Schatten classes and commutators of singular integral operators

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Let T be a Calderon—Zygmund transform — a singular integral operator with kernel $K(x-y)$, where K is homogeneous of degree $-n$ with mean value zero on spheres centered at the origin. We assume K is C^∞ except at the origin and not identically zero. If $f\in L^1_{\text{loc}}(R^n)$ let M_f be pointwise multiplication by f and consider the commutator $C_f=M_fT-TM_f$. Explicitly this is

$$C_f\,\varphi(x)=\int_{R^n}K(x-y)\big(f(x)-f(y)\big)\,\varphi(y)\,dy.$$

It follows from the Calderon—Zygmund theory ([13], Ch. 2) that the (principal-value) integral converges a.e. if φ is a bounded function with compact support. Uchiyama [14] proved that C_f extends to a bounded operator on $L^2(R^n)$ precisely when $f\in\text{BMO}\,(R^n)$, sharpening a result by Coifman, Rochberg and Weiss [3].

We will characterize functions f for which C_f belongs to the Schatten class S^p in the case $n\geqq 2$ as those in a certain Besov space. We recall the definition of S^p. If R is any compact operator on Hilbert space then R^*R is compact, positive and therefore diagonalizable. Let $\{S_n(R)\}$ be the sequence of square roots of eigenvalues of R^*R, counted according to multiplicity. For $0<p<\infty$ one says that $R\in S^p$ if $\{S_n(R)\}\in l^p$. In this paper, the endpoint class S^∞ is the class of bounded operators. For the theory of S^p classes, see [12]. We will require the following facts.

(1) If $p\geqq 1$, then $R\in S^p$ if and only if $\sum\left|\langle Re_n|f_n\rangle\right|^p<\infty$ for all choices of orthonormal bases $\{e_n\}$, $\{f_n\}$.

(2) If $p\geqq 2$ and $R\in S^p$ then $\sum\|Re_n\|^p<\infty$ for all choices of orthonormal basis $\{e_n\}$.

(3) If $0<p<\infty$, $0<q\leqq\infty$ let $S^{pq}=\{R\in B(H)|S_n(R)\in l^{pq}\}$ where l^{pq} is the Lorentz space [1]. We take $S^{\infty\infty}=S^\infty=B(H)$ but do not define $S^{\infty q}$ if $q<\infty$.

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The classes S^{pq} may then be interpolated by the real method; in fact $[S^{p_1 q_1}, S^{p_2 q_2}]_{\theta q} = S^{pq}$ if $p_1 \neq p_2$ and $\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}$ ([1], [12]). Note $S^{pp} = S^p$.

(4) For $p \geqq 2$ there is a sufficient condition due to Russo [11] for an integral operator to belong to S^p. If $G\colon M \times M \to C$ for some measure space (M, μ), let $G^*(x, y) = \overline{G(y, x)}$. Let $L^p(L^q)$ be the mixed norm space

$$\left\{G \middle| \left(\int |G(x, y)|^q \, d\mu(y)\right)^{p/q} d\mu(x) < \infty\right\}$$

and define

$$L^p(L^q)^{\text{symm}} = L^p(L^q) \cap L^p(L^q)^*, \text{ i.e. } G \in L^p(L^q)^{\text{symm}} \text{ iff } G, G^* \in L^p(L^q).$$

We have then

Theorem A (Russo). *If* $G \in L^p(L^{p'})^{\text{symm}}$ with $2 \leqq p \leqq \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$ *and if* $R\varphi(x) = \int G(x, y)\varphi(y)\, d\mu(y)$ *for* $\varphi \in L^2$, *then* $R \in S^p$. ∎

For $p=2$ (Hilbert—Schmidt operators) or $p=\infty$ this is classical.

The Besov spaces we use are the homogeneous ones. We give the definition of these spaces and refer to [4], [7], [13] for further discussion. Note that [13] only treats the analogous non-homogeneous spaces.

Definition. *Suppose* $1 \leqq p, q \leqq \infty$ *and* $0 < \alpha < 1$. *Let* $f \in L^1_{\text{loc}}(R^n)$. *Then* f *belongs to the Besov space* Λ^{pq}_α *if and only if*

$$\int_{R^n} \frac{\|f(x+t) - f(x)\|^q_{L^p(R^n, dx)}}{|t|^{n+q\alpha}}\, dt < \infty.$$

When $q=\infty$ this becomes $\|f(x+t) - f(x)\|_{L^p(R^n, dx)} = O(|t|^\alpha)$. One makes analogous definitions for $\alpha \geqq 1$ with the first difference $f(x+t) - f(x)$ replaced by a difference of order $[\alpha]+1$. We will need to interpolate Besov spaces by the real method. This is discussed in [1] and [7].

In case $n=1$, the question of when the commutator operators C_f belong to S^p has been considered by Howland [5], Peller [8], Coifman—Rochberg [2] and Rochberg [10]. In the one-dimensional case the Hilbert transform is the only Calderon—Zygmund operator and the commutators are

$$C_f \varphi(x) = \text{P.V.} \int_R \frac{f(x) - f(y)}{x - y} \varphi(y)\, dy.$$

Because of the characterization of Hilbert—Schmidt operators in terms of their kernels it is immediate that $C_f \in S^2$ if and only if $f \in \Lambda^{22}_{1/2}$. For others values of p the answer is due to Peller [8] who proved the following.

Theorem B (Peller). *If* $n=1$ *and* $1\leqq p<\infty$ *then* $C_f\in S^p$ *if and only if* $f\in\Lambda^{pp}_{1/p}$. ∎

Peller actually worked on the circle. Coifman and Rochberg [2] (when $p=1$) and Rochberg [10] (for arbitrary $p\geqq 1$) gave another proof on the line based on a molecular decomposition of $\Lambda^{pp}_{1/p}$. Peller [8] and Rochberg [10] also showed that $f\in\Lambda^{pp}_{1/p}$ implies $C_f\in S^p$ when $p<1$, but it is not known whether the converse is true.

Rochberg asked whether there is an n-dimensional version of Theorem B. The result of this paper is the following.

Theorem 1. *Suppose* $n\geqq 2$, $0<p<\infty$ *and* $f\in L^1_{\text{loc}}(R^n)$. *Then necessary and sufficient conditions for* $C_f\in S^p$ *are*

$$f \text{ constant, if } p\leqq n$$

$$f\in\Lambda^{pp}_{n/p}, \quad \text{if} \quad p>n. \quad ∎$$

Our methods do not work when $n=1$ and in fact our result differs somewhat from Peller's and Rochberg's. If $n=1$ and $f\in C_0^\infty$, C_f will belong to all S^p classes, $p>0$, while if $n>1$, C_f will belong to S^p only when $p>n$. To see why this is so consider the somewhat simpler periodic case and let $f(t)=e^{ik\cdot t}$. Then $C_f e^{ij t}=(m(j)-m(k+j))e^{i(k+j)\cdot t}$ with $m=\hat{K}$ and thus $\{S_j(C_f)\}=\{|m(j)-m(k+j)|\}_{j\in Z^n}$. If $n=1$, $m(j)=c\operatorname{sign}(j)$ and only finitely many $S_j(C_f)$ are non-zero, but if $n\geqq 2$, $|m(j)-m(k+j)|$ is usually about $|j|^{-1}$, and $\sum|j|^{-p}=\infty$ for $p\leqq n$.

In Section 1 we prove that $p>n$ and $f\in\Lambda^{pp}_{n/p}$ imply $C_f\in S^p$. In Section 2 we prove the converse for $p>n$. The case $p\leqq n$ requires an additional argument (based on the preceding example) which we give in Section 3.

1.

If X_0 and X_1 are (compatible) Banach spaces, then $(X_0, X_1)_{\theta q}$ is the interpolation space obtained by the real method as described in [1]. L^{qr} is the Lorentz space, $L^p(L^{qr})$ the corresponding mixed norm space. $L^p(L^{qr})^{\text{symm}}=L^p(L^{qr})\cap L^p(L^{qr})^*$. The letter C will denote a constant.

Since we are assuming $p>n\geqq 2$ it is natural to use Russo's Theorem A to prove sufficiency in Theorem 1. In fact we don't use Theorem A as it stands but rather a variant involving weak type spaces. Russo proved Theorem A by complex interpolation and we use the analogous real interpolation argument.

Lemma 1. *If* $p>2$ *and* $\dfrac{1}{p}+\dfrac{1}{p'}=1$ *then*

$$L^p(L^{p'\infty})^{\text{symm}}\subset\big(L^\infty(L^1)^{\text{symm}},\ L^2(L^2)\big)_{\theta\infty}, \text{ where } \theta=2/p.$$

Proof. Fix $f\in L^p(L^{p'\infty})^{\text{symm}}$. For $t>0$ let

$$K(t) = \inf\left(\|b\|_{L^\infty(L^1)^{\text{symm}}}+t\,\|g\|_{L^2(L^2)}: b+g=f\right).$$

We must show $t^{-2/p}K(t)$ is bounded as t varies. Let $f_x=f(x,\cdot)$ etc., and take $g(x,y)=\operatorname{sign} f(x,y)\cdot\min\left(|f(x,y)|,\lambda\right)$ where

$$\lambda = t^{-2/p'}\max\left(\|f_x\|^p_{L^{p'\infty}}, \|f_y\|^p_{L^{p'\infty}}\right), \text{ and } b=f-g.$$

To estimate b, fix x and let E_x be the distribution function of $|f_x|$; then

$$\|b_x\|_{L^1} \leqq \int_{s>t^{-2/p'}\|f_x\|^p_{L^{p'\infty}}} E_x(s)\,ds$$

$$\leqq \|f_x\|^{p'}_{L^{p'\infty}}\int_{s>t^{-2/p'}\|f_x\|^p_{L^{p'\infty}}} s^{-p'}\,ds$$

$$\leqq \frac{1}{p'-1}\|f_x\|^{p'}_{L^{p'\infty}}\left(t^{-2/p'}\|f_x\|^p_{L^{p'\infty}}\right)^{1-p'} = \frac{t^{2/p}}{p'-1}.$$

One obviously has the same estimate for $\|b_y\|_{L^1}$ so $\|b\|_{L^\infty(L^1)^{\text{symm}}}\leqq Ct^{2/p}$. As to g,

$$\iint |g|^2\,dx\,dy \leqq \int\int_{s<t^{-2/p'}\|f_x\|^p_{L^{p'\infty}}} 2s\,E_x(s)\,ds\,dx+\int\int_{s<t^{-2/p'}\|f_y\|^p_{L^{p'\infty}}} 2s\,E_y(s)\,ds\,dy$$

and

$$\int\int_{s<t^{-2/p'}\|f_x\|^p_{L^{p'\infty}}} 2s\,E_x(s)\,ds\,dx \leqq 2\int \|f_x\|^{p'}_{L^{p'\infty}}\int_{s<t^{-2/p'}\|f_x\|^p_{L^{p'\infty}}} s^{1-p'}\,ds\,dx$$

$$= \frac{2}{2-p'}\,t^{-\frac{2}{p'}(2-p')}\int \|f_x\|^p_{L^{p'\infty}}\,dx.$$

By symmetry

$$\iint |g|^2\,dx\,dy \leqq \frac{2}{2-p'}\,t^{-\frac{2}{p'}(2-p')}\left(\int \|f_x\|^p_{L^{p'\infty}}\,dx+\int \|f_y\|^p_{L^{p'\infty}}\,dy\right)$$

so that $\|g\|_{L^2(L^2)} \leqq Ct^{\frac{2}{p}-1}\,\|f\|^{p/2}_{L^p(L^{p'\infty})^{\text{symm}}}$, completing the proof of the lemma. ∎

The following version of Russo's theorem now follows immediately by interpolation between $p=2$ and $p=\infty$.

Lemma 2. *Suppose* (M,μ) *is a measure space,* $2<p<\infty$, $\frac{1}{p}+\frac{1}{p'}=1$, *and* $G\colon M\times M\to C$ *belongs to* $L^p(L^{p'\infty})^{\text{symm}}$. *Then the integral operator on* $L^2(\mu)$ *with kernel* G *belongs to* $S^{p\infty}$.

Proof of sufficiency in Theorem 1. Fix $p>n$ and $f\in\Lambda^{pp}_{n/p}$. By definition of $\Lambda^{pp}_{n/p}$, $\frac{|f(x)-f(y)|}{|x-y|^{2n/p}}\in L^p(L^p)$, and clearly $|x-y|^{\frac{2n}{p}-n}\in L^\infty(L^{q\infty})$ where $\frac{1}{q}=1-\frac{2}{p}$. For any functions g and h, $\|gh\|_{p',\infty}\leqq C\|g\|_p\|h\|_{q\infty}$. So $\frac{f(x)-f(y)}{|x-y|^n}\in L^p(L^{p'\infty})$. Hence $(f(x)-f(y))K(x-y)\in L^p(L^{p'\infty})$ and by Lemma 2, $C_f\in S^{p\infty}$.

We use another interpolation argument to prove $C_f\in S^p$. Given $p>n$ choose p_1, p_2 with $n<p_1<p<p_2<\infty$ and let θ satisfy $\frac{1-\theta}{p_1}+\frac{\theta}{p_2}=\frac{1}{p}$. Then $(S^{p_1\infty}, S^{p_2\infty})_{\theta p}=S^p$ ([1], [12]) and $(\Lambda^{p_1p_1}_{n/p_1}, \Lambda^{p_2p_2}_{n/p_2})_{\theta p}=\Lambda^{pp}_{n/p}$ ([1]) so the map $f\mapsto C_f$, which is bounded from $\Lambda^{p_jp_j}_{n/p_j}$ to $S^{p_j\infty}$ $(j=1,2)$ is also bounded from $\Lambda^{pp}_{n/p}$ to S^p. ∎

2.

Assume that $C_f\in S^p$ and $p>n$. We will adapt the proof in [6] and estimate the mean oscillation on all cubes simultaneously. For any $v\in R^n$ and $k\in Z$ let $\mathscr{I}^v_k$ be the dyadic partition of R^n into cubes with vertices at $\{v+2^{-k}m, m\in Z^n\}$. Let $f^v_k(x)=2^{nk}\int_Q f(y)\,dy$, if $x\in Q\in\mathscr{I}^v_k$. For $Q\in\mathscr{I}^v_k$, choose s_Q among the functions that are O off Q, $+1$ on exactly 2^{n-1} of the 2^n subcubes of Q belonging to $\mathscr{I}^v_{k+1}$, and -1 on the others such that $\left|\int_Q f(x)s_Q(x)\,dx\right|$ is maximal. Then

$$2^{nk}\int_Q|f^v_k-f^v_{k+1}|^p\leqq C\left(2^{nk}\max\left\{\left|\int fs\right|:s \text{ as above}\right\}\right)^p=C\left(2^{nk}\left|\int_Q fs_Q\right|\right)^p,$$

where C depends only on p and n, since $2^{nk}\max_s\left|\int fs\right|$ and $\left(2^{nk}\int_Q|f^v_{k+1}-f^v_k|^p\right)^{1/p}$ are norms on the same 2^n-1-dimensional vector space.

Choose z, $0<|z|<1$, such that $K(z)\neq 0$. There exists a neighborhood $|x-z|<\delta\sqrt{n}$ where $1/K(x)$ can be expressed as an absolutely convergent Fourier series $\sum c_m e^{iv_m\cdot x}$ for some vectors v_m.

For $Q\in\mathscr{I}^v_k$, let $t_Q(x)=\chi_Q(x+2^{-k}\delta^{-1}z)$, where χ_Q is the characteristic function. Then, since $\int s_Q(x)\,dx=0$,

$$\begin{aligned}\int fs_Q &= 2^{nk}\iint(f(x)-f(y))s_Q(x)t_Q(y)\,dx\,dy\\ &= 2^{nk}\iint(f(x)-f(y))\frac{\delta^{-n}2^{-nk}K(x-y)}{K(\delta 2^k(x-y))}s_Q(x)\,t_Q(y)\,dx\,dy\\ &= C\iint(f(x)-f(y))K(x-y)\textstyle\sum_m c_m e^{i\delta 2^k v_m\cdot(x-y)}s_Q(x)\,t_Q(y)\,dx\,dy.\end{aligned}$$

Let $g_{Qm}(x)=2^{nk/2}e^{i\delta 2^k v_m\cdot x}s_Q(x)$ and $h_{Qm}(y)=2^{nk/2}e^{-i\delta 2^k v_m\cdot y}t_Q(y)$. Then

$$\begin{aligned}2^{nk}\int fs_Q &= C\iint(f(x)-f(y))K(x-y)\textstyle\sum_m c_m g_{Qm}(x)\,h_{Qm}(y)\,dx\,dy\\ &= C\textstyle\sum_m c_{Qm}\langle g_m|C_f\,h_{Qm}\rangle.\end{aligned}$$

For each m, $\{g_{Qm}\}$ and $\{h_{Qm}\}$ are orthonormal sequences as Q ranges over $\mathcal{I}_k^v$ The condition $C_f\in S^p$ and Minkowski's inequality give

$$\Big(\sum_{Q\in\mathcal{I}_k^v}\Big|2^{nk}\int fs_Q\Big|^p\Big)^{1/p}\leq C\sum_m|c_m|\big(\sum_Q|\langle g_{Qm}|C_f h_{Qm}\rangle|^p\big)^{1/p}\leq C.$$

Thus

$$2^{nk}\int|f_{k+1}^v-f_k^v|^p=\sum_{Q\in\mathcal{I}_k^v}2^{nk}\int_Q|f_{k+1}^v-f_k^v|^p\leq C,$$

$$\text{i.e. } \|f_{k+1}^v-f_k^v\|_p\leq C2^{-nk/p}.$$

Since $f_k^v\to f$ a.e. as $k\to\infty$, summation of the geometric series yields

$$\|f-f_k^v\|_p\leq C2^{-nk/p}. \tag{1}$$

This estimate only implies that $f\in\Lambda_{n/p}^{p\infty}$. (Cf. the final step below of the proof that $f\in\Lambda_{n/p}^{pp}$.) To improve it we use the S^p property once more. Let N be a large number to be chosen later. For any cube $Q\in\mathcal{I}_k^v$, let $a_Q=(2^{nk}\int_Q|f-f_k^v|^p)^{1/p}$ (the L^p mean oscillation) and let $g_Q(x)=2^{nk/2}s_Q(x)$ and $h_Q(y)=2^{nk/2}\chi_Q(y+2^{N-k}z)$. Also, for $j=0,\dots,N-1$, let $Q_j=2^jQ-(2^j-1)2^{-k}z$ and $Q_j'=2^jQ-(2^N-2^j+1)2^{-k}z$, where 2^jQ has the same center as Q and side 2^{j-k}. Then

$$\begin{aligned}
&\Big|\int C_f g_Q(x)h_Q(x)\,dx+K(z)\,2^{n(k-N)}\int f(x)s_Q(x)\,dx\Big|\\
&=\Big|\iint(f(x)-f(y))(K(x-y)-K(2^{N-k}z))\,g_Q(y)\,h_Q(x)\,dx\,dy\Big|\\
&\leq C2^{-(n+1)(N-k)}\iint|x-y-2^{N-k}z|\,|f(x)-f(y)|\,|g_Q(y)\,h_Q(x)|\,dx\,dy\\
&\leq C2^{-k}2^{(n+1)(k-N)}2^{nk}\int_Q\int_{Q_0'}|f(x)-f(y)|\,dx\,dy\\
&\leq C2^{-(n+1)N}\sum_0^{N-1}(a_{Q_j}+a_{Q_j'})
\end{aligned}$$

where the last inequality follows from standard arguments with the mean oscillation. Hence

$$\Big|2^{nk}\int fs_Q\Big|\leq C2^{nN}\big|\langle C_f g_Q|h_Q\rangle\big|+C2^{-N}\sum_0^{N-1}(a_{Q_j}+a_{Q_j'}).$$

Fix L, M and v and let Q range over $\mathcal{I}_k^v$ for $L\leq k\leq M$. $\{g_Q\}$ is an orthonormal sequence. $\{h_Q\}$ is not, but $\|h_Q\|=1$ and thus

$$\sum\big|\langle C_f g_Q|h_Q\rangle\big|^p\leq\|C_f g_Q\|^p\leq C.$$

(Here we use $p\geq 2$). Taking the l^p norms over these Q we obtain by Minkowski's inequality

$$\begin{aligned}
\Big(\sum_L^M 2^{nk}\|f_{k+1}^v-f_k^v\|_p^p\Big)^{1/p}&=\Big(\sum_Q 2^{nk}\int_Q|f_{k+1}^v-f_k^v|^p\Big)^{1/p}\\
&\leq C\Big(\sum_Q\Big|2^{nk}\int fs_Q\Big|^p\Big)^{1/p}\\
&\leq C_N+C2^{-N}\sum_{j=0}^{N-1}\Big(\sum_{k=L}^M\sum_{Q\in\mathcal{I}_k^v}(a_{Q_j}^p+a_{Q_j'}^p)\Big)^{1/p}.
\end{aligned}$$

As Q ranges over $\mathscr{I}_k^v$, $\{Q_j\}$ and $\{Q_j'\}$ cover R^n 2^{nj} times each (for fixed j, k), and there exist $v_1^j \dots v_{2^{nj+1}}^j$ translates of v by fixed vectors) such that Q_j and Q_j' range over $\bigcup \mathscr{I}_{k-j}^{v_i^j}$. Hence we obtain

$$(2)\quad \begin{aligned} \left(\sum_L^M 2^{nk}\|f-f_k^v\|_p^p\right)^{1/p} &\leqq \left(\sum_L^M 2^{nk}\|f-f_{k+1}^v\|_p^p\right)^{1/p}+\left(\sum_L^M 2^{nk}\|f_{k+1}^v-f_k^v\|_p^p\right)^{1/p} \\ &\leqq \left(\sum_L^M 2^{nk}\|f-f_{k+1}^v\|_p^p\right)^{1/p}+C_N+C2^{-N}\sum_{j=0}^{N-1}\left(\sum_{k=L}^M\sum_{i=1}^{2^{nj+1}}\sum_{Q\in\mathscr{I}_{k-j}^{v_i^j}} a_Q^p\right)^{1/p}. \end{aligned}$$

The next step is to average over all dyadic partitions. Define

$$A_k = |I|^{-1}\int_I \|f-f_k^v\|_p^p\,dv = |I|^{-1}\int_I 2^{-nk}\left(\sum_{Q\in\mathscr{I}_k^v} a_Q^p\,dv\right)$$

for any dyadic cube I of side $\geqq 2^{-k}$. A_k is independent of the choice of I, since $\mathscr{I}_k^v=\mathscr{I}_k^w$ if $v-w\in 2^{-k}Z^n$. Choose I large enough and take the L^p norms with respect to $|I|^{-1}\chi_I(v)dv$ in (2):

$$\begin{aligned} \left(\sum_L^M 2^{nk}A_k\right)^{1/p} &\leqq \left(\sum_L^M 2^{nk}A_{k+1}\right)^{1/p}+C_N+C2^{-N}\sum_{j=0}^{N-1}\left(\sum_{K=L}^M 2^{nj+1}2^{n(k-j)}A_{k-j}\right)^{1/p} \\ &= \left(\sum_{L+1}^{M+1}2^{n(k-1)}A_k\right)^{1/p}+C_N+C2^{-N}\sum_0^{N-1}\left(2^{nj}\sum_{L-j}^{M-j}2^{nk}A_k\right)^{1/p} \\ &\leqq 2^{-n/p}\left(\sum_{L+1}^M 2^{nk}A_k\right)^{1/p}+\left(2^{nM}A_{M+1}\right)^{1/p}+C_N \\ &\quad + C2^{-N}\sum_0^{N-1}2^{nj/p}\left(\left(\sum_L^{M-j}2^{nk}A_k\right)^{1/p}+\left(\sum_{L-j}^{L-1}2^{nk}A_k\right)^{1/p}\right) \\ &\leqq \left(2^{-n/p}+C2^{-N}2^{nN/p}\right)\left(\sum_L^M 2^{nk}A_k\right)^{1/p}+C_N, \end{aligned}$$

since $2^{nk}A_k\leqq C$ by (1). If N is chosen large enough then $2^{-n/p}+C2^{-N(1-n/p)}<1$. (Here we use $p>n$.) Consequently $\sum_L^M 2^{nk}A_k\leqq C$, C independent of L and M — i.e., $\sum_{-\infty}^{\infty}2^{nk}A_k<\infty$.

Finally we show that this condition on the mean oscillation of f over cubes implies $f\in A_{n/p}^{pp}$. A similar characterization of Lipschitz spaces has been given by Ricci and Taibleson [9].

Again, let I be a large dyadic cube. If $|x-y|<2^{-k-1}$, the probability that x and y belong to the same cube in $\mathscr{I}_k^v$ (for random v) is at least 2^{-n}. Hence

$$\begin{aligned} \iint_{|x-y|<2^{-k-1}}|f(x)-f(y)|^p\,dx\,dy &\leqq 2^n|I|^{-1}\int_I\sum_{Q\in\mathscr{I}_k^v}\int_Q\int_Q|f(x)-f(y)|^p\,dx\,dy\,dv \\ &\leqq C|I|^{-1}\int_I\sum_{\mathscr{I}_k^v}\int_Q\int_Q\left(|f(x)-f_k^v(x)|^p+|f_k^v(y)-f(y)|^p\right)dx\,dy\,dv \\ &= C2^{-nk}|I|^{-1}\int_I\int_{R^n}|f(x)-f_k^v(x)|^p\,dx\,dv \\ &= C2^{-nk}A_k. \end{aligned}$$

Therefore

$$\iint \frac{|f(x)-f(y)|^p}{|x-y|^{2n}}\,dx\,dy \leqq C \iint \sum_{2^k<|x-y|^{-1}} 2^{2nk}\,|f(x)-f(y)|^p\,dx\,dy$$
$$= C\sum 2^{2nk} \iint_{|x-y|<2^{-k}} |f(x)-f(y)|^p\,dx\,dy$$
$$\leqq C\sum 2^{nk}A_k < \infty.$$

This completes the proof that $f\in \Lambda^{pp}_{n/p}$.

3.

We need to prove that $C_f\in S^p$ only when f is constant. Suppose f is not constant. Since non-constant polynomials do not belong to BMO there is a point $z_0\neq 0$ in the support of the Fourier transform $\hat{f}$. For some constants k, $M<\infty$ we will have $|\langle\varphi|\hat{f}\rangle|\leqq M\|\varphi\|_{C^k}$ whenever φ is a test function supported in $\{y|\,|y-z_0|<1\}$. Choose $0<\varepsilon<\min\left(\frac{1}{2}, \frac{|z_0|}{2}\right)$. Let ψ be a test function with $\|\psi\|_{C^k}=1$, $\operatorname{supp}\psi\subset\{y|\,|y-z_0|<\varepsilon\}$, and $\langle\psi|\hat{f}\rangle=B>0$. Let δ be some number to be chosen later, but small enough so that $|z|<\delta$ implies $z+\operatorname{supp}\psi\subset\{|y-z_0|<\varepsilon\}$ and $\langle\psi(y+z)|\hat{f}(y)\rangle>\frac{B}{2}$.

Let $m=\hat{K}$. Then m is homogeneous of degree zero and nonconstant so the derivative $D_{z_0}m$ is not identically zero. Let V be an open cone with vertex at the origin, such that the real part (say) of $D_{z_0}m$ is bounded away from zero on $V\cap$(unit ball). Let $\{x_j\}$ be the set of all lattice points whose distance to the complement of V is at least $2|z_0|$. Then $|x_j-x_k|\geqq 1$ $(j\neq k)$, $|m(x_j)-m(x_j-z_0)|>\frac{A}{|x_j|}$ for some constant A, and $\sum_j |x_j|^{-n}=\infty$. Define Schwartz functions φ_j by $\hat{\varphi}_j(y)=|x_j|^{-1}\frac{\psi(x_j-y)}{m(y)-m(x_j)}$. These φ_j are orthogonal since the supports of the $\hat{\varphi}_j$ are disjoint, and if ε is small enough they will satisfy $\|\varphi_j\|_2\leqq 1$.

Claim. If δ is small enough and $|x-x_j|<\delta$ then $|(C_f\varphi_j)\,\hat{}\,(x)|>\frac{C}{|x_j|}$.

In fact,

$$(C_f\varphi_j)\,\hat{}\,(x) = |x_j|^{-1}\left\langle \frac{m(x-y)-m(x)}{m(x-y)-m(x_j)}\psi(x_j-x+y)|\hat{f}(y)\right\rangle$$

$$(*)\qquad = |x_j|^{-1}\langle\psi(y+x_j-x)|\hat{f}(y)\rangle$$
$$+|x_j|^{-1}(m(x)-m(x_j))\left\langle\frac{\psi(y+x_j-x)}{m(x_j)-m(x-y)}|\hat{f}(y)\right\rangle.$$

Now $\psi(y+x_j-x)$ is supported on $|y-z_0|<\varepsilon+\delta$ and on this set, all y-derivatives of $\dfrac{1}{m(x_j)-m(x-y)}$ are bounded by constants times $|x_j|$. So

$$\left|\left\langle \frac{\psi(y+x_j-x)}{m(x_j)-m(x-y)}, |\hat{f}(y)\right\rangle\right| \leqq M \left\| \frac{\psi(y+x_j-x)}{m(x_j)-m(x-y)} \right\|_{C^k} \leqq CM|x_j|.$$

This proves the second term in $(*)$ is bounded by $B/4|x_j|$ provided δ is small. The first term is at least $B/2|x_j|$, which proves the claim.

It follows that $\sum \|C_f\varphi_j\|_2^n = \sum \|(C_f\varphi_j)\hat{}\,\|_2^n \geqq C\sum |x_j|^{-n} = \infty$, so $C_f \notin S^n$. ▮

Remarks. The above argument shows that even if $f\in C_0^\infty$ the best one can hope is that $C_f\in S^{n\infty}$. For $n\geqq 3$, one can show using Lemma 2 that $f\in \Lambda_1^{n1}$ implies $C_f\in S^{n\infty}$. We do not know whether this result is true for $n=2$ because Lemma 2 is false for $p=2$.

A colleague of the referee pointed out that a pseudodifferential operator $(\not\equiv 0)$ of symbol class S^{-1} is never of Schatten class S^n, and that the case $p\leqq n$ of Theorem 1 follows from this.

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