On the comparison principle in the calculus of variations

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1. Introduction

A well-known phenomenon in classic Potential Theory can be regarded as a prototype for the variational problem to be studied in this paper. Recall that the **harmonic functions** in a domain $G \subset R^n$, $n \geqq 2$, are precisely the **free extremals** for Dirichlet's integral $\int |\nabla u|^2 dm$.

The basic fact is that the following two conditions are equivalent for a function u with continuous first partial derivatives $\nabla u = (\partial u/\partial x_1, \ldots, \partial u/\partial x_n)$ in G:

1° For every non-negative η in $C_0^\infty(G)$
$\int |\nabla u|^2 dm \leqq \int |\nabla (u-\eta)|^2 dm$
where the integrals are taken over the set $\operatorname{spt} \eta = \overline{\{x \mid \eta(x) \neq 0\}}$.

2° Given any domain D with compact closure $\bar{D}$ in G and any function h that is harmonic in D and continuous in $\bar{D}$, the boundary inequality $h|\partial D \geqq u|\partial D$ implies that $h \geqq u$ in D.

These conditions express that u is **subharmonic** in G. (Condition 1° is usually formulated as the familiar inequality $\int \nabla u \cdot \nabla \eta \, dm \leqq 0$ for all $\eta \geqq 0$ in $C_0^\infty(G)$.)

The object of our paper is the proper analogue to the above situation for variational integrals of the form

$$I(u, D) = \int_D F(x, \nabla u(x))\, dx, \quad D \subset G. \tag{1.1}$$

Here the integrand is assumed to satisfy certain natural conditions about measurability, strict convexity, and growth: $F(x, w) \approx |w|^p$, $1 < p < \infty$.

If $u \in C(G) \cap W^1_{p,\mathrm{loc}}(G)$ satisfies the inequality

$$I(u, \operatorname{spt} \eta) \leqq I(u-\eta, \operatorname{spt} \eta) \tag{1.2}$$

for every non-negative η in $C_0^\infty(G)$, then u necessarily obeys the comparison principle with respect to the free extremals for the integral (1.1). The corresponding fact is well-known in the theory of partial differential equations [3, 9.5, pp. 211—213]. Our main result, Theorem 4.1, states that, if $u\in C(G)\cap W^1_{p,\mathrm{loc}}(G)$, then the comparison principle is also sufficient to guarantee the validity of (1.2). The direct proof given in § 4 avoids the difficult question about the continuity of the solution to an "obstacle problem".

As an application we mention that the maximum of two free extremals satisfies (1.2).

We use merely standard notation.

2. Assumptions and preliminaries

Let G denote a fixed domain in the Euclidean n-dimensional space R^n, $n\geqq 2$. Consider the variational integral

$$I(u, D) = \int_D F(x, \nabla u(x))\, dx, \quad D\subset G. \tag{2.1}$$

The integrand $F: G\times R^n \to R$ is assumed to satisfy the following conditions.

(i) Given $\varepsilon>0$ and a set $D\subset\subset G$, there is a compact set $K_\varepsilon\subset D$, $m(K_\varepsilon)> m(D)-\varepsilon$, such that the restriction $F|K_\varepsilon\times R^n$ is continuous.

(ii) The mapping $w\mapsto F(x, w)$ is strictly convex for a.e. fixed $x\in G$.

(iii) There are constants $0<\alpha\leqq\beta<\infty$, and an exponent p, $1<p<\infty$, such that for a.e. $x\in G$

$$\alpha|w|^p\leqq F(x, w)\leqq\beta|w|^p \tag{2.2}$$

when $w\in R$.

2.3. Remark. 1° The strict convexity (ii) guarantees the uniqueness of extremals with given boundary values [4, Corollary 4.19, p. 31]. If $\varphi\in C(D)\cap W^1_p(D)$, D being a domain with compact closure in G, then there is a unique **extremal** $h\in C(D)\cap W^1_p(D)$ with $h-\varphi\in W^1_{p,0}(D)$ such that $I(h, D)\leqq I(v, D)$ for all similar v, c.f. [2, Ch. I. 3, pp. 29—31] and [1].

2° If $u_1, u_2, u_3, \dots W^1_p(D)$, D being a domain in G, and if $\nabla u_i\to\nabla u$, $u\in W^1_p(D)$, weakly in $L^p(D)$, then $I(u, D)\leqq\underline{\lim}\, I(u_i, D)$. For this lower-semicontinuity result, we refer the reader to [8] or [5].

We say that $u\in C(D)\cap W^1_{p,\mathrm{loc}}(D)$, D being a domain in G, is a **free extremal** in D, if

$$I(u, \operatorname{spt}\eta)\leqq I(u-\eta, \operatorname{spt}\eta) \tag{2.4}$$

for all $\eta\in C^\infty_0(D)$. Analogously, we say that u is a **free subextremal** in D, if (2.4) holds for all non-negative $\eta\in C^\infty_0(D)$. Of course, an extremal (in the ordinary sense) is also a free extremal.

We say that a function $u \in C(G) \cap W^1_{p,\mathrm{loc}}(G)$ obeys the **comparison principle** in G, if the following implication is true for every domain D with compact closure in G:

Whenever $h \in C(\bar{D}) \cap W^1_{p,\mathrm{loc}}(D)$ *is a free extremal in* D *and* $h|\partial D \geqq u|\partial D$, *then* $h \geqq u$ *in* D.

3. Necessity for the comparison principle

For the sake of completeness we shall give a simple proof of the fact that the (free) subextremals obey the comparison principle.

3.1. Theorem. Suppose that $u \in C(G) \cap W^1_{p,\mathrm{loc}}(G)$ **is a free subextremal in** G. **Then** u **obeys the comparison principle in** G.

Proof. Suppose that $h \in C(\bar{D}) \cap W^1_{p,\mathrm{loc}}(D)$, $h|\partial D \geqq u|\partial D$, is a free extremal in a domain D with compact closure in G. Given $\varepsilon > 0$, $h_\varepsilon = h + \varepsilon$ is a free extremal in D. The set $D_\varepsilon = \{x | h_\varepsilon(x) < u(x)\}$ is open and $\bar{D}_\varepsilon \subset D$. If $m\{x \in D_\varepsilon | \nabla h_\varepsilon(x) \neq \nabla u(x)\} > 0$, then the subextremality of u and the strict convexity (ii) imply that

$$I(u, D_\varepsilon) \leqq I\left(\frac{u+h_\varepsilon}{2}, D_\varepsilon\right) < \frac{1}{2} I(u, D_\varepsilon) + \frac{1}{2} I(h_\varepsilon, D_\varepsilon),$$

i.e. that

$$I(u, D_\varepsilon) < I(h_\varepsilon, D_\varepsilon). \tag{3.2}$$

On the other hand

$$I(h_\varepsilon, D_\varepsilon) \leqq I(u, D_\varepsilon). \tag{3.3}$$

Obviously, (3.2) and (3.3) are incompatible, and so $\nabla h_\varepsilon = \nabla u$ a.e. in D_ε. This implies that $h_\varepsilon = u$ in D_ε in virtue of $h_\varepsilon|\partial D_\varepsilon = u|\partial D_\varepsilon$. Hence the set D_ε is empty, and so $h + \varepsilon \geqq u$ in D. Since $\varepsilon > 0$ was arbitrary, we obtain the desired inequality $h \geqq u$ in D.

4. Sufficiency for the comparison principle

We are now going to prove our main result, viz.:

4.1. Theorem. Suppose that $u \in C(G) \cap W^1_{p,\mathrm{loc}}(G)$ **obeys the comparison principle in** G. **Then** u **is a free subextremal in** D.

Proof. Fix a domain D with compact closure in G. Consider the class

$$\mathscr{F} = \{v \in C(\bar{D}) \cap W^1_p(D) | v \leqq u,\ v|\partial D = u|\partial D\}.$$

Obviously, $u\in\mathcal{F}$. There is a function $\mathbf{u}\in W_p^1(D)$ such that $\mathbf{u}\leqq u$, $u-\mathbf{u}\in W_{p,0}^1(D)$, and

$$I(\mathbf{u}, D) \leqq I(v, D) \quad \text{for all} \quad v\in\mathcal{F}. \tag{4.2}$$

The existence of $\mathbf{u}$ is established by the aid of a minimizing sequence $u_1, u_2, u_3, \dots$ in $\mathcal{F}$ that converges weakly in $W_p^1(D)$ to $\mathbf{u}$. Especially,

$$\lim I(u_k, D) = I(\mathbf{u}, D). \tag{4.3}$$

It is easily seen that $\mathbf{u}$ is minimizing the integral in a somewhat broader class than $\mathcal{F}$, i.e.

$$I(\mathbf{u}, D) \leqq I(v, D) \tag{4.2$'$}$$

for all $v\in W_p^1(D)$, $v\leqq u$, $v-u\in W_{p,0}^1(D)$.

We have to show that $\mathbf{u}=u$. If $\mathbf{u}$ were known to be continuous, this question were trivial, c.f. [4, Theorem 5.17]. In order to avoid this difficult regularity question we shall construct another minimizing sequence from $u_1, u_2, \dots$ As in the classical "méthode de balayage" we aim at modifying u_k in some **regular** set close to D.

To this end, fix $\varepsilon>0$ and note that the sets $\{x\in D|u_k(x)<u(x)-\varepsilon\}$ and $\{x\in D|u_k(x)<u(x)-2\varepsilon\}$ are open. Using a standard method we can construct open sets D_k, as regular as we please (e.g. "polyhedrons"), such that 1°

$$\{x\in D|u_k(x) < u(x)-2\varepsilon\} \subset D_k \subset \{x\in D|u_k(x) < u(x)-\varepsilon\}$$

and 2° there is a unique extremal $h_k\in C(\bar{D}_k)\cap W_p^1(D_k)$ with boundary values $h_k|\partial D_k=u_k|\partial D_k$, provided that D_k is not empty [1, end of § 3]. If $x\in\partial D_k$, then $u_k(x)\geqq u(x)-2\varepsilon$, and so $u|\partial D_k\leqq h_k|\partial D_k+2\varepsilon=(h_k+2\varepsilon)|\partial D_k$. By the comparison principle, which u is assumed to obey, this inequality holds in D_k, i.e.

$$h_k \geqq u-2\varepsilon \quad \text{in} \quad \bar{D}_k.$$

Define

$$H_k = \begin{cases} h_k & \text{in} \quad D_k, \\ u_k & \text{in} \quad D\setminus D_k. \end{cases} \tag{4.4}$$

Then $H_k\in W_p^1(D)$ and $H_k-u\in W_{p,0}^1(D)$. Obviously, $I(H_k, D)\leqq I(u_k, D)$. Moreover, we have

$$H_k \geqq u-2\varepsilon \tag{4.5}$$

in D. So far we have imitated the classical "méthode de balayage", but the functions constructed are not necessarily in the class $\mathcal{F}$. Giving due care to the obstacle in $\mathcal{F}$, we are therefore forced to adjust H_k.

Note that the uniform bound

$$\sup_{D_k} h_k = \max_{\partial D_k} h_k < \sup_D u = M$$

is trivially valid, c.f. [4, Remark 4.6, p. 25]. Fix $\lambda\in(0, 1)$ so that $0<\lambda<\varepsilon/|M|$.

The convex combination

$$w_k = \lambda H_k + (1-\lambda) u_k \tag{4.6}$$

is in $W_p^1(D)$ and $w_k - u \in W_{p,0}^1(D)$. Moreover, $w_k \leqq u$ in D, since $w_k = u_k$ in $D \setminus D_k$ and in D_k we have $w_k = (1-\lambda)u_k + \lambda h_k \leqq (1-\lambda)(u-\varepsilon) + \lambda M \leqq (u-\varepsilon) + \lambda M < u$. Thus w_k is admissible in (4.2)′.

Hence $I(w_k, D) \geqq I(u, D)$ by (4.2)′, and by the construction $I(w_k, D) \leqq \leqq \lambda I(H_k, D) + (1-\lambda) I(u_k, D) \leqq \lambda I(u_k, D) + (1-\lambda) I(u_k, D) = I(u_k, D)$, i.e.

$$I(\mathbf{u}, D) \leqq I(w_k, D) \leqq I(u_k, D), \quad k = 1, 2, 3, \ldots .$$

Thus

$$\lim I(w_k, D) = I(\mathbf{u}, D). \tag{4.7}$$

Since $\int |\nabla w_k|^p dm \leqq \frac{1}{\alpha} I(w_k, D) \leqq \frac{1}{\alpha} I(u_k, D)$, the sequence $\int |\nabla w_k|^p dm$, $k = 1, 2, 3, \ldots$, is uniformly bounded in virtue of (4.3). Thus there are indices $k_1 < k_2 < k_3 < \ldots$ and a function $w_\varepsilon \in W_p^1(D)$ such that $w_{k_i} \to w_\varepsilon$ weakly in $W_p^1(D)$. We have $w_\varepsilon \leqq u$ and $w_\varepsilon - u \in W_{p,0}^1(D)$. Now

$$I(\mathbf{u}, D) \leqq I(w_\varepsilon, D) \leqq \underline{\lim}\, I(w_{k_i}, D) = I(\mathbf{u}, D)$$

by the minimizing property (4.2)′ of $\mathbf{u}$, the lower-semicontinuity of the integral [Remark 2.3], and (4.7). Hence $I(w_\varepsilon, D) = I(\mathbf{u}, D)$. As in the proof of Theorem 3.1, the strict convexity (ii) implies the uniqueness

$$\mathbf{u} = w_\varepsilon$$

(this independence of ε indicates that the sets D_{k_i} are empty sooner or later).

The weak convergences

$$u_{k_i} \to \mathbf{u}, \quad \lambda H_{k_i} + (1-\lambda) u_{k_i} \to \mathbf{u}$$

imply that $H_{k_i} \to \mathbf{u}$ weakly in $W_p^1(D)$. By (4.5) $\mathbf{u} = \lim H_{k_i} \geqq u - 2\varepsilon$ in D. Since $\varepsilon > 0$ was arbitrary, $\mathbf{u} \geqq u$ in D. On the other hand $\mathbf{u} \leqq u$, whence $\mathbf{u} = u$. This means that u has the desired minimizing property.

5. The maximum of two extremals

As a simple application we mention the following result, difficult to prove without the comparison principle.

5.1. Theorem. If $u, v \in C(G) \cap W_{p,\mathrm{loc}}^1(G)$ **are (free) extremals, then** $\max\{u, v\}$ **is a (free) subextremal.**

Proof. The function $\max\{u, v\}$ is in $C(G) \cap W_{p,\mathrm{loc}}^1(G)$. Since u and v obey the comparison principle in G, so does $\max\{u, v\}$. The result follows from Theorem 4.1.

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