On approximation by translates and related problems in function theory

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1. Introduction

From Wiener's approximation theorem we know that the set of finite linear combinations of translates of a function $f \in L(R)$ is dense in $L(R)$ if and only if its Fourier transform is never zero. What can be said if we only allow translates $f(\cdot - \lambda)$ with λ belonging to some fixed set Λ? Problems of this type have been studied by Edwards [3], [4], Ganelius [6], Landau [8], Lönnroth [9], and Zalik [10], [11] among others.

Several approximation problems can be transformed to problems about approximation by translates. We take the Müntz—Szász theorem as an example. Consider approximation in $L(0, 1)$ by linear combinations of monomials x^{μ_k}, where μ_k are distinct numbers greater than -1. Take $g \in L(0, 1)$. Under the transformation $x = \exp(-\exp(-t))$ the expression

$$\int_0^1 \left| \sum a_k x^{\mu_k} - g(x) \right| dx$$

converts to

$$\int_{-\infty}^{\infty} \left| \sum \frac{a_k}{1+\mu_k} f(t - \log(1+\mu_k)) - g(e^{-e^{-t}}) e^{-e^{-t}-t} \right| dt$$

where $f(t) = \exp(-\exp(-t) - t)$. Putting $\lambda_k = \log(1+\mu_k)$ this can be written

$$\int_{-\infty}^{\infty} \left| \sum b_k f(t - \lambda_k) - h(t) \right| dt$$

where $h \in L(R)$.

We will relate the approximation properties of the translates of f to its Fourier transform. In the example above the transform is $\Gamma(1+it) \sim t^{1/2} \exp\left(-\frac{\pi}{2}|t|\right)$, and the corollary to theorem 5 gives the precise answer that approximation is possible if and only if $\sum \exp(-|\lambda_k|)=\infty$, that is

$$\sum \min\left(\frac{1}{1+\mu_k},\ 1+\mu_k\right) = \infty.$$

We will study approximation in the spaces $L(R)$, $L^2(R)$, and $C_0(R)$. In section 3 we consider the case $\hat{f}(t)=O(\exp(-\theta(t)))$, where θ is even, convex, and $\theta(t)/t \to \alpha\neq 0, \infty$ as $t\to\infty$. Edwards [3] contains a result on approximation in $C_0(R)$ for the case $\hat{f}(t)\exp(\alpha|t|)\in L(R)$, a more restrictive hypothesis.

In section 4 we consider the case $\hat{f}(t)=O(\exp(-\alpha|t|^p))$, $p>1$. Assuming that Λ satisfies a separation condition we obtain a rather sharp theorem. Zalik [10], [11], deals with this problem with no separation condition, but the results do not involve the same degree of precision as ours.

I wish to thank Professor Tord Ganelius for suggesting the topic and for his support and kind interest in my work.

2. Zeros of functions analytic in a strip

The following theorem will be important in the next section. We give the proof at the end of this paper. By S_α we denote the strip $\{z=x+iy\colon |y|<\alpha\}$. The convex conjugate of a convex function ϱ is defined as usual; $\varrho^*(y)=\sup_x (xy-\varrho(x))$.

Theorem 1. *Let f be analytic in the strip S_α, let ϱ be an even, convex function on $(-\alpha, \alpha)$, and let ϱ^* be its convex conjugate. Suppose that*

$$(*) \qquad \sup_x |f(x+iy)| \leqq \exp(\varrho(y)), \quad |y|<\alpha.$$

Given $\lambda\in R$, define λ^* by

$$\frac{2}{\pi}\int_1^{\lambda^*} \frac{\varrho^*(x)-\varrho^*(0)+1}{x^2}\,dx = |\lambda|.$$

If $\{\lambda_n\}_1^\infty$ is a sequence of real zeros of f (counting multiplicities) and

$$\sum 1/\lambda_n^* = \infty$$

then $f=0$.

Conversely, if the series converges there is a function f analytic in the strip S_α, satisfying (*), with precisely the zeros $\{\lambda_n\}_1^\infty$. In particular

$$\int_0^\alpha \log^+(\varrho(y))\,dy < \infty$$

is a necessary and sufficient condition for the implication

$$\sum \exp\left(-\frac{\pi}{2\alpha}|\lambda_n|\right) = \infty \Rightarrow f=0$$

to hold.

3. Müntz—Szász type theorems

In this section Λ is an indexed set of real numbers. Given a function f we let $\Lambda(f)$ denote the set of linear combinations of functions $f(\cdot-\lambda_k)$, $\lambda_k\in\Lambda$. The class of functions θ that are even, convex, and satisfy $\theta(t)/t\to\alpha\neq 0, \infty$, as $t\to\infty$ is denoted by A. Given $\theta\in A$ and $\lambda\in\Lambda$ we define λ^* by

$$\frac{2}{\pi}\int_1^{\lambda^*}\frac{\theta(t)-\theta(0)+1}{x^2}\,dt=|\lambda|.$$

Fourier transforms will be taken without the factor $(2\pi)^{-1/2}$.

We first treat approximation in L^2.

Theorem 2. *Let* $f\in L^2$, $\theta\in A$, *and suppose that* $\hat f(t)\neq 0$ a.e. *and*

$$|\hat f(t)|\leqq\exp(-\theta(t)).$$

If $\sum_\Lambda 1/\lambda_k^*=\infty$ *then* $\Lambda(f)$ *is dense in* L^2.

On the other hand, if

$$\exp(-\theta(t))/\hat f(t)\in L^2$$

and $\sum_\Lambda 1/\lambda_k^*<\infty$ *then* $\Lambda(f)$ *is not dense in* L^2.

Proof. Take $g\in L^2$ such that $g\perp\Lambda(f)$.

Put

$$h(\lambda)=\int_{-\infty}^{\infty}f(x-\lambda)\,g(x)\,dx.$$

Then h is obviously the inverse Fourier transform of $\hat f(-t)\hat g(t)$. Since

$$\hat f(-t)\hat g(t)\exp(yt)\in L$$

for $|y|<\alpha$, it is clear that h can be analytically extended to S_α. If we let θ_1 be the largest convex minorant to $\theta(t)-\log(1+t^2)$ the inversion formula gives

$$|h(x+iy)|\leqq C_0\int_{-\infty}^{\infty}\exp(t|y|-\theta_1(t))\frac{|\hat g(t)|}{1+t^2}\,dt.$$

Hence

$$|h(x+iy)|\leqq C_1\exp(\theta_1^*(y)), \tag{3.1}$$

Defining λ_k^+ with respect to θ_1 it easily follows that there are $c_1>0$, $c_2>0$, such that $c_1<\lambda_k^+/\lambda_k^*<c_2$. This implies

$$\sum 1/\lambda_k^+=\infty. \tag{3.2}$$

By the elementary theory of convex functions $(\theta_1^*)^*=\theta_1$. Obviously $h(\lambda_k)=0$. By theorem 1 (3.1) and (3.2) imply $h=0$. Since $\hat f(t)\neq 0$ a.e. we conclude $g=0$ which proves the first part of the theorem.

Now take $h \neq 0$, holomorphic in S_α with $h(\lambda_k)=0$ for all k, and

$$|h(x+iy)| \leqq \exp(\theta^*(y)).$$

This is possible by theorem 1. Put $h_0(z)=h(z)/(z^2+2\alpha^2)$. The Fourier transform of h_0 is given by

$$\hat{h}_0(t) = \int h_0(x) e^{-itx}\, dx.$$

The path of integration can be taken to be $\operatorname{Im}(z)=-y$. One easily obtains

$$|\hat{h}_0(t)| \leqq C \exp(-(ty-\theta^*(y))).$$

Since $\theta^{**}=\theta$, minimizing over y gives

$$|\hat{h}_0(t)| \leqq C \exp(-\theta(t)).$$

Define $G(t)=\hat{h}_0(t)/\hat{f}(-t)$. The assumptions on f show that $G \in L^2$ and by Plancherel's theorem G is the Fourier transform of some $g \in L^2$. By inversion

$$h_0(\lambda) = \int f(x-\lambda)\, g(x)\, dx.$$

It is clear that $g \perp \Lambda(f)$. Since $g \neq 0$ this completes the proof.

In the special case $\hat{f}(t)=O(\exp(-\alpha|t|))$, the second part of theorem 2 does not give much information. We cover this case separately.

Theorem 3. *Let β be an even, positive function, concave for $t \geqq 0$, such that*

$$\int_1^\infty \frac{\beta(t)}{t^2}\, dt < \infty.$$

Suppose $f \in L^2$, $|\hat{f}(t)| \geqq C \exp(-\alpha|t|-\beta(t))$, $C>0, \alpha>0$. If $\sum_\Lambda \exp\left(-\frac{\pi}{2\alpha}|\lambda_k|\right)<\infty$, then $\Lambda(f)$ is not dense in L^2.

Proof. Put $\beta_0(t)=\beta(t)+\log(1+|t|)$ and $p(x)=e^x/\beta_0(e^x)$ for $x \geqq 0$. Then p will be increasing. Define $a=\alpha+1/p(0)$. For $z \in \bar{S}_a$ the function defined by

$$R(z) = \exp\left(-\left\{\exp\left(\frac{\pi z}{4a}\right)+\exp\left(-\frac{\pi z}{4a}\right)\right\}\right)$$

has modulus less than $\exp(-c\exp(b|x|))$ for some positive constants b and c $(x=\operatorname{Re}(z))$. Put $p_0(x)=p(bx)$ and $\psi(x)=\alpha+1/p_0(x)$. Let D be the region $\{z=x+iy\colon |y|<\psi(|x|)\}$ and map it conformally onto S_α by φ such that $\varphi(0)$, $\varphi'(0)>0$. Define $\mu_k=\varphi(\lambda_k)$. By Ahlfors' distortion theorem, [1],

$$\mu_k = \alpha \int_0^{\lambda_k} \frac{dx}{\psi(x)} + O(1) = \lambda_k - \int_0^{\lambda_k} \frac{dx}{1+\alpha p_0(x)} + O(1).$$

The last integral is less than

$$\frac{1}{\alpha}\int_0^\infty \frac{dx}{p_0(x)} = \frac{1}{\alpha b}\int_1^\infty \frac{\beta_0(t)}{t^2}\,dt < \infty.$$

Hence $\sum \exp\left(-\frac{\pi}{2\alpha}|\mu_k|\right)<\infty$, and we can find $g\neq 0$, holomorphic in S_α with $g(\mu_k)=0$ and $|g|<1$.

Define

$$h(z) = g(\varphi(z))\,R(z)/(z^2+2a^2), \quad z\in D.$$

Its Fourier transform will be estimated by integration along the lower boundary of D. Suppose $t\geqq 0$. We substitute $x=s-i\alpha-i/p_0(s)$ and easily verify $|dx|\leqq C ds$, for some $C>0$. Using the estimate for $|R|$ and the fact that $|g|$ is bounded we obtain

$$|\hat{h}(t)| \leqq C_0\int_0^\infty \exp\left(-\alpha t-\frac{t}{p_0(s)}-ce^{bs}\right)\frac{ds}{s^2+a^2}. \tag{3.3}$$

Choosing σ such that $ce^{b\sigma}=t/p_0(\sigma)$, which is possible for all $t\geqq M$ for some $M>0$, we have

$$\frac{t}{p_0(s)}+ce^{bs}\geqq\frac{t}{p_0(\sigma)} \tag{3.4}$$

for all $s\geqq 0$, since at least one of the terms on the left hand side is not smaller than the right member. Since $p_0(\sigma)\to\infty$ as $t\to\infty$ there is $M_1>M$ such that $t\geqq M_1$ implies $p_0(\sigma)>1/c$, and by the definition of σ, $\sigma<\log(t)/b$. Hence

$$t/p_0(\sigma) > t/p_0\left(\frac{\log t}{b}\right) = t/p(\log t) = \beta_0(t). \tag{3.5}$$

Using (3.4) and (3.5) in (3.3) yields

$$|\hat{h}(t)| \leqq C_1\exp(-\alpha t-\beta_0(t)), \quad t\geqq M_1.$$

By the corresponding estimate for $t\leqq 0$ and suitable choice of C we obtain

$$|\hat{h}(t)| \leqq C\frac{\exp(-\alpha|t|-\beta(t))}{1+|t|}.$$

Proceeding in the same way as in the proof of theorem 2 we find that $\Lambda(f)$ is not dense in L^2. This finishes the proof.

Corollary. *Let $f\in L^2$ and suppose that*

$$C_1(|t|+1)^{-n} \leqq |\hat{f}(t)|\exp(\alpha|t|) \leqq C_2(|t|+1)^m$$

for some $n\geqq 0$, $m\geqq 0$, $\alpha>0$.

Then $\Lambda(f)$ *is dense in* L^2 *if and only if* $\sum_\lambda \exp\left(-\frac{\pi}{2\alpha}|\lambda_k|\right)=\infty$.

Remark. With $f(x)=\exp\left(-\exp(-t)-\frac{t}{2}\right)$ one has

$$\hat{f}(t)=\Gamma\left(\tfrac{1}{2}+it\right)\sim C\exp\left(-\tfrac{\pi}{2}|t|\right)$$

and the corollary gives Müntz—Szász theorem for $L^2(0, 1)$, if one makes the transformation shown in the introduction.

We now turn to approximation in L. Here it seems that stronger conditions on $\hat{f}$ are needed.

Theorem 4. *Let* $f\in L$, $\theta\in A$, *and suppose that* $\hat{f}(t)\neq 0$ *for all* t, *and*

$$|D\hat{f}(t)|\leqq C\exp(-\theta(t)).$$

If $\sum_\Lambda 1/\lambda_k^*=\infty$, *then* $\Lambda(f)$ *is dense in* L.

On the other hand, if

$$\exp(-\theta(t))/\hat{f}(t)\in L,$$

and $\sum_\Lambda 1/\lambda_k^*<\infty$, *then* $\Lambda(f)$ *is not dense in* L.

Proof. Put $F=\hat{f}$. First observe that $F(t)=O(\exp(-\theta(t)))$: Since $\theta'(t)>\alpha/2$ for t sufficiently large,

$$|F(t)|\leqq\int_{|t|}^{\infty}\exp(-\theta(x)\,dx)\leqq\frac{2}{\alpha}\int_{|t|}^{\infty}\exp(-\theta(x))\theta'(x)\,dx=\frac{2}{\alpha}\exp(-\theta(t)).$$

It is no restriction to assume that, for all x,

$$f(x)=\frac{1}{2\pi}\int e^{ixt}F(t)\,dt.$$

It follows that f has an analytic extension to S_α. We define f_y and F_y in the following way

$$f_y(x)=f(x+iy),\quad |y|<\alpha.$$
$$F_y(t)=\exp(-yt)F(t).$$

We have to show that $f_y\in L$. The L-norm can be estimated by Carlson's inequality. Since f_y is the inverse Fourier transform of F_y we have

$$\|f_y\|_1^2\leqq C_0\|F_y\|_2\|F_y'\|_2.$$

Let $\theta_1(t)$ be the largest convex minorant to $\theta(t)-\log(1+t^2)$.

$$\|F_y\|_2^2\leqq C_1\int(\exp(|y|t-\theta_1(t)))^2\frac{dt}{(1+t^2)^2}\leqq C_2\exp(2\theta_1^*(y)).$$

Moreover

$$\|F_y'\|_2^2=\|\exp(-yt)F'(t)-yF_y(t)\|_2^2\leqq C_3\exp(2\theta_1^*(y)),$$

since both terms can be handled as in the preceding inequality. Hence

$$\|f_y\|_1 \leqq C_4 \exp(\theta_1^*(y)),$$

and it is not difficult to see that the norm depends continuously on y.

Suppose that $g \in L^\infty$ and $g \perp A(f)$. Put

$$h(z) = \int f(x-z)\, g(x)\, dx.$$

To see that h is holomorphic in S_α, first note that h is continuous, then use Fubini's and Morera's theorems. Since $|h(x+iy)| \leqq C_5 \exp(\theta_1^*(y))$, and $h(\lambda_k)=0$, theorem 1 gives $h=0$, and we conclude $g=0$ by Wiener's approximation theorem.

The second part of the theorem follows by a small change in the proof of the corresponding part in theorem 2.

Theorem 5. *If $f \in L$ satisfies the conditions in theorem 3 and*

$$\sum_A \exp\left(-\frac{\pi}{2\alpha}|\lambda_k|\right) < \infty,$$

then $A(f)$ is not dense in L.

The proof is almost the same as for the theorem 3.

Corollary. *Let $f \in L$ and suppose that for some $m \geqq 0$, $n \geqq 0$, $\alpha > 0$,*

$$|D\hat{f}(t)| \exp(\alpha|t|) \leqq C_1(|t|+1)^m, \quad C_1 > 0,$$

$$|\hat{f}(t)| \exp(\alpha|t|) \geqq C_2(|t|+1)^{-n}, \quad C_2 > 0.$$

Then $A(f)$ is dense in L if and only if $\sum_A \exp\left(-\frac{\pi}{2\alpha}|\lambda_k|\right) = \infty$.

Remark. If we let f be as in the example in the introduction it follows, from properties of the gamma function, that $|D\hat{f}(t)| \leqq C|t|^{3/2} \exp\left(-\frac{\pi}{2}|t|\right)$. Hence the corollary is applicable.

The following theorem on approximation in C_0 has a proof similar to the proofs of theorems 2 and 4.

Theorem 6. *Let f be the Fourier transform of $g \in L$ and let $\theta \in A$.*

(a) *Suppose that $|g(t)| \leqq C \exp(-\theta(t))$ and that g is not zero a.e. on any open interval. If $\sum_A 1/\lambda_k^* = \infty$, then $A(f)$ is dense in C_0.*

(b) *Suppose g is differentiable, $\exp(-\theta(t))/g(t) \in L^2$, and*

$$|Dg/g| \leqq M, \quad M > 0. \tag{3.6}$$

If $\sum_A 1/\lambda_k^* < \infty$, then $A(f)$ is not dense in C_0.

Differentiability is important in (b) as the following proposition shows. Let us call a strictly increasing sequence of positive numbers, $\{\lambda_k\}_1^\infty$, *regular* if its counting function $n(\lambda)$ coincides for $\lambda=\lambda_k$ with a function $h(\lambda)$ such that $h'(e^x)$ is convex and increasing. A doubly infinite sequence $\{\lambda_k\}_{-\infty}^\infty$ will be called regular if $\{\lambda_k\}_1^\infty$ is regular and $\lambda_{-k}=-\lambda_k$.

Proposition. *Let f be the Fourier transform of $g\in L$ satisfying*

(a) $|g(t)|\leqq C\exp(-\alpha|t|)$ *for some* $\alpha>0$, $C>0$,

(b) *the set of points where g:s left and right hand limits exist and are different has a finite accumulation point,* ξ,

(c) *g is not zero a.e. on any open interval.*

If $\{\lambda_k\}_{-\infty}^\infty$ is regular and

$$\sum_{-\infty}^{\infty}(1+\lambda_k^2)^{-1}=\infty, \tag{3.7}$$

then $A(f)$ is dense in C_0.

Proof. It is easily seen that the number of λ_k in the interval $[x, x+1]$ is between $h'(x)-1$ and $h'(x+1)+1$ for $x\geqq 0$. Take $d\mu\perp A(f)$, put $F(\lambda)=\int f(x-\lambda)d\mu(x)$ and extend F analytically to S_α. Since F is uniformly bounded in, say, $S_{\alpha/2}$ and $F(\lambda_k)=0$, Schwartz's lemma yields

$$\log|F(x)|\leqq\sum\log\left|\frac{\exp(\pi x/\alpha)-\exp(\pi\lambda_k/\alpha)}{\exp(\pi x/\alpha)+\exp(\pi\lambda_k/\alpha)}\right|+C_0 \tag{3.8}$$

$$\leqq -2\sum\exp\left(-\frac{\pi}{\alpha}|x-\lambda_k|\right)+C_0\leqq -C_1h'(x)-C_2,\quad (C_1, C_2>0).$$

Izumi and Kawata [7], have proved that a function $F\in L$ satisfying $\log|F(x)|\leqq -\omega(|x|)$, where $\omega(e^x)$ is convex, and

$$\int_0^\infty\frac{\omega(x)}{1+x^2}\,dx=\infty,$$

has its Fourier transform in a quasi-analytic class. Now

$$\int_0^\infty\frac{h'(x)}{1+x^2}\,dx=\int_{h(0)}^\infty\frac{dt}{1+\lambda(t)^2},$$

where λ is the inverse function to h. Since λ is increasing it follows from (3.7), by a comparison argument, that the integral to the right is divergent. Then, by (3.8) and the cited theorem, $\hat{F}$ is quasi-analytic. We claim that all derivatives of $\hat{F}$ are zero at ξ. This clearly would give the conclusion of the theorem. Observe that, since $F\in L$,

$$\hat{F}(t)=g(t)\hat{\mu}(t)\quad\text{a.e.} \tag{3.9}$$

Let t_i be a discontinuity point of g, as described in (b), and let t tend to t_i avoiding points where (3.9) does not hold. Since $\hat{F}$ and $\hat{\mu}$ are continuous

$$\hat{\mu}(t_i)\lim_{t\uparrow t_i} g(t) = \hat{F}(t_i) = \hat{\mu}(t_i)\lim_{t\downarrow t_i} g(t).$$

But the limits are different, hence $\hat{\mu}(t_i)=0$, so $\hat{F}(t_i)=0$. Since $t_i\to\xi$ repeated application of Rolle's theorem shows $D^n\hat{F}(\xi)=0$, all n. The proof is complete.

4. Functions with rapidly decreasing transforms

The theorem in this section is stated for L^2, but it has analogues in L and C_0 in the same way as the theorems in section 3. The constant in front of $\log(r)$ is sharp for $p>1$, we give a simple example for $p=2$.

Theorem 7. *Let $f\in L^2$, suppose $\alpha>0, p>1$ and $\hat{f}(t)=O(\exp(-\alpha|t|^p))$, $\hat{f}(t)\neq 0$ a.e. Let q be the conjugate exponent to p and suppose that for some $\delta>0$*

$$\lambda_{n+1}^q-\lambda_n^q\geqq\delta,\quad (\lambda_n>0)$$

and

$$\limsup_{r\to\infty}\left\{\sum\nolimits_{0<\lambda_n<r}\lambda_n^{-q}-\frac{(p\alpha)^{1-q}}{\pi}\left(\sin\left(\frac{\pi}{2q}\right)\right)^q\log r\right\}=\infty.$$

Then $\Lambda(f)$ is dense in L^2.

In the proof we use the following lemma.

Lemma. *Suppose G is an entire function such that*

$$\limsup_{r\to\infty}\frac{\log|G(re^{i\theta})|}{r^q}\leqq\beta\,|\sin\theta|^q,$$

$q>1$, $\beta>0$. Suppose that for $n\geqq 0$, $\lambda_n>0$, $\lambda_{n+1}^q-\lambda_n^q\geqq\delta>0$, and $G(\lambda_n)=0$. If

$$\limsup_{r\to\infty}\left\{\sum\nolimits_{\lambda_n<r}\lambda_n^{-q}-\frac{\beta q}{\pi}\left(\sin\left(\frac{\pi}{2q}\right)\right)^q\log r\right\}=\infty,$$

then $G=0$.

Proof. Define $H(z)=G(z^{1/q})$ for $|\arg(z)|\leqq\frac{\pi}{2}$. Then

$$\limsup_{r\to\infty}\frac{\log|H(re^{i\theta})|}{r}=\limsup_{r\to\infty}\frac{\log|G(r^{1/q}e^{i(\theta/q)})|}{r}$$

$$\leqq\beta(\sin(\theta/q))^q\leqq\beta\left(\sin\left(\frac{\pi}{2q}\right)\right)^q=c,\quad\text{say}.$$

We now use Fuchs' theorem concerning zeros of functions of exponential type, [5]. Evidently H is of exponential type c, has zeros at the points $\mu_n=\lambda_n^q$, $\mu_{n+1}-\mu_n\geqq\delta>0$ and

$$\limsup_{R\to\infty}\left\{\sum_{\mu_n<R}\frac{1}{\mu_n}-\frac{c}{\pi}\log R\right\}=\limsup_{r\to\infty}\left\{\sum_{\lambda_n<r}\lambda_n^{-q}-\frac{cq}{\pi}\log r\right\}=\infty.$$

Hence $H=0$ and the lemma follows.

Proof of theorem 7. Take $g\in L^2$, $g\perp\Lambda(f)$ and define F by convolution as before. Then F will be entire and

$$|F(x+iy)|\leqq C\int_0^\infty\exp\left(-\alpha t^p+\log(1+t^2)+|y|\,t\right)\frac{\hat{g}(t)\,dt}{1+t^2}$$

$$\leqq C'\exp\left(\max_{t\geqq 0}\{|y|\,t-\alpha t^p+\log(1+t^2)\}\right).$$

If the maximum is attained at $t=\tau$, then

$$|y|=\alpha p\tau^{p-1}-\frac{2\tau}{1+\tau^2}>\alpha p\tau^{p-1}-1,$$

hence $|y|+1\geqq\alpha p\tau^{p-1}\geqq|y|$ and the maximum value is

$$|y|\,\tau-\alpha\tau^p+\log(1+\tau^2)\leqq\alpha(p-1)\tau^p+\log(1+\tau^2)$$

$$\leqq\alpha^{1-q}(p-1)p^{-q}(|y|+1)^q+o(|y|).$$

Hence

$$\limsup_{r\to\infty}\left(\frac{\log|F(re^{i\theta})|}{r}\right)\leqq\alpha^{1-q}(p-1)p^{-q}(|\sin\theta|)^q.$$

Now $F=0$ follows from the lemma since $F(\lambda_n)=0$, and $\alpha^{1-q}(p-1)p^{-q}q=(p\alpha)^{1-q}$. It then follows that $g=0$ and the theorem is proved.

For $p=2$ it is easy to see that the constant in front of $\log(r)$ cannot be smaller. Take e.g. $F(x)=\sin(x^2/2)\exp(-x^2/2)/x^2$. Using the fact that $|F(z)|\leqq\exp(y^2)/(x^2+y^2)$ one obtains

$$|\hat{F}(t)|\leqq C\exp(-t^2/4)/(1+|t|).$$

Hence, given $f\in L^2$ with $|\hat{f}(t)|\sim\exp(-t^2/4)$ there is $g\in L^2$ such that the convolution equation $F=f*g$ is satisfied. This shows that $\Lambda(f)$ is not dense in L^2 if $\Lambda=\{(2\pi n)^{1/2}\}_1^\infty$. On the other hand, by theorem 7, if $c<2\pi$ and $\Lambda=\{(cn)^{1/2}\}_1^\infty$, then $\Lambda(f)$ is dense in L^2.

5. Proof of theorem 1

It is no restriction to assume ϱ twice continuously differentiable with $\varrho''>0$. Furthermore we may assume $\varrho(y)\to\infty$ as $y\to\alpha$, since otherwise f is bounded and the theorem follows from Blaschkes theorem by a transformation to the unit disc.

By definition of convex conjugate,

(5.1) $$\varrho^*(t) = st-\varrho(s), \quad \text{where} \quad \varrho'(s) = t.$$

Differentiation of (5.1) yields

(5.2) $$\frac{d}{dt}\varrho^*(t) = s.$$

Put $\theta(t)=\varrho^*(t)$. By (5.2) and (5.1)

$$\varrho(\theta'(t)) = \varrho(s) = st-\varrho^*(t) = t\theta'(t)-\theta(t).$$

Hence, by (*) in the statement of the theorem,

(5.3) $$|f(x+i\theta'(t))| \leqq \exp(t\theta'(t)-\theta(t))$$

independently of x. Note that, by convexity, the right side increases with t. Also note that $\theta'(t)>0$ for $t>0$ and $\theta(t)/t\to\alpha$ as $t\to\infty$.

We shall now prove the first part of theorem 1. Suppose that f is not identically zero. We can assume $f(0)\neq 0$. Let D be the band-shaped domain bordered by the four curves

(5.4) $$t\curvearrowright\pm\frac{2}{\pi}\int_0^t\theta'(e^u)\,du\pm i\theta'(e^t), \quad t\in[0,\infty[.$$

The domain D is illustrated in figure 1.

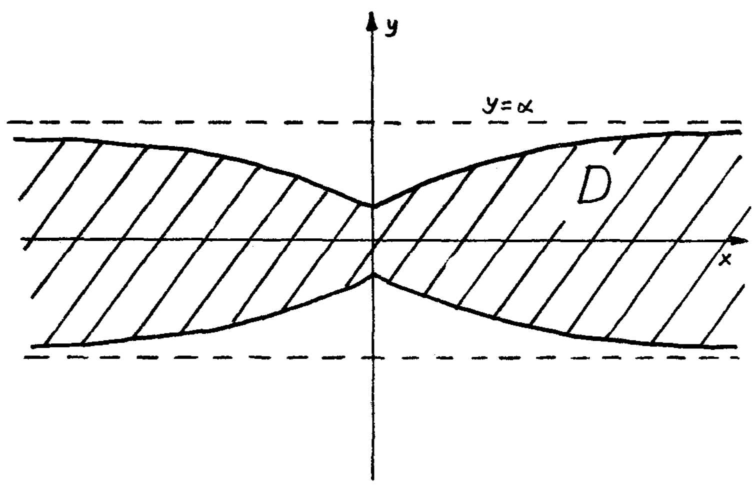

Fig. 1

Let φ map $S_{\pi/2}$ conformally onto D, such that $\varphi(0), \varphi'(0)>0$. Then φ preserves symmetry with respect to the axes. Putting $h(z)=f(\varphi(z))$ we obtain a function holomorphic in $S_{\pi/2}$. We will use Ahlfors' distortion theorem [1] to show that $h(0)\neq 0$ leads to a contradiction.

Let $\psi(u)$ be half the length of the intersection between the line $x=u$ and D. Given $\mu\in R$, put $\lambda=\varphi(\mu)$. By the distortion theorem there is a constant k such that for all λ

$$\left|\varphi^{-1}(\lambda)-\frac{\pi}{2}\int_0^\lambda \frac{dt}{\psi(t)}\right|<k. \tag{5.5}$$

By symmetry it is sufficient to consider $\mu>0$, hence $\lambda>0$. Define

$$\Phi(\lambda)=\frac{\pi}{2}\int_0^\lambda \frac{dt}{\psi(t)}. \tag{5.6}$$

Making the substitution $t=\frac{\pi}{2}\int_0^u \theta'(e^x)\,dx$ and observing that, by construction, $\psi(t)=\theta'(e^u)$ we find $\Phi(t)=u$, hence

$$\Phi^{-1}(u)=\frac{2}{\pi}\int_0^u \theta'(e^x)\,dx. \tag{5.7}$$

The function h extends continuously to $\bar{S}_{\pi/2}$. We shall estimate it at the boundary. For $x\geqq 0$ we define x' by $\varphi(x')=\operatorname{Re}\varphi\left(x+i\frac{\pi}{2}\right)$. By the distortion theorem, there is a constant k_1 such that $|x'-x|<k_1$. Put $u=\Phi(\varphi(x'))$. By (5.5) $|x'-u|<k$, hence $|u-x|<k_2$. By the definition of u and (5.7)

$$h\left(x+i\frac{\pi}{2}\right)=f(\varphi(x')+i\psi(\varphi(x')))=f\left(\frac{2}{\pi}\int_0^u \theta'(e^t)\,dt+i\theta'(e^u)\right),$$

and, by (5.3),

$$\left|h\left(x+i\frac{\pi}{2}\right)\right|\leqq \exp\left(e^{x+k_2}\theta'(e^{x+k_2})-\theta(e^{x+k_2})\right). \tag{5.8}$$

Note also that, by the same argument, $|h(x+iy)|$ is majorized by the right side of (5.8), for $|y|\leqq\frac{\pi}{2}$.

We now use the inequality

$$\log|h(0)|+\sum\log\left|\frac{\exp(\pi\mu_n/2\beta)+1}{\exp(\pi\mu_n/2\beta)-1}\right| \tag{5.9}$$

$$\leqq\frac{1}{2\beta}\int_{-\infty}^{\infty}\frac{\log|h(x+i\beta)h(x-i\beta)|\,dx}{\exp(\pi x/2\beta)+\exp(-\pi x/2\beta)},$$

where $\mu_n=\varphi^{-1}(\lambda_n)$ are zeros of h and $0<\beta<\pi/2$. This inequality is obtained by transformation to the strip S_β of the following well known inequality for $F\in H^\infty(U)$, where U is the unit disc.

$$\log|F(0)|+\sum\log\left(\frac{1}{r_k}\right)\leqq\frac{1}{2\pi}\int_{-\infty}^{\infty}\log|F^*(e^{i\theta})|\,d\theta,$$

with F^* the radial limit function and $r_k e^{i\theta_k}$ zeros of F.

That h is bounded in each strip S_β follows from

$$\text{(5.10)}\qquad |\mathrm{Im}\,\varphi(x+iy)|\leqq\frac{2\alpha}{\pi}|y|.$$

To see this, observe that $\mathrm{Im}\,\varphi(z)$ is harmonic in $\bar{D}$, and that the inequality holds for $y=0$ and $y=\pi/2$. By a Phragmén—Lindelöf argument it follows for $0\leqq y\leqq\pi/2$ which is sufficient, by symmetry.

Using (5.8) we find that the right side of (5.9), for $\beta>\pi/4$ say, is not larger than

$$C\int_0^\infty\frac{e^x\theta'(e^x)-\theta(e^x)}{e^x}\,dx=C\left[\frac{\theta(t)}{t}\right]_1^\infty<C\alpha.$$

Since the terms in the sum in (5.9) are not smaller than $2\exp(-\pi|\mu_n|/2\beta)$, it follows, by letting β tend to $\pi/2$, that $\sum\exp(-|\mu_n|)<\infty$. But $\Phi^{-1}(\log\lambda_n^*)=O(1)+|\lambda_n|$ as is easily seen. Hence $\log\lambda_n^*=O(1)+\Phi(|\lambda_n|)=|\mu_n|+O(1)$ and we can conclude $\sum 1/\lambda_n^*<\infty$, which contradicts the hypothesis.

We now turn to the converse part of theorem 1.

Lemma 1. *Suppose that ϱ is an even, convex function on $(-\alpha,\alpha)$, such that ϱ^* has an analytic continuation with uniformly bounded derivative in the region $|z|>R$, $|\arg(z)|<\delta$ for some $R>0$, $\delta>0$.*

If $\{\lambda_n\}_{-\infty}^{\infty}$ is a real sequence such that $\sum 1/\lambda_n^*<\infty$ then there is a function f analytic in the strip S_α such that f has precisely the zeros λ_n (counting multiplicities) and $\sup_x|f(x+iy)|\leqq\exp(\varrho(y))$.

Proof. Let θ be the continuation of ϱ^* and let D, φ, and ψ be defined as on pp. 281—282. Let $\zeta=\varphi^{-1}\colon D\to S_{\pi/2}$.

The idea of the construction is to take a Blaschke product with suitably located zeros in $S_{\pi/2}$, compose it with ζ to obtain a function F defined in D with zeros λ_n. If we could continue ζ across the boundary of D to the strip S_α we would have a candidate for the function f. However, by the construction of D, the function ζ can only be extended to a region $D_0=D\cup\{w=u+iv\colon |u|>x_0,\ |v|<\alpha\}$ for some $x_0>0$. This is no serious limitation, but we have to introduce an auxiliary mapping $w\colon S_\alpha\to D_0$ to obtain the desired function $f(z)=F(w(z))$.

We now claim

(* *) It is possible to continue ζ analytically to a region D_0 as above, in such a way that for some $M>0$ one has $|\zeta'(w)|\leqq M$ in $|\mathrm{Re}\,(w)|>x_0$, $|\mathrm{Im}\,(w)|<\alpha$.

We prove this fact showing that f has the desired properties. Note that it follows that for x_0 sufficiently large the image of D_0 under ζ is contained in, say, the strip $S_{3\pi/4}$.

Let w map S_α conformally onto D_0 with $w(0)=0$, $w'(0)>0$, so that symmetry with respect to the axes is preserved. Put $\sigma(z)=\zeta(w(z))$ and

$$f(z)=\prod\frac{\exp(\sigma(z))-\exp(\sigma(\lambda_n))}{\exp(\sigma(z))+\exp(\sigma(\lambda_n))},\quad z\in S_\alpha. \tag{5.11}$$

By the distortion theorem $w(\lambda_n)=\lambda_n+O(1)$ hence

$$|\sigma(\lambda_n)|=|\zeta(\lambda_n)|+O(1)=\log(\lambda_n^*)+O(1) \tag{5.12}$$

where the last equality follows as in the proof of the first part of theorem 1. Thus $\sum'\exp(-|\sigma(\lambda_n)|)<\infty$, hence the product (5.11) converges if $\sigma(z)\in S_\pi$, a fortiori if $z\in S_\alpha$.

By reasons of symmetry it is sufficient to consider $z=x+iy$, $x\geqq 0$, $y\geqq 0$. Put $\theta=\mathrm{Im}\,\sigma(z)=\mathrm{Im}\,(\zeta(u+iv))=\mathrm{Im}\,(\zeta(w(x+iy)))$. As in (5.10) it follows that $0\leqq v\leqq y$. We shall use the inequality

$$\theta-\frac{\pi}{2}\leqq\begin{cases}M(y-\psi(u)) & \text{if}\quad \psi(u)\leqq y\\ 0 & \text{if}\quad \psi(u)>y.\end{cases} \tag{5.13}$$

To prove (5.13) we note that if $\psi(u)\leqq v$, then $\theta-\frac{\pi}{2}=\mathrm{Im}\,(\zeta(u+iv))-\mathrm{Im}\,\zeta(u+i\psi(u))\leqq M(v-\psi(u))\leqq M(y-\psi(u))$. If $\psi(u)>v$, then $\theta<\frac{\pi}{2}$, hence (5.13) is trivially satisfied.

Using the easily proved inequality

$$\log\left|\frac{1-re^{i\theta}}{1+re^{i\theta}}\right|\leqq\begin{cases}4r\left(\theta-\frac{\pi}{2}\right) & \text{if}\quad \frac{\pi}{2}\leqq\theta\leqq\frac{3\pi}{4},\quad 0\leqq r\leqq 1,\\ 0 & \text{if}\quad 0\leqq\theta\leqq\frac{\pi}{2},\end{cases}$$

and putting $r_k=\exp(-|\mathrm{Re}\,\sigma(z)-\sigma(\lambda_n)|)$ we get

$$\log|f(z)|=\sum\log\left|\frac{1-r_ne^{i\theta}}{1+r_ne^{i\theta}}\right|\leqq\begin{cases}4\left(\theta-\frac{\pi}{2}\right)\sum r_n & \text{if}\quad \frac{\pi}{2}\leqq\theta\leqq\frac{3\pi}{4}\\ 0 & \text{if}\quad 0\leqq\theta\leqq\frac{\pi}{2}.\end{cases} \tag{5.14}$$

With Φ as in (5.6) we have, by (5.12) and (5.5),

$$\left|\operatorname{Re}(\sigma(z))-\sigma(\lambda_n)\right| = |\operatorname{Re}\zeta(u+iv)-\zeta(\lambda_n)|+O(1) = |\Phi(u)-\Phi(\lambda_n)|+O(1).$$

Thus

$$\log|f(z)| \leqq \begin{cases} 4MC(y-\psi(u))\sum\exp(-|\Phi(u)-\Phi(\lambda_n)|) & \text{if} \quad y \geqq \psi(u) \\ 0 \quad \text{if} \quad y<\psi(u). \end{cases} \tag{5.15}$$

It is no restriction to assume $\lambda_n \leqq \lambda_{n+1}$ and $\lambda_0=0$. We shall prove that one can choose N such that

$$\sum\nolimits_{|\lambda_n| \geqq N} \exp\{-|\Phi(u)-\Phi(\lambda_n)|\} \leqq \frac{1}{4MC}\, e^{\Phi(u)} \quad \text{for} \quad u \geqq 0. \tag{5.16}$$

Since $\sum \exp(-|\Phi(\lambda_n)|)$ converges and has decreasing terms, there is a number n_0 such that $\exp(-\Phi(\lambda_n)) \leqq 1/(8MCn)$ for $n \geqq n_0$, hence $\lambda_{n_0} \leqq \lambda_n \leqq u$ implies $n \leqq \exp(\Phi(u))/(8MC)$. The contribution to the sum in (5.16) from the corresponding terms therefore cannot exceed $e^{\Phi(u)}/(8MC)$. If $N \geqq \lambda_{n_0}$ the remaining part of the sum in (5.16) is

$$e^{-\Phi(u)} \sum\nolimits_{\lambda_n \leqq -N} e^{-\Phi(|\lambda_n|)} + e^{\Phi(u)} \sum\nolimits_{\lambda_n \geqq \max(u,N)} e^{-\Phi(\lambda_n)}$$

which for N sufficiently large is less than $\exp(\Phi(u))/(8MC)$.

We assume that the product (5.11) is taken over indices n such that $|\lambda_n| \geqq N$. This is no restriction since the remaining zeros can be added by multiplication with a finite product with modulus less than 1.

By (5.15) we have $\log|f(z)| \leqq \sup_{u \geqq 0}(y-\psi(u))\exp(\Phi(u))$. Putting $\Phi(u)=s$ we get $\psi(u)=\theta'(e^x)$. Hence

$$\log|f(x+iy)| \leqq \sup_{s \geqq 0}(y-\theta'(e^s))e^s \leqq \sup_{t \geqq 0}(y-\theta(t)+\theta(t)-t\theta'(t))$$

$$\leqq \sup_{t \geqq 0}(yt-\theta(t))+C_0 = \theta^*(y)+C_0$$

since $\theta(t)-t\theta'(t)$ is decreasing.

It remains to prove $(**)$. It is sufficient to treat the case $\operatorname{Re}(w)>0$, $\operatorname{Im}(w)>0$. The boundary curve of D is given by

$$g(x) = \frac{2}{\pi}\int_0^x \theta'(e^u)\,du + i\theta'(e^x), \quad x \geqq 0.$$

The assumptions on θ permit us to extend g analytically to the half-strip

$$\{z = x+iy \colon x > \log R,\ |y|<\delta\}$$

by taking the integral along a path to z. Since $\theta'(e^x)\to\alpha$ as $x\to\infty$, and $|\theta'|$ is bounded, it follows by a standard argument that $\theta'(e^z)\to\alpha$ and $\dfrac{d}{dz}(\theta'(e^x))\to 0$

uniformly for $|y| \leqq \delta_0 < \delta$ as $x \to \infty$. It follows that, given $\varepsilon > 0$, there is a number K such that for $x > K$, $|y| \leqq \delta_0$, we have

$$(5.17) \qquad \left| g'(z) - \frac{2\alpha}{\pi} \right| < \varepsilon.$$

Hence, for K large enough, g is a conformal mapping defined on

$$B = \{z = x+iy\colon\ x > K,\ |y| \leqq \delta_0\}$$

and

$$\begin{cases} g(z) \in \bar{D} & \text{if} \quad \operatorname{Im}(z) \leqq 0 \\ g(z) \notin D & \text{if} \quad \operatorname{Im}(z) > 0. \end{cases}$$

Because of (5.17) it follows that $g(B)$ is a "road" with a certain width in the w-plane. Hence, for some $x_0 > 0$ and $r > 0$ the half-strip

$$D' = \{w = u+iv\colon\ u > x_0 - r,\ 0 < v < \alpha + r\}$$

is contained in $D \cup g(B)$.

Put $h(z) = \zeta(g(z)) - i\dfrac{\pi}{2}$ for $z \in B$, $\operatorname{Im} z < 0$. Then h can be continuously extended to real $z \in B$ and h takes real values for z real. By Schwarz's reflection principle h can be extended to all of B. We denote this extension by $\tilde{h}$. For $w \in g(B)$ define $\tilde{\xi}(w) = \tilde{h}(g^{-1}(w)) + i\dfrac{\pi}{2}$. Then $\tilde{\xi}$ and ζ coincide on $g(B) \cap D$, hence ζ can be continued to D'. By the construction $|\operatorname{Im}(\zeta)|$ is uniformly bounded in D'. It easily follows that $|\zeta'|$ is uniformly bounded in the smaller half-strip $D'' = \{w = u+iv\colon u > x_0,\ 0 < v < \alpha\}$. The proof of lemma 1 is complete.

To remove the analyticity assumption in lemma 1 we need the following result on one-sided approximation of concave functions.

Lemma 2. *Let F be an increasing, concave function defined for $x \geqq 0$. There is a function G, analytic in a sector $|\arg z| < v$, $v > 0$, such that*

(a) *$|G'|$ is uniformly bounded in the sector,*

(b) *for $x \geqq 0$, G is real, increasing and concave,*

(c) *$G(x) \leqq F(x)$,*

(d′) $\displaystyle\int_1^\infty (F(x) - G(x)) \frac{dx}{x^2} < \infty.$

If, in addition, $F'(0) < \infty$, then (d′) *can be sharpened to*

(d″) $\displaystyle\int_0^\infty (F(x) - G(x)) \frac{dx}{x^2} < \infty.$

Proof. We assume $F'(0)<\infty$ and prove (a)—(d″). The other case is then trivial. Further we assume that F is twice differentiable and that $F(0)=0$. Put $A=\lim_{x\to\infty} F'(x)$. The representation

$$F(x) = Ax - \int_0^\infty \min(t, x) F''(t)\, dt \tag{5.18}$$

is easily verified by differentiation. Let $p(t)=F(e^t)e^{-t}$ and $P=p*h$ where $h(t)=\pi^{-1/2}e^{-t^2}$. Since p is bounded it follows that P and P' have bounded analytic extensions to the strip $|\mathrm{Im}\,(z)|<v$, $v>0$.

We define G in the sector so that

$$G(e^z) = e^z P\left(z+\tfrac{1}{4}\right).$$

Differentiation yields (a). Differentiating again and observing that $F''\leqq 0$ we get

$$e^t G''(t) = \left((p''+p')*h\right)\left(t+\tfrac{1}{4}\right) = (F''*h)\left(t+\tfrac{1}{4}\right) \leqq 0,$$

hence G is concave. Of course it is also real and increasing since F is.

Put $q(t)=\min(1, e^{-t})$. By (5.18)

$$p(u) = A + \int_0^\infty (-F''(s))\, q(u-\log s)\, ds. \tag{5.19}$$

By Fubini's theorem

$$P(t) = A + \int_0^\infty (-F''(s)) Q(t-\log s)\, ds \tag{5.20}$$

where $Q=q*h$. Putting $R(t)=\int_{-\infty}^t h(u)\,du$ we get after an elementary calculation

$$Q(t) = e^{-t+\frac{1}{4}} R\left(t-\tfrac{1}{4}\right) + R(-t).$$

A rough estimate for R shows that

$$\int_{-\infty}^\infty \left|Q(t) - q\left(t-\tfrac{1}{4}\right)\right| dt < \infty$$

and it is easy to see that $Q(t)<e^{-t+1/4}$. Obviously $Q(t)<1$ for all t, hence

$$Q(t) < q\left(t-\tfrac{1}{4}\right). \tag{5.21}$$

By (5.19)—(5.21) $e^{-t}G(e^t)=P\left(t+\tfrac{1}{4}\right)\leqq p(t)=e^{-t}F(e^t)$ which proves (c).

To prove (d″) we substitute $x=e^t$ and use Fubini's theorem

$$\int_0^\infty (F(x)-G(x))\frac{dx}{x^2} = \int_{-\infty}^\infty \left(p(t)-P\left(t+\frac{1}{4}\right)\right)dt$$

$$= \int_0^\infty (-F''(s))\,ds \cdot \int_{-\infty}^\infty \left(q(t)-Q\left(t+\tfrac{1}{4}\right)\right)dt = (f'(0)-A)C < \infty.$$

Lemma 2 is proved.

To prove the converse part of theorem 1 we put $F(x)=\alpha x-\varrho^*(x)$ and apply lemma 2. Putting $\sigma(x)=\alpha|x|-G(|x|)$ we obtain a convex function such that $\lim_{|x|\to\infty}\sigma(x)/x=\alpha$. Define $\varrho_1(y)=\sigma^*(y)$. Then ϱ_1 is a convex function on $(-\alpha, \alpha)$ and $\varrho_1^*=\sigma^{**}=\sigma$, hence, by the properties of G, ϱ_1^* satisfies the analyticity condition in lemma 1.

In order to apply lemma 1 we must show that $\sum 1/\lambda_n^+<\infty$ where

$$\frac{2}{\pi}\int_1^{\lambda_n^+} \frac{\varrho_1^*(x)-\varrho_1^*(0)+1}{x^2}\,dx = |\lambda_n|.$$

By (d′) of lemma 2 it follows that for some $c>0$, $\lambda_n^+/\lambda_n^*>c$. Hence $\sum 1/\lambda_n^*<\infty$ implies $\sum 1/\lambda_n^+<\infty$.

Now lemma 1 gives us a function f with zeros λ_n such that $\sup_x |f(x+iy)|\leqq \exp(\varrho_1(y))$. Since $G(x)\leqq F(x)$ we have $\sigma(x)\geqq\varrho^*(x)$. Hence $\varrho_1(y)=\sigma^*(y)\leqq\varrho(y)$ and the proof is complete.

The last statement in theorem 1 is a consequence of a lemma, proved in [2], and the first part of theorem 1. Put $\beta(x)=\alpha x-\varrho^*(x)$, $x\geqq 0$. From the cited lemma it follows that

$$\int_1^\infty \frac{\beta(x)}{x^2}\,dx<\infty,$$

if and only if

$$\int_0^\alpha \log^+(\varrho(y))\,dy<\infty.$$

Suppose that the integrals converge. Then it is not difficult to see that $|\lambda_n|= \frac{2\alpha}{\pi}\log(\lambda_n^*)+O(1)$. Hence $\sum\exp\left(-\frac{\pi}{2\alpha}|\lambda_n|\right)=\infty$ implies $\sum 1/\lambda_n^*=\infty$, which, as we have proved, leads to $f=0$.

If the integrals diverge, then $1/\lambda_n^*=o\left(\exp\left(-\frac{\pi}{2\alpha}|\lambda_n|\right)\right)$, and it is possible to find a sequence $\{\lambda_n\}_1^\infty$ such that $\sum\exp\left(-\frac{\pi}{2\alpha}|\lambda_n|\right)=\infty$, but $\sum 1/\lambda_n^*<\infty$. From the converse part of theorem 1 it follows that there exists $f\neq 0$ with zeros at λ_n, satisfying (*).

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