# Homeomorphisms of the line which preserve BMO

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Suppose $u\colon \mathbf{R}\to\mathbf{R}$ is an increasing homeomorphism. For a function $f$ we then define $(Uf)(x)=f(u^{-1}(x))$. The purpose of this paper is to classify those $u$ for which the operator $U$ is bounded on $BMO$, the space of functions of bounded mean oscillation. Our theorem answers a question of Coifman and Meyer, and was announced in [2]. Though some time has passed since then, the result still seems to be of some interest as it has been used in several papers. The corresponding problem for $\mathbf{R}^n$, $n\geqq 2$, was solved by Reimann [5], who showed that $U$ is bounded on $BMO$ if and only if $u$ is quasiconformal. The translation of this statement to $\mathbf{R}$, i.e. that $u$ is quasiregular, is false.

In order to understand our result, we must first recall the definition of the Muckenhoupt class $A_\infty$. A positive Baire measure $\mu$ is said to be in $A_\infty$ if there are constants $C$, $\delta>0$ such that whenever $I$ is an interval and $E\subset I$,

$$\frac{\mu(E)}{\mu(I)}\leqq C\left(\frac{|E|}{|I|}\right)^{\delta},$$

where $|\cdot|$ denotes Lebesgue measure. Thus, every $\mu\in A_\infty$ satisfies $d\mu(x)=\omega(x)\,dx$. For such a positive weight function $\omega$, we put $\omega(E)=\int_E \omega\,dx$. We will need to use the fact that whenever $u$ is an increasing homeomorphism, the measure $u'\in A_\infty$ if and only if the same is true for $u^{-1}$. See [1] for a proof.

**Theorem.** *The following conditions are equivalent:*

a) $U\varphi\in BMO$ *whenever* $\varphi\in BMO$ *is lower semicontinuous.*
b) $U$ *is a bounded mapping from* $BMO$ *to* $BMO$.
c) $U$ *is a bijection from* $BMO$ *to* $BMO$.
d) $u'\in A_\infty$.

The proof of the theorem relies upon two useful facts. The first is the theorem of John and Nirenberg [4]. Let $I$ denote a generic (bounded) interval, and let

$\|\cdot\|_*$ denote the *BMO* norm. Then

$$\sup_I \frac{1}{|I|}\left|\{x\in I\colon\ |\varphi(x)-\varphi_I|>\lambda\}\right| \leqq c_1 \exp\left\{\frac{-c_2\lambda}{\|\varphi\|_*}\right\},$$

where $\varphi_I=\frac{1}{|I|}\int_I \varphi\,dx$ is the mean value of $\varphi$ on $I$. The second tool is due to Coifman and Rochberg [3]. Let $M$ denote the Hardy—Littlewood maximal operator. Suppose $\mu$ is a Baire measure on $\mathbf{R}$. Then if there exists a point $x$ such that $M\mu(x)<\infty$, the function $\log(M\mu)\in BMO$ and $\|\log(M\mu)\|_*\leqq c_3$.

With the John—Nirenberg theorem in hand, it is an easy exercise to show that condition (d) of the theorem implies condition (c). We must therefore only show that (a)$\Rightarrow$(b)$\Rightarrow$(d).

To show that (a)$\Rightarrow$(b), let $u'=\mu$ and suppose there is a set $E$ such that $\mu(E)>0$, while $|E|=0$. Let $f\in L^1$ be such that $\{x\colon f(x)>n\}$ is an open set containing $E$, for all integers $n$. Then $\varphi=\log(Mf)$ is in $BMO$ and $\varphi\equiv+\infty$ on $E$. Since any function of the form $M\mu$ is lower semicontinuous, so is $\varphi$. This is a contradiction because $U\varphi\notin L^1_{\mathrm{loc}}$, let alone $BMO$. Therefore $d\mu=\omega(x)\,dx$ where $\omega\geqq 0$ is in $L^1_{\mathrm{loc}}$, and $Uf$ is Lebesgue measurable whenever $f$ is.

By the results of [3], every $\varphi\in BMO$ is of the form $\varphi(x)=\alpha\log(M\mu_1)(x)+\beta\log(M\mu_2)(x)+b(x)$, where $\mu_1$ and $\mu_2$ are Baire measures and $b\in L^\infty$. Then since $Ub$ is bounded and Lebesgue measurable, $U\varphi$ is Lebesgue measurable and in $BMO$. The closed graph theorem now shows that $U$ is a bounded operator on $BMO$. Thus, (a)$\Rightarrow$(b).

Now suppose that (b) holds with, say, $\|U\|=K$, and suppose that (d) fails. Let $u'(x)=\omega(x)$, and fix $\varepsilon,\delta>0$. Then there is an interval $I$ and a (measurable) set $E\subset I$ with $0<|E|<\varepsilon|I|$ and $\omega(E)/\omega(I)\geqq(|E|/|I|)^\delta$. By a translation and a change of scale, we may assume that $I=u(I)=[0,1]$. We may also assume by symmetry that $E\subset[0,1/2]$. Let $\alpha>1$ be such that $u(\alpha)=2$, and let $\varphi=\log(M\chi_E)$. Then $\varphi\equiv 0$ a.e. on $E$, $\varphi(x)\leqq\log(2|E|)$ on $[1,\alpha]$ and $\|\varphi\|_*\leqq c_3$. Since $\|U\varphi\|_*\leqq c_3K$ and $U\varphi\leqq\log(2|E|)$ on $[1,2]$, the John—Nirenberg theorem shows $(U\varphi)_{[0,2]}\leqq c_4K+\log|E|$. If $\varepsilon$ is small enough, $c_4K+\log|E|<0$ and we thus have

$$|F|\equiv\left|\{x\in[0,2]\colon\ |U\varphi(x)-(U\varphi)_{[0,2]}|>-c_4K-\log|E|\}\right|\geqq|U(E)|\geqq|E|^\delta.$$

But by the John—Nirenberg theorem,

$$|F|\leqq 2c_1\exp\left\{\frac{c_2}{c_4K}(c_4K+\log|E|)\right\}=c_3|E|^{c_6}.$$

If $\delta<c_6$, we then obtain $(c_5)^{-1}\leqq\varepsilon^{c_6-\delta}$, which fails as soon as $\varepsilon$ is small enough.

## References


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*Received September 1, 1982*

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