|(T_h^\gamma\partial_h^\alpha)^*u|_{\vartheta/2}\leqq C|Ru|_{m-\vartheta/2}|u|_{m+\vartheta/2-1}=\eta^2_{m+\vartheta/2}(u)\end{aligned}$$

[cf. (3.6b), (3.1)] follows from Lemma 3.1 and $c_{\alpha\beta\gamma\delta}\in C^\varkappa(\mathbf{R}^d)$. In the case of $|\alpha|\leqq m$ and $|\beta|<m$, the boundedness of $c_{\alpha\beta\gamma\delta}$ yields

$$|(T_h^\gamma\partial_h^\alpha c_{\alpha\beta\gamma\delta}(\cdot,h)T_h^\delta\partial_h^\beta Ru,u)|\leqq C|Ru|_{m-1}|u|_m\leqq C|u|_{m-1+\vartheta}|u|_m=\eta^2_{m+\vartheta/2}(u)$$

[cf. (3.1)]. If $|\alpha|=|\beta|=m$, the Hölder condition

$$|c_{\alpha\beta\gamma\delta}(x,h)-c_{\alpha\beta\gamma\delta}(x,0)|\leqq Ch^{\vartheta/2}$$

implies

$$\begin{aligned}&|(T_h^\gamma\partial_h^\alpha(c_{\alpha\beta\gamma\delta}(\cdot,h)-c_{\alpha\beta\gamma\delta}(\cdot,0))T_h^\delta\partial_h^\beta Ru,u)|\leqq Ch^{\vartheta/2}|Ru|_m|u|_m\\ &\leqq C'h^{\vartheta/2}|u|_{m+\vartheta}|u|_m\leqq C''|u|_{m+\vartheta/2}|u|_m=\eta^2_{m+\vartheta/2}(u)\end{aligned}$$

by virtue of (3.6b), $h^s|\cdot|_{t+s}\leqq C|\cdot|_t$, and (3.1). Hence, (3.8a) is proved.

Define L_k by (3.7). (3.6d) implies

(3.8b) $$|(L_h^P Ru,u)-\sum_{k\in\mathbf{Z}}(L_kR_ke_ku,e_ku)|\leqq\varepsilon(\varrho)|u|^2_{m+\vartheta/2}+\eta^2_{m+\vartheta/2}(u)$$

for all $u\in\mathscr{H}^{m-\Theta}$, where $\varepsilon(\varrho)\searrow 0$ as $\varrho\to 0$. (3.6e) and Lemma 3.3 result in

(3.8c) $$\mathrm{Re}\sum_k(L_kR_ke_ku,e_ku)\geqq\sum_k[\varepsilon'|e_ku|^2_{m+\vartheta/2}-C|e_ku|_0^2]\geqq\varepsilon''|u|^2_{m+\vartheta/2}-C|u|_0^2$$

with $\varepsilon',\varepsilon''>0$.

Choose ϱ so small that $\varepsilon(\varrho)\leqq\varepsilon:=\varepsilon''/3$. The estimates (3.8a, b, c) yield

$$\mathrm{Re}(L_hRu,u)\geqq 2\varepsilon|u|^2_{m+\vartheta/2}+\eta^2_{m+\vartheta/2}(u)\quad(\varepsilon>0)\quad\text{for all}\quad u\in\mathscr{H}^{m+\vartheta/2}.$$

By definition of $\eta_t^2(u)$ the right-hand side can be replaced with $\varepsilon|u|^2_{m+\vartheta/2}-C|u|_0^2$. Restriction of this inequality to $u\in\mathscr{H}_0^{m+\vartheta/2}\subset\mathscr{H}^{m+\vartheta/2}$ concludes the proof of Theorem 3.1. ∎

By repeated applications of Lemma 3.6 we finally prove:

Note 3.4. Let L_h be stable with respect to $\mathscr{H}_0^0$. Then Theorem 3.1 implies Theorem 2.2.

Proof. 1) Case of $\Theta=0$. Use $R_0=I$ and apply the Lemmata 3.7, 3.6.

2) Case of $\Theta<0$. Set $\vartheta=-2\Theta\in(0,1)$. (3.6a, c) and Lemma 3.7 yield

$$\|(R_\vartheta+\lambda I)^{-1}\|_{\mathscr{H}_0^{-\vartheta/2}\to\mathscr{H}_0^{\vartheta/2}}\leqq C\quad(\lambda\text{ sufficiently large}).$$

In particular, $R_\vartheta+\lambda I$ is stable with respect to $\mathscr{H}_0^0$.

Since $|(L_hu,u)|\leqq C|u|_m^2=\eta^2_{m+\vartheta/2}(u)$, Theorem 3.1 yields

$$\mathrm{Re}(L_h(R_\vartheta+\lambda I)u,u)\geqq\frac{\varepsilon}{2}|u|^2_{m+\vartheta/2}-C'|u|_0^2\quad(u\in\mathscr{H}_0^{m+\vartheta/2}).$$

Applying Lemma 3.6 to this estimate one obtains

$$\|[L_h(R_\vartheta+\lambda I)+\lambda' I]^{-1}\|_{\mathscr{H}_0^{-m-\vartheta/2}\to\mathscr{H}_0^{m+\vartheta/2}} \leqq C \quad (\lambda' = C').$$

Since L_h (more precisely $L_{h,0}$) and $R_\vartheta+\lambda I$ are stable, Lemma 3.6 yields

$$\|[L_h(R_\vartheta+\lambda I)]^{-1}\|_{\mathscr{H}_0^{-m-\vartheta/2}\to\mathscr{H}_0^{m+\vartheta/2}} \leqq C.$$

By virtue of

$$\|L_h^{-1}\|_{\mathscr{H}_0^{-m-\vartheta/2}\to\mathscr{H}_0^{m-\vartheta/2}}$$

$$\leqq \|R_\vartheta+\lambda I\|_{\mathscr{H}_0^{m+\vartheta/2}\to\mathscr{H}_0^{m-\vartheta/2}}\|[L_h(R_\vartheta+\lambda I)]^{-1}\|_{\mathscr{H}_0^{-m-\vartheta/2}\to\mathscr{H}_0^{m+\vartheta/2}} \leqq C$$

[cf. (3.6a, b)] the estimate (2.5) of Theorem 2.2 follows.

3) Case of $\Theta>0$. Since $\mathscr{H}_0^{-t}$ is the dual space of $\mathscr{H}_0^{t}$ the estimate (2.5) is equivalent to the estimate

$$\|(L_h^*)^{-1}\|_{\mathscr{H}_0^{-m-\Theta}\to\mathscr{H}_0^{m-\Theta}} \leqq C \tag{3.9}$$

involving the adjoint operator. L_h^* is again of the form (2.2) with $c_{\alpha\beta\gamma\delta}$ replaced by

$$c^*_{\alpha\beta\gamma\delta} = (-1)^{|\alpha|+|\beta|}\bar{c}_{\beta,\alpha,\beta-\delta,\alpha-\gamma}.$$

Applying the foregoing part 2) to L_h^* we obtain (3.9). ∎

3.5. Proof of Lemmata 2.1, 2.2

Proof of Lemma 2.2. i) Choose ε_0, ε_1, ε_2 suitably. ii) By the arguments of the proof of [17, Lemma 3.4] the estimate follows for the case of $s=0$, $t=2m$. The result is trivial for $s=t=0$. Noting $\gamma=\gamma^*$ and applying interpolation (cf. [9, Lemma 5]) we obtain the general estimate. ∎

Proof of Lemma 2.1. 1) It suffices to prove the inequalities for $s\in[0, m]$, since they imply the same estimates for $-s$.

2) Set $\sigma=s/(2m)$. At first we prove the first inequality, $|u|_{s,0}\leqq C|L_{h,0}^\sigma u|_{0,0}$. Denote the extension by zero outside of Ω_h by ω: $\mathscr{H}_0^0\to\mathscr{H}^0$. ω^*: $\mathscr{H}^0\to\mathscr{H}_0^0$ is the restriction to Ω_h. Since $|v|_s=|L_h^\sigma v|_0$ [$v\in\mathscr{H}^s$, cf. (2.4)] the assertion holds if and only if $\|L_h^\sigma\omega L_{h,0}^{-\sigma}\|\leqq C$, where $0\leqq\sigma\leqq 1/2$ and $\|\cdot\|=\|\cdot\|_{\mathscr{H}_0^0\to\mathscr{H}^0}$. This inequality becomes $\|\omega\|\leqq 1$ for $\sigma=0$ and

$$\|L_h^{1/2}\omega L_{h,0}^{-1/2}\| = \|L_{h,0}^{-1/2}\omega^* L_h\omega L_{h,0}^{-1/2}\|^{1/2} = \|I\|^{1/2} = 1$$

for $\sigma=1/2$ because of $L_{h,0}=\omega^* L_h\omega$. By interpolation the estimate is valid for all $\sigma\in[0, 1/2]$ with $C=1$ (cf. [11, p. 19]).

3) In part 4) we shall show the existence of Γ: $\mathscr{H}^s\to\mathscr{H}_0^s$ $(0\leqq s\leqq m)$ with $\|L_{h,0}^\sigma\Gamma L_h^{-\sigma}\|\leqq C$ $(\sigma=s/2m)$ and $\Gamma\omega=$identity on $\mathscr{H}_0^s$. Thus, $|L_{h,0}^\sigma\Gamma v|_{0,0}\leqq C|L_h^\sigma v|_0$

holds for all $v\in\mathscr{H}^s$. Substituting $v=\omega u$ we obtain

$$|L^\sigma_{h,0}u|_{0,0}=|L^\sigma_{h,0}\Gamma\omega u|_{0,0}\leqq C|L^\sigma_h\omega u|_0=C|\omega u|_s=C|u|_{s,0}.$$

Thus, the second inequality is proved, too.

4) Define $\Gamma=\omega^* r\gamma p$ as follows. By means of finite elements of sufficiently high order define the prolongation $p\colon \mathscr{H}^s\to H^s(\mathbf{R}^d)$ $(0\leqq s\leqq m)$. The restriction $r\colon H^s(\mathbf{R}^d)\to\mathscr{H}^s$ is the projection defined by $\|pru-u\|_{H^0(\mathbf{R}^d)}\leqq\|pw-u\|_{H^0(\mathbf{R}^d)}$ (for all $w\in\mathscr{H}^0$, $u\in H^0(\mathbf{R}^d)$).

Note that $rp=I$. p can be chosen so that $p\colon\mathscr{H}^s\to H^s(\mathbf{R}^d)$ and $r\colon H^s(\mathbf{R}^d)\to\mathscr{H}^s$ $(0\leqq s\leqq m)$ are uniformly bounded (i.e. independently of $h\in I_0$). If $u\in\mathscr{H}^0_0$ the support of pu is contained in

$$\Omega'=\{x\in\mathbf{R}^d\colon \text{distance}\,(x,\Omega_h)\leqq C'h\}\quad\text{for some}\quad C'=C'(m)\geqq\sqrt{d}/2.$$

By 'property C' of Ω_h there is a grid Ω''_h with

$$\Omega_h\subset G_h\cap\Omega'\subset\Omega''_h\subset\{x\in G_h\colon \text{distance}\,(x,\Omega_h)\leqq C^*h\}\quad(C^*\geqq C')$$

such that Ω''_h has 'property C' with $\varepsilon_0=0$. Then there is

$$\Omega''\subset\{x\in\mathbf{R}^d\colon \text{distance}\,(x,\Omega_h)\leqq C''h\}\quad\text{with}\quad\Omega''_h=\Omega''\cap G_h\quad\text{and}\quad\Omega''\supset\Omega'$$

such that $\mathbf{R}^d\setminus\Omega''$ satisfies the requirements of the Calderón extension theorem (cf. [1, p. 91]) uniformly for all $h\in I_0$. Thus, there is an extension operator $E\colon H^k(\mathbf{R}^d\setminus\Omega'')\to H^k(\mathbf{R}^d)$ with

$$Eu=u\quad\text{in}\quad\mathbf{R}^d\setminus\Omega'',$$

$$\|Eu\|_{H^k(\mathbf{R}^d)}\leqq C\|u\|_{H^k(\mathbf{R}^d\setminus\Omega'')}\quad\text{for}\quad k=0,m,\quad u\in H^k(\mathbf{R}^d\setminus\Omega''),\quad h\in I_0.$$

Hence, $\gamma:=$restriction of $I-E$ on Ω'' is a uniformly bounded mapping from $H^k(\mathbf{R}^d)$ onto $H^k_0(\Omega'')$ $(k=0,m)$ with $\gamma u=u$ for $u\in H^k_0(\Omega'')$. If $u\in H^s_0(\Omega'')$, the support of $r\tilde{u}$ ($\tilde{u}=u$ in Ω'', $\tilde{u}=0$ otherwise) is contained in

$$\Omega'''_h\subset\{x\in G_h\colon \text{distance}\,(x,\Omega_h)\leqq C'''h\}$$

for some C'''. Let $\mathscr{H}^s_0(\Omega'''_h)$ be defined as $\mathscr{H}^s_0$ but with Ω'''_h instead of Ω_h. By Lemma 2.2

$$|v|_k\leqq Ch^{-k}|v|_0\leqq\tilde{C}|u|_{\mathscr{H}^k_0(\Omega'''_h)}$$

holds for $u\in\mathscr{H}^k_0(\Omega'''_h)$ $(k=0,m)$ and $v:=u-\omega^*u$, i.e. $v_\nu=u_\nu$ at $\nu h\in\Omega'''_h\setminus\Omega_h$ and $v_\nu=0$ otherwise. Therefore, $\omega^*\colon\mathscr{H}^k_0(\Omega'''_h)\to\mathscr{H}^k_0$ $(k=0,m)$ is uniformly bounded for all $h\in I_0$. It follows that $\Gamma\colon\mathscr{H}^k\to\mathscr{H}^k_0$ $(k=0,m)$ is uniformly bounded, i.e. $\|L^\sigma_{h,0}\Gamma L_h^{-\sigma}\|\leqq C$ holds for $\sigma=0$ and $\sigma=1/2$ with $C\neq C(h)$. Interpolation yields the same bound for all $\sigma\in[0,1/2]$. The proof is concluded by the observation that $u\in\mathscr{H}^k_0$ implies $\gamma p\omega u=p\omega u\in H^k_0(\Omega'')$ and therefore $\Gamma u=\omega^*rp\omega u=\omega^*\omega u=u$. ■

3.6. Proof of Note 2.2 and Note 2.3

Proof of Note 2.2. Let $\Omega_h = \Omega'_h \cup \Gamma_h$ and $\Gamma'_h \subset \Omega'_h$ as in Example 2.2 and define $\mathscr{H}_0^s$ by means of Ω'_h. In order to apply Criterion 2.1 we denote the five-point formula (restricted to Ω'_h) by L_h and define $l_h = \tilde{L}_h - L_h$. By Note 2.1 and Lemma 2.2, $\{\Omega'_h\}_{h\in I_0}$ has 'property C'. Thus, L_h satisfies the assumptions of Theorem 2.2.

Since Ω is bounded, $|\cdot|_1$ and $|||\cdot|||_1 = (L_h u, u)^{1/2}$ are equivalent norms on $\mathscr{H}_0^1$. The support of $l_h u$ is contained in Γ'_h. Hence, Lemma 2.2 implies

$$\|l_h\|_{\mathscr{H}_0^2 \to \mathscr{H}_0^0} \leqq C,$$

i.e. the estimate required in Criterion 2.1 holds with $s=1$.

Let $x = \nu h \in \Gamma'_h$, $y = x \pm h e_j \in \Gamma_h$ and $y \pm \varkappa h e_j \in \partial\Omega$. Then the term $-h^{-2}u(y)$ of $(L_h u)(x)$ is replaced in $(\tilde{L}_h u)(x)$ by $-h^{-2}\varkappa u(x)/(1+\varkappa)$. A more general representation is $-h^{-2}\sum_{\mu h\in\Gamma'_h} w_{\nu\mu} u_\mu$, where $w_{\nu\mu} = w_{\nu\mu}(\pm e_j)$. Thus, the estimates

$$(3.10)\quad \sum\nolimits_\nu |w_{\nu\mu}(\pm e_j)| \leqq C_1, \quad \sum\nolimits_\mu |w_{\nu\mu}(\pm e_j)| \leqq C_2, \quad 2C_1C_2 < 1 \quad (j = 1, 2)$$

hold with $C_1 = C_2 = 1/2$.

Finally we prove that (3.10) implies $|(l_h u, v)| \leqq \varkappa |||u|||_1 |||v|||_1$ with $\varkappa = \sqrt{2C_1C_2} < 1$. Split l_h into $l_{h1} + l_{h,-1} + l_{h2} + l_{h,-2}$, where

$$(l_{h,\pm j}u)_\nu = \sum\nolimits_{\nu h\in\Gamma'_h} w_{\nu\mu}(\pm e_j) u_\mu \quad \text{if} \quad \nu h \in \Gamma'_h, \quad (l_{h,\pm j}u)_\nu = 0 \quad \text{otherwise},$$

and let $\|\cdot\|_p$ $(p=1, 2, \infty)$ be the matrix norm corresponding to the vector norm $|u|_{\ell_p}^p = \sum_\nu |u_\nu|^p$, $\|u\|_{\ell_\infty} = \sup_\nu |u_\nu|$. Since $(l_{hj}u)_\nu = 0$ or $(\nu + e_j)h \notin \Omega'_h$ we have

$$(3.11)\quad \begin{aligned} (l_{hj}u)_\nu \bar{v}_\nu &= -(l_{hj}u)_\nu h(\partial_{hj}\bar{v})_{\nu+e_j} \\ (l_{h,-j}u)_\nu \bar{v}_\nu &= (l_{h,-j}u)_\nu h(\partial_{hj}\bar{v})_\nu \end{aligned} \quad (j = 1, 2).$$

The inequalities (3.10) imply

$$h^2\|l_{h,\pm j}\|_2 \leqq h^2\{\|l_{h,\pm j}\|_1 \|l_{h,\pm j}\|_\infty\}^{1/2} \leqq \sqrt{C_1C_2}.$$

Therefore, summation of (3.11) yields

$$|(l_h u, v)| \leqq \sqrt{C_1C_2}\, h^{-1}|\gamma' u|_0 \quad (|\partial_{h1}v|_0 + |\partial_{h2}v|_0),$$

where $\gamma' u$ is the restriction of u to Γ'_h: $(\gamma' u)_\nu = u_\nu$ if $\nu h \in \Gamma'_h$ and $(\gamma' u)_\nu = 0$ otherwise. From $|\partial_{h1}v|_0^2 + |\partial_{h2}v|_0^2 = |||v|||_1^2$ and $|\gamma' u|_0 \leqq h|||u|||_1$ it follows that

$$|(l_h u, v)| \leqq \sqrt{2C_1C_2}\,|||u|||_1 |||v|||_1, \quad \sqrt{2C_1C_2} < 1.$$

Thus, all conditions of Criterion 2.1 are satisfied and Note 2.2 is proved. ∎

Proof of Note 2.3. Split the right-hand side of $L_h u = f$ into $f' + \gamma f$ where support$(f') \subset \Omega'_h$ and γ=restriction to Γ_h: $(\gamma u)_\nu = u_\nu$ if $\nu h \in \Gamma_h$, $(\gamma u)_\nu = 0$ otherwise.

By $\mathscr{H}_0'^s$ we denote the discrete Sobolev spaces corresponding to Ω_h' (instead of Ω_h). By the same arguments as in the proof of Note 2.2 we show

(3.12) $|\tilde{L}_h^{-1}f'|_{1+\Theta}' \leqq C|f'|_{\Theta-1,0}'$ for $\Theta\in[0,\Theta_0)$, $0<\Theta_0\leqq\frac{1}{2}$, $\text{support}(f')\subset\Omega_h'$,

where $|\cdot|_{s,0}'$ is the norm of $\mathscr{H}_0'^s$. The only difference to the proof of Note 2.2 is the fact that the equations $(\tilde{L}_h u)_\nu$ $(\nu h\in\Gamma_h)$ involve not only components u_μ $(\mu h\in\Gamma_h')$ but also u_μ $(\nu\neq\mu,\ \mu h\in\Gamma_h)$. Therefore, we obtain a general representation

$$(l_{h,\pm j}u)_\nu = -h^{-2}\sum_{\mu h\in\Gamma_h'} w_{\nu\mu}(\pm e_j)u_\mu \quad (\nu h\in\Gamma_h')$$

as mentioned in the foregoing proof. The coefficients $w_{\nu\mu}$ are non-negative. Let $x=\nu h\in\Gamma_h'$ and $x+he_1\in\Gamma_h$. The sums $\sum_\nu w_{\nu\mu}(e_1)$ and $\sum_\mu w_{\nu\mu}(e_1)$ are maximal in the case of the following shape of the boundary: $x+\alpha he_2\in\Gamma_h'$, $x+he_1+\alpha he_2\in\Gamma_h$, $x+2he_1+\alpha he_2\in\partial\Omega$ for all $\alpha\in\mathbf{Z}$. Then $\sum_\nu w_{\nu\mu}=\sum_\mu w_{\nu\mu}=1/2$ follows. Therefore, (3.10) is fulfilled with $C_1=C_2=1/2$.

Let γ' be the restriction to Γ_h'. Since $|\gamma L_h^{-1}f'|_0\leqq C|\gamma' L^{-1}f'|_0'$, Lemma 2.2 and (3.12) yield

(3.13a)
$$\begin{aligned}|\gamma\tilde{L}_h^{-1}f'|_{1+\Theta} &\leqq Ch^{-1-\Theta}|\gamma\tilde{L}_h^{-1}f'|_0 \leqq C'h^{-1-\Theta}|\gamma'\tilde{L}_h^{-1}f'|_0' \\ &\leqq C''|\tilde{L}_h^{-1}f'|_{1+\Theta}' \leqq C'''|f'|_{\Theta-1,0}'.\end{aligned}$$

Define $\gamma'':=\gamma+\gamma'$ and note that $L_h\tilde{L}_h^{-1}\gamma=\gamma L_h\gamma''L_h^{-1}\gamma$, where L_h is the five-point formula (restricted to Ω_h). The inequality $0\leqq\tilde{L}_h^{-1}\leqq L_h^{-1}$ holds for all entries of the matrices and implies

$$|\gamma''L_h^{-1}\gamma f|_0 \leqq |\gamma''L_h^{-1}\gamma g|_0, \quad \text{where} \quad g_\nu=|f_\nu|.$$

Hence a repeated application of Lemma 2.2 shows

(3.13b)
$$\begin{aligned}|\tilde{L}_h^{-1}\gamma f|_s &= |L_h^{-1}L_h\tilde{L}_h^{-1}\gamma f|_s = |L_h^{-1}\gamma L_h\gamma''L_h^{-1}\gamma f|_s \\ &\leqq C_1|\gamma L_h\gamma''\tilde{L}_h^{-1}\gamma f|_{s-2,0} \leqq C_2|L_h\gamma''\tilde{L}_h^{-1}\gamma f|_{s-2,0} \\ &\leqq C_3|\gamma''\tilde{L}_h^{-1}\gamma f|_s \leqq C_4h^{-s}|\gamma''\tilde{L}_h^{-1}\gamma f|_0 \leqq C_4h^{-s}|\gamma''L_h^{-1}\gamma g|_0 \\ &\leqq C_5|\gamma''L_h^{-1}\gamma g|_s \leqq C_6|L_h^{-1}\gamma g|_s \leqq C_7|\gamma g|_{s-2,0} \leqq C_8h^{2-s}|\gamma g|_0 \\ &= C_8h^{2-s}|\gamma f|_0 \leqq C_9|\gamma f|_{s-2,0} \leqq C_{10}|f|_{s-2,0},\end{aligned}$$

where $s=1+\Theta$. Finally, we note that

(3.13c)
$$|f'|_{s,0}' \leqq |f'|_{s,0} \quad (s\in\mathbf{R},\ f'\in\mathscr{H}_0'^s).$$

By virtue of (3.12) and (3.13a, b, c) the desired estimate (2.5*) follows:

$$\begin{aligned}|\tilde{L}_h^{-1}f|_{1+\Theta} &\leqq |\tilde{L}_h^{-1}f'|_{1+\Theta}+|\tilde{L}_h^{-1}\gamma f|_{1+\Theta} \\ &\leqq |\gamma\tilde{L}_h^{-1}f'|_{1+\Theta}+|\tilde{L}_h^{-1}f'|_{1+\Theta}'+|\tilde{L}_h^{-1}\gamma f|_{1+\Theta} \\ &\leqq C[|f'|_{\Theta-1,0}'+|f|_{\Theta-1,0}] \leqq C'|f|_{\Theta-1,0}.\end{aligned}$$ ■

4. Applications

4.1. Optimal Error Estimates

The estimate $\|u-u_h\|_{H_0^s(\Omega)} \leqq Ch^{t-s}\|u\|_{H_0^t(\Omega)}$ $(m \leqq s \leqq t \leqq t_{\max})$ is well-known for finite element approximations u_h to the exact solution of $Lu=f$. Using $\|u\|_{H_0^t(\Omega)} \leqq C\|f\|_{H^{t-2m}(\Omega)}$ for $t=m+\Theta$ (cf. Theorem 2.1) we obtain the optimal estimate

$$\|u-u_h\|_{H_0^{m-\Theta'}(\Omega)} \leqq Ch^{\Theta+\Theta'}\|f\|_{H^{\Theta-m}(\Omega)}.$$

A similar estimate can be obtained for difference approximations, too.

Let P_h: $\mathscr{H}_0^m \to H_0^m(\Omega)$ be a suitable prolongation satisfying

(4.1) $$\|P_h\|_{\mathscr{H}_0^s \to H_0^s(\Omega)} \leqq C \quad \text{for} \quad 0 \leqq s \leqq m+\Theta_0 \quad (\Theta_0 \in (0, \tfrac{1}{2}) \text{ fixed}).$$

If we define the restriction R_h: $H^{-m}(\Omega) \to \mathscr{H}_0^{-m}$ by $R_h = P_h^*$ also (4.2) is fulfilled:

(4.2) $$\|R_h\|_{H^{-s}(\Omega) \to \mathscr{H}_0^{-s}} \leqq C \quad \text{for} \quad 0 \leqq s \leqq m+\Theta_0.$$

For a suitable choice of P_h and R_h the product $P_h R_h$ approximates the identity:

(4.3) $$\|I - P_h R_h\|_{H_0^s(\Omega) \to H_0^t(\Omega)} \leqq Ch^{s-t} \quad \text{for} \quad s, t \in [0, m+\Theta_0].$$

The Galerkin approximation related to the subspace $P_h \mathscr{H}_0^m \subset H_0^m(\Omega)$ leads us to the scheme $R_h L P_h$ (if $R_h = P_h^*$). Since L_h must be consistent, the difference

(4.4a) $$\delta_h := R_h L P_h - L_h$$

should satisfy

(4.4b) $$\|\delta_h\|_{\mathscr{H}_0^{m+\Theta} \to \mathscr{H}_0^{-m-\Theta'}} \leqq Ch^{\Theta+\Theta'}, \quad \Theta, \Theta' \in [0, \Theta_0].$$

Note 4.1. Assume (2.5) and $L^{\pm 1}$: $H_0^{\pm m+\Theta}(\Omega) \to H_0^{\mp m+\Theta}(\Omega)$ for $|\Theta| \leqq \Theta_0$ (cf. Theorems 2.1, 2.2). Define the right-hand side of (1.2) by $f_h = R_h f$ with f from (1.1). Then (4.1), (4.2), (4.3), and (4.4a, b) imply

(4.5) $$\|u - P_h u_h\|_{H_0^{m-\Theta'}(\Omega)} \leqq Ch^{\Theta+\Theta'}\|f\|_{H^{\Theta-m}(\Omega)}$$

for $\Theta, \Theta' \in [0, \Theta_0]$ and $f \in H^{\Theta-m}(\Omega)$, where u and u_h are the solutions of (1.1) and (1.2), respectively: $Lu=f$, $L_h u_h = f_h$.

Proof. Abbreviate $\|\cdot\|_{H_0^s(\Omega) \to H_0^t(\Omega)}$, $\|\cdot\|_{H_0^s(\Omega) \to \mathscr{H}_0^t}$, $\|\cdot\|_{\mathscr{H}_0^s \to H_0^t(\Omega)}$, and $\|\cdot\|_{\mathscr{H}_0^s \to \mathscr{H}_0^t}$ by $\|\cdot\|_{s,t}$. The estimate (4.5) can be rewritten as

$$\|L^{-1} - P_h L_h^{-1} R_h\|_{s,t} \leqq Ch^{\Theta+\Theta'}, \quad \text{where} \quad s = \Theta - m \quad \text{and} \quad t = m - \Theta'.$$

Noting $[I - P_h L_h^{-1} R_h L] P_h = -P_h L_h^{-1} \delta_h$ we obtain the result by means of the following splitting:

$$\|L^{-1}-P_h L_h^{-1} R_h\|_{s,t} = \|[I-P_h L_h^{-1} R_h L]L^{-1}\|_{s,t}$$
$$= \|[I-P_h L_h^{-1} R_h L][I-P_h R_h]L^{-1}-P_h L_h^{-1}\delta_h R_h L^{-1}\|_{s,t}$$
$$\leqq \{1+\|P_h\|_{t,t}\|L_h^{-1}\|_{t-2m,t}\|R_h\|_{t-2m,t-2m}\|L\|_{t,t-2m}\}$$
$$\times\|I-P_h R_h\|_{s+2m,t}\|L^{-1}\|_{s,s+2m}$$
$$+\|P_h\|_{t,t}\|L_h^{-1}\|_{t-2m,t}\|\delta_h\|_{s+2m,t-2m}\|R_h\|_{s+2m,s+2m}\|L^{-1}\|_{s,s+2m} \leqq Ch^{\Theta+\Theta'}. \quad \blacksquare$$

4.2. Convergence Proof of Multi-Grid Methods

The multi-grid method is a widely applicable and very fast iterative process for solving systems of linear (or nonlinear) equations arising from difference or Galerkin approximations (cf. [7, 8, 9]). It consists of a smoothing step and a correction by means of approximations corresponding to coarser grids. Accordingly the proof of convergence requires a 'smoothing property' and a 'approximation property'. The latter is similar to (4.5):

(4.6) $$|u_h-p_{hh'}u_{h'}|_{m-\Theta'} \leqq Ch^{\Theta+\Theta'}|f_h|_{\Theta-m}, \quad \Theta\geqq 0, \quad \Theta'\geqq 0,$$

where $u_h=L_h^{-1}f_h$, $h'>h$, $u_{h'}=L_{h'}^{-1}f_{h'}$, $f_{h'}=r_{h'h}f$. $p_{hh'}$ and $r_{h'h}$ are prolongations and restrictions acting on the discrete Sobolev spaces with grid widths h and h'. It turns out that the convergence of the multi-grid method requires $\Theta+\Theta'>0$. Thus, the introduction of $\mathscr{H}_0^s$ with $s\notin\mathbf{Z}$ cannot be avoided.

4.3. Stability with respect to ℓ_∞

As mentioned above $\mathscr{H}_0^0$ is usually denoted by ℓ_2. ℓ_∞ is the space endowed with the supremum norm

$$\|u\|_{\ell_\infty} = \sup\{|u_\nu|:\ \nu h\in\Omega_h\}.$$

A scheme L_h is called stable with respect to ℓ_∞ if $\|L_h^{-1}\|_{\ell_\infty\to\ell_\infty}\leqq C$. ℓ_∞ stability can be proved by virtue of the M-matrix property or related properties (cf. [2], [12, p. 197]). Here we show:

Note 4.2. Assume $m\geqq d/2$ and let $\Omega\subset\mathbf{R}^d$ be a bounded domain. Under the requirements of Theorem 2.2 (for some $\Theta>0$) stability with respect to ℓ_2 implies stability with respect to ℓ_∞.

Proof. By assumption the number of grid points of Ω_h is proportional to h^{-d}. Hence

$$|f|_{\Theta-m,0} \leqq |f|_0 = \|f\|_{\ell_2} \leqq C\|f\|_{\ell_\infty}$$

is valid. Thus, ℓ_2 stability implies

$$|L_h^{-1}f|_{m+\Theta} \leqq C|f|_{\Theta-m,0} \leqq C'\|f\|_{\ell_\infty}.$$

The proof is concluded by $\|u\|_{\ell_\infty}\leqq C(s)|u|_s$ $(s>d/2)$ since we may choose $u=L_h^{-1}f$ and $s=m+\Theta>d/2$. $\blacksquare$

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