Polynomially convex hulls and analyticity

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Introduction

We denote by z, w the coordinates in $\mathbf{C}^2$ and we write π for the projection which sends $(z, w) \to z$. Let Y be a compact subset of $\mathbf{C}^2$ with $\pi(Y)$ contained in the unit circle. We denote by $\hat{Y}$ the polynomially convex hull of Y. For λ in $\mathbf{C}$ we put

$$\pi^{-1}(\lambda) = \{(z, w) \in \hat{Y} \mid \pi(z, w) = \lambda\}.$$

We assume that $\pi^{-1}(\lambda) \neq \emptyset$ for some λ with $|\lambda| < 1$. Then $\pi^{-1}(\lambda) \neq \emptyset$ for each λ in the open unit disk.

Under various conditions $\hat{Y} \setminus Y$ has been shown to possess analytic structure. In particular we have ([4], [5]):

Theorem. *If $\pi^{-1}(\lambda)$ is finite or countably infinite for each λ in $|\lambda| < 1$, then $\hat{Y} \setminus Y$ contains an analytic variety of dimension* 1.

The object of this note is to show that no such conclusion holds in general.

Theorem 1. *There exists a compact subset Y of $\mathbf{C}^2$ with $\pi(Y) \subseteqq \{|z| = 1\}$ such that $\pi(\hat{Y}) = \{|z| \leqq 1\}$ and $\hat{Y} \setminus Y$ contains no analytic variety of positive dimension.*

Our construction proceeds by modifying the idea which was used by Brian Cole in [1] (see also [3], Theorem 20.1) to prove the infinite-dimensional analogue of Theorem 1.

In the famous example of a hull without analytic structure given by Stolzenberg in [2] the set whose hull is taken and the hull have the same coordinate projections. In our example the projection $\pi(Y)$ is a proper subset of the projection $\pi(\hat{Y})$.

Note. By a change of variable we may replace the unit circle and unit disk by the circle $|z| = 1/2$ and the disk $|z| \leqq 1/2$, and we shall prove Theorem 1 for this case. The convenience that results is that for $|a|$, $|b| \leqq 1/2$, $|a-b| \leqq 1$.

Notations. Let $a_1, a_2, \ldots$ denote the points in the disk $|z|<1/2$ both of whose coordinates are rational numbers. Fix an n-tuple of positive constants $c_1, c_2, \ldots, c_n$. For each j we denote by B_j the algebraic function

$$B_j(z) = (z-a_1)(z-a_2)\ldots(z-a_{j-1})\sqrt{(z-a_j)}$$

and by g_n the algebraic function

$$g_n(z) = \sum_{j=1}^n c_j B_j(z).$$

We denote by $\sum(c_1, \ldots, c_n)$ the subset of the Riemann surface of g_n which lies in $|z|\leqq 1/2$. In other words,

$$\sum(c_1, \ldots, c_n) = \{(z, w)| |z| \leqq 1/2, \ w = w_j, \ j = 1, 2, \ldots, 2^n\},$$

where w_j, $j=1, \ldots, 2^n$ are the values of g_n at z.

Lemma 1. *There exists a sequence c_j, $j=1, 2, \ldots$ of positive constants with $c_1=1/10$ and $c_{n+1}\leqq(1/10)c_n$, $n=1, 2, \ldots$ and there exists a sequence $\{\varepsilon_j | j=1, 2, \ldots\}$ of positive constants, and there exists a sequence of polynomials $\{P_n\}$ in z and w such that*

$$\{P_n = 0, \ |z| \leqq 1/2\} = \sum(c_1, \ldots, c_n), \quad n = 1, 2, \ldots \tag{1}$$

$$\{|P_{n+1}| \leqq \varepsilon_{n+1}, |z| \leqq 1/2\} \subseteqq \{|P_n| < \varepsilon_n, \ |z| \leqq 1/2\}, \quad n = 1, 2, \ldots \tag{2}$$

(3) *If $|a|\leqq 1/2$ and $|P_n(a, w)|\leqq\varepsilon_n$, then there exists w_n with $P_n(a, w_n)=0$ and $|w-w_n|<1/n$, $n=1, 2, \ldots$.*

Proof. For $j=1$, we take $c_1=1/10$, $\varepsilon_1=1/4$, $P_1(z, w)=w^2-(1/100)(z-a_1)$. Then (1) and (3) hold. Suppose now that c_j, ε_j, P_j have been chosen for $j=1, 2, \ldots, n$ in such a way that (1), (2), (3) are satisfied. We shall choose $c_{n+1}, \varepsilon_{n+1}, P_{n+1}$.

Denote by $w_j(z)$, $j=1, 2, \ldots, 2^n$, the roots of $P_n(z, \cdot)=0$. To each constant $c\geqq 0$ we assign a polynomial P_c by putting

$$P_c(z, w) = \prod_{j=1}^{2^n} \left[(w-w_j(z))^2 - c^2(B_{n+1}(z))^2\right].$$

Then $P_c(z, \cdot)=0$ has the roots $w_j(z)\pm cB_{n+1}(z), j=1, \ldots, 2^n$, and so $\{P_c(z, w)=0\}\cap\{|z|\leqq 1/2\}=\sum(c_1, c_2, \ldots, c_n, c)$. Also,

$$P_c = P_n^2 + c^2 Q_1 + \ldots + (c^2)^{2^n} Q_{2^n},$$

where the Q_j are polynomials in z and w, not depending on c.

Claim: for sufficiently small positive c,

$$\left\{|P_c| < \frac{\varepsilon_n^2}{2}\right\} \cap \{|z| \leqq 1/2\} \subset \{|P_n| < \varepsilon_n\} \cap \{|z| \leqq 1/2\}. \tag{4}$$

Denote by Δ_M the bidisk: $|z|\leqq 1/2$, $|w|\leqq M$. For M sufficiently large, Δ_M contains

$$\left\{|P_c| < \frac{\varepsilon_n^2}{2}\right\}\cap\{|z| \leqq 1/2\}$$

for all c, $0\leqq c\leqq 1$.

Suppose the Claim is false. Then for arbitrarily small c $\exists\zeta_c$ in Δ_M with $|P_c(\zeta_c)|<\frac{\varepsilon_n^2}{2}$ and $|P_n(\zeta_c)|\geqq\varepsilon_n$. Since Δ_M is compact, ζ_c has an accumulation point ζ^* in Δ_M. Then $|P_n^2(\zeta^*)|\leqq\frac{\varepsilon_n^2}{2}$ and $|P_n(\zeta^*)|\geqq\varepsilon_n$. This is impossible, and hence the Claim is true.

Fix c such that (4) holds and such that $c<(1/10)c_n$. Then choose ε_{n+1} such that $\varepsilon_{n+1}<\frac{\varepsilon_n^2}{2}$ and such that $|P_c(z, w)|<\varepsilon_{n+1}$ and $|z|\leqq 1/2$ implies that there exists w_{n+1} with $P_c(z, w_{n+1})=0$ and $|w-w_{n+1}|<1/(n+1)$. Putting $c_{n+1}=c$, then, putting $P_{n+1}=P_c$, and choosing ε_{n+1} as above, we have that (1), (2), (3) hold for $j=1, 2, \ldots, n+1$. This completes the proof of Lemma 1 by induction.

Definition. With P_n, ε_n, $n=1, 2, \ldots$ chosen as in Lemma 1, we put

$$X=\bigcap_{n=1}^{\infty}\left[\{|P_n| \leqq \varepsilon_n\}\cap\{|z| \leqq 1/2\}\right].$$

It follows at once from this definition that X is a compact polynomially convex subset of the bidisk $\{|z|\leqq 1/2, |w|\leqq 1\}$. For each n we put

$$\Sigma_n = \{P_n = 0\}\cap\{|z| \leqq 1/2\} = \sum(c_1, \ldots, c_n)$$

where $c_1, c_2, \ldots$ is the sequence obtained in Lemma 1.

Lemma 2. *A point (z, w) belongs to X if and only if $|z|\leqq 1/2$ and there exists a sequence (z, w_n) with $(z, w_n)\in\sum_n$ and $w_n\to w$ as $n\to\infty$.*

Proof. Consider (z, w) with $|z|\leqq 1/2$ and assume there exists such a sequence (z, w_n). Fix n_0. Because of (2), if $k>n_0$, then

$$\{|P_k| \leqq \varepsilon_k,\ |z| \leqq 1/2\} \subseteqq \{|P_{n_0}| \leqq \varepsilon_{n_0}\}.$$

Since $P_k(z, w_k)=0$ for each k, $(z, w_k)\in\{|P_{n_0}|\leqq\varepsilon_{n_0}\}$ for each $k>n_0$. Hence $(z, w)\in\{|P_{n_0}|\leqq\varepsilon_{n_0}\}$. This holds for all n_0, and so $(z, w)\in X$.

Conversely, assume $(z, w)\in X$. Fix n. Then $|P_n(z, w)|\leqq\varepsilon_n$. Hence by (3) there exists w_n with (z, w_n) in $\sum_n$ and $|w-w_n|<1/n$. Hence $\{(z, w_n)\}$ is a sequence as required. Lemma 2 is proved.

We now go on to show that X contains no analytic disk. Suppose first that D is an analytic disk contained in X with z non-constant on D. Assuming this, we shall arrive at a contradiction.

z is one-one on some subdisk of D and so it is no loss of generality to suppose that D is given by an equation: $w=f(z)$, where f is a single-valued analytic function defined in some plane region contained in $|z|<1/2$. In that region we choose a rectangle defined by inequalities: $S_1 \leqq \mathrm{Re}\, z \leqq S_2$, $t_1 \leqq \mathrm{Im}\, z \leqq t_2$, with S_1, S_2, t_1, t_2, irrational numbers. We denote the boundary of this rectangle by γ. Then γ is a simple closed curve such that none of the points a_i lies on γ. We note the following:
(5) f is a continuous function defined on γ and $(z, f(z)) \in X$ for $z \in \gamma$.

We denote by z_1 the midpoint of the left-hand edge of γ and we denote by γ_1 the punctured curve $\gamma \setminus \{z_1\}$. For each j $B_j(z)=(z-a_1)(z-a_2)\dots(z-a_{j-1})\sqrt{(z-a_j)}$ has two single-valued continuous branches defined on γ_1. If a_j lies outside γ, then each branch extends continuously to γ, while for a_j inside γ each branch has a jump-discontinuity at z_1. We choose one of these branches, arbitrarily, and denote it β_j. Then $|\beta_j|$ is single-valued.

Let n be the smallest index such that a_n lies inside γ. The algebraic function $\sum_{j=1}^{n} c_j B_j$ has on γ_1 the 2^n branches

$$\sum_{j=1}^{n} c_j \varrho_j \beta_j$$

where each ϱ_j is a constant $=1$ or $=-1$. We denote by $\mathfrak{K}$ the collection of these 2^n functions on γ_1.

Assertion 1: Fix z in γ_1. There exists k in $\mathfrak{K}$, where k depends on z, such that

$$|f(z)-k(z)| < (1/4)|\beta_n(z)|c_n. \tag{6}$$

In view of (5) and Lemma 2, we can find w_N such that (z, w_N) lies on Σ_N and

$$f(z) = w_N + R(z), \quad \text{where} \quad |R(z)| \leqq (1/10)|\beta_n(z)|c_n.$$

Thus

$$f(z) = \sum_{\nu=1}^{N} c_\nu \varrho_\nu(z)\beta_\nu(z) + R(z) = k(z) + \sum_{\nu=n+1}^{N} c_\nu \varrho_\nu(z)\beta_\nu(z) + R(z)$$

where each $\varrho_\nu(z)=1$ or -1 and

$$k = \sum_{\nu=1}^{n} c_\nu \varrho_\nu(z)\beta_\nu \in \mathfrak{K}.$$

Then $|f(z)-k(z)| \leqq \sum_{\nu=n+1}^{N} c_\nu |\beta_\nu(z)| + R(z)$. Note that for all j,

$$\begin{aligned} |\beta_{j+1}(z)| &= |(z-a_1)\dots(z-a_j)|\sqrt{|z-a_{j+1}|} \\ &\leqq |(z-a_1)\dots(z-a_j)| \\ &\leqq |z-a_1|\dots|z-a_{j-1}|\sqrt{|z-a_j|} = |\beta_j(z)| \end{aligned}$$

So $\sum_{\nu=n+1}^{N} c_\nu |\beta_\nu(z)| \leqq \sum_{\nu=n+1}^{N} c_\nu |\beta_n(z)| \leqq |\beta_n(z)| \left[\frac{c_n}{10} + \frac{c_n}{10^2} + \dots\right] = \frac{1}{9}|\beta_n(z)|c_n$.
We thus get (6), as asserted.

Assertion 2. Let g, h be two distinct functions in $\mathfrak{K}$. Fix z in γ_1. Then

(7) $$|g(z)-h(z)| \geqq (3/2)|\beta_n(z)|c_n.$$

$$g = \sum_{j=1}^{n} c_j \varrho_j \beta_j, \quad h = \sum_{j=1}^{n} c_j \varrho_j' \beta_j$$

where ϱ_j, ϱ_j' are constants $=1$ or -1. For some j, $\varrho_j \neq \varrho_j'$. Let j_0 be the first such j. Then

$$g(z)-h(z) = \pm 2c_{j_0}\beta_{j_0}(z) + \sum_{j=j_0+1}^{n} c_j(\varrho_j-\varrho_j')\beta_j(z).$$

So

$$\begin{aligned} |g(z)-h(z)| &\geqq 2c_{j_0}|\beta_{j_0}(z)| - 2\sum_{j=j_0+1}^{n} c_j|\beta_j(z)| \\ &\geqq 2c_{j_0}|\beta_{j_0}(z)| - 2|\beta_{j_0}(z)|\left[\sum_{j=j_0+1}^{n} c_j\right] \\ &\geqq 2|\beta_{j_0}(z)|\left[c_{j_0} - \sum_{j=j_0+1}^{n} c_j\right] \geqq \frac{16}{9}|\beta_{j_0}(z)|c_{j_0} \\ &\geqq (3/2)|\beta_n(z)|c_n, \end{aligned}$$

proving (7).

Fix z_0 in γ_1. By Assertion 1 there is some k_0 in $\mathfrak{K}$ with

(8) $$|f(z_0)-k_0(z_0)| \leqq (1/4)|\beta_n(z_0)|c_n.$$

Assertion 3. Let k_0 satisfy (8). Then for all z in γ_1:

(9) $$|f(z)-k_0(z)| < (1/3)|\beta_n(z)|c_n.$$

$\{z|(9)$ holds at $z\}$ is an open subset $\mathcal{O}$ of γ_1 containing z_0. If $\mathcal{O} \neq \gamma_1$, then there is a boundary point p of $\mathcal{O}$ on γ_1. Then

(10) $$|f(p)-k_0(p)| = (1/3)|\beta_n(p)|c_n.$$

By Assertion 1, there is some k_1 in $\mathfrak{K}$ such that

(11) $$|f(p)-k_1(p)| \leqq (1/4)|\beta_n(p)|c_n.$$

Thus $|k_0(p)-k_1(p)| \leqq (7/12)|\beta_n(p)|c_n$. Also $k_0 \neq k_1$, in view of (10) and (11). This contradicts (7). Thus $\mathcal{O}=\gamma_1$ and so Assertion 3 is true.

For each continuous function u defined on γ_1 which has a jump at z_1 let us write $L^+(u)$ and $L^-(u)$ for the two limits of $u(z)$ as $z \to z_1$ along γ_1. Then by (9)

$$|L^+(f)-L^+(k_0)| \leqq (1/3)|\beta_n(z_1)|c_n,$$

and

$$|L^-(f)-L^-(k_0)| \leqq (1/3)|\beta_n(z_1)|c_n,$$

and so

$$\left|\left(L^+(f)-L^+(k_0)\right)-\left(L^-(f)-L^-(k_0)\right)\right| \leqq (2/3)|\beta_n(z_1)|c_n.$$

Since f is continuous on γ, this gives that the jump of k_0 at z_1 is in modulus $\leqq (2/3)|\beta_n(z_1)|c_n$. But k_0 is in $\mathfrak{K}$, so its jump at z_1 has modulus $2c_n|\beta_n(z_1)|$. This is a contradiction.

The assumption that there is an analytic disk contained in X with z non-constant on it must therefore be rejected. The assumption that X contains an analytic disk on which z is constant must also be rejected, for the following reason:

Assertion 4. Fix z_0 with $|z_0| \leqq 1/2$. Put $F_{z_0} = \{z = z_0\} \cap X$. Each connected component of F_{z_0} is a single point. Assume first that $z_0 \neq a_j$ for all j. Fix an integer N. Consider the 2^N points

$$w_j = \sum_{\nu=1}^{N} c_\nu \varrho_\nu^{(j)} B_\nu(z_0), \quad j = 1, 2, \ldots, 2^N, \ \varrho_\nu^{(j)} = \pm 1,$$

where for each ν $B_\nu(z_0)$ denotes one of the two values of B_ν at z_0, chosen arbitrarily. Then w_j, $j = 1, \ldots, 2^N$, are the w-coordinates of the points on $\sum_N$ lying over z_0. By calculations like those in the proof of Assertion 2, we find that

$$|w_j - w_k| \geqq (3/2)|B_N(z_0)| c_N. \tag{12}$$

By hypothesis, $B_N(z_0) \neq 0$.

Consider the closed disks with centers w_j, $j = 1, 2, \ldots, 2^N$, and radius $(1/2)|B_N(z_0)| c_N$. Because of (12) these disks are disjoint.

Fix (z_0, b) in F_{z_0}. We claim that b belongs to the union of these 2^N disks. By Lemma 2 there exists $M > N$ and there exists (z_0, w') in $\sum_M$ such that

$$|b - w'| < (1/9) c_N |B_N(z_0)|.$$

Then

$$w' = \sum_{\nu=1}^{M} c_\nu \varrho_\nu B_\nu(z_0)$$

with $\varrho_\nu = \pm 1$.

Hence for some j, $1 \leqq j \leqq 2^N$,

$$w' = w_j + \sum_{\nu=N+1}^{M} c_\nu \varrho_\nu B_\nu(z_0).$$

So

$$|w' - w_j| \leqq \sum_{\nu=N+1}^{M} c_\nu |B_\nu(z_0)| \leqq \frac{1}{9} c_N |B_N(z_0)|.$$

Hence $|b - w_j| \leqq (2/9) c_N |B_N(z_0)|$. Thus b belongs to the disk with center w_j and radius $(1/2) c_N |B_N(z_0)|$ and so to the union of the 2^N disks, as claimed. Since b was arbitrary with (z_0, b) in F_{z_0}, it follows that each connected component K of F_{z_0} is contained in a disk of radius $(1/2) c_N |B_N(z_0)|$ and hence has diameter $\leqq c_N$. This holds for arbitrary N. Hence K is a single point. This proves our Assertion, in this case. If $z_0 = a_j$, then $B_N(z_0) = 0$ for $N > j$ and so F_{z_0} is a finite set, hence totally disconnected.

We can thus conclude that X contains no analytic disk.

Lemma 3. *Put* $Y = X \cap \{|z| = 1/2\}$. *Then* $X = \hat{Y}$.

Proof. Since X is polynomially convex, $\hat{Y} \subseteq X$.

Fix now (z_0, w_0) in X. We shall show that (z_0, w_0) is in $\hat{Y}$. Let Q be a polynomial in z and w. By Lemma 2 we can find a sequence $\{(z_0, w_N)\}$ converging to (z_0, w_0) with (z_0, w_N) in Σ_N. Σ_N is a finite Riemann surface whose boundary lies on $\{|z|=1/2\}$. Hence $|Q(z_0, w_N)| \leqq |Q(z'_N, w'_N)|$, where (z'_N, w'_N) is a point of $\Sigma_N \cap \{|z|=1/2\}$. Let (z', w') be an accumulation point of the sequence $\{(z'_N, w'_N)\}$. Arguing as in the proof of Lemma 2, we see that (z', w') is in X. Also $|z'|=1/2$. Then denoting by $\{n_j\}$ a sequence of integers such that (z'_{n_j}, w'_{n_j}) converges to (z', w'), we have

$$|Q(z_0, w_0)| = \lim_{j\to\infty} |Q(z_0, w_{n_j})| \leqq \lim_{j\to\infty} |Q(z'_{n_j}, w'_{n_j})| = |Q(z', w')|.$$

Since (z', w') is in Y, $|Q(z_0, w_0)| \leqq \max_Y |Q|$. Thus (z_0, w_0) is in $\hat{Y}$, as claimed. Thus $X \subseteqq \hat{Y}$, and so $X = \hat{Y}$.

The set Y thus has the properties asserted in Theorem 1.

References

1. B. Cole, One point parts and the peak point conjecture, PhD dissertation, Yale University (1968).
2. G. Stolzenberg, A hull with no analytic structure, *Jour. of Math. and Mech.* **12** (1963).
3. J. Wermer, Banach algebras and several complex variables, 2nd edition, Graduate Texts in Mathematics 35, Springer-Verlag (1976).
4. E. Bishop, Holomorphic completions, analytic continuations and the interpolation of semi-norms, *Ann. of Math.* **78** (1963).
5. R. Basener, A condition for analytic structure, *Proc. Amer. Math. Soc.* **36** (1972).

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