On the postulation of canonical curves in $\mathbf{P}^3$

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Introduction

In this paper we consider a very particular problem. We study the postulation of a general smooth curve C in $\mathbf{P}^3$ with $\mathcal{O}_C(1) \cong K_C$, when C has genus 7, 8, 9 or 11.

We call canonical curve in $\mathbf{P}^n$ a smooth, connected curve $C \subset \mathbf{P}^n$ with $\mathcal{O}_C(1) \cong K_C$.

Definition. We say that a subscheme Z of $\mathbf{P}^3$ has maximal rank if foı every integer t the restriction map $\varrho_Z(t): H(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}}(t)) \to H^0(Z, \mathcal{O}_Z(t))$ is either injective or surjective.

It is useful to know that a curve C in $\mathbf{P}^3$ has maximal rank since in this case for every k we know completely the dimension of the vector space of surfaces of degree k containing C. For example if C is a canonical curve of genus g with maximal rank, it is contained in a surface of degree k if and only if $\binom{k+3}{3} > (2k-1)(g-1)$.

We prove the following theorem:

Theorem 1. *The general canonical curve of genus* $g=7, 8, 9$ *or* 11 *in* $\mathbf{P}^3$ *has maximal rank.*

The proof is really an existence proof. We construct a reducible curve Z with the expected postulation, i.e. with maximal rank. Then we show that Z is a limit of a flat family of canonical curves in $\mathbf{P}^3$. By semicontinuity a general canonical curve of that genus has maximal rank. The existence of the flat family follows from 2 theorems about degeneration of curves; we will state them in section 1. One of them is due to Hartshorne and Hirschowitz [13]; the other one is contained in [1]. We use often very particular cases of the Brill—Noether theory proved by Griffiths and Harris in [6]. However it seems that sometimes the use of this deep theory could be avoided by ad hoc argument. Theorem 1 is proved separately for each genus $g=7, 8, 9, 11$.

In [7] Gruson and Peskine gave a striking counterexample to a conjecture by Hartshorne [11], showing that no canonical curve of genus 5 or 6 has maximal rank.

It remained open the problem for canonical curves of higher genera. We hope that the existence theorem given here will clarify the situation. We tried invain to extend the result to other genera.

In the last section we consider a strange example of non-smoothability of reducible curves. Let C be a smooth curve of genus 5 and degree 7 in $\mathbf{P}^3$. We have $\mathcal{O}_C(1)\cong K_C(-P)$ for a unique point P of C. Think of $\mathbf{P}^3$ as a hyperplane of $\mathbf{P}^4$. Fix a point x of C and let D be a general line in $\mathbf{P}^4$ through x, D not contained in $\mathbf{P}^3$. Then $C\cup D$ can be deformed to a smooth curve in $\mathbf{P}^4$ if and only if $x=P$.

§ 1.

In this section we consider the tools for the proof of Theorem 1. The proof will be a case by case check. The cases $g=7, 8, 9, 11$ will be considered respectively in sections 3, 4, 5, 6.

The notations below were introduced in [3].

Definition 1. *Define $Z(d, g; n)$ as the closure in the Hilbert scheme Hilb $\mathbf{P}^n$ of the set of smooth non-degenerate, irreducible curves of degree d and genus g. Then define $Z'(d, g; n)$ as the closure in $Z(d, g; n)$ of the set of smooth, irreducible, non degenerate curves C of genus g and degree d with $h^1(C, \mathcal{O}_C(1))=1$.*

It is well-known that $Z'(d, g; n)$ is irreducible (eventually empty). It follows from the irreducibility of the moduli scheme of curves of genus f and the fact that a line bundle L of degree d with $h^1(C, L)=1$ corresponds to a non-zero section of $K_C\otimes L^*$, i.e. to $2g-2-d$ general points of C.

The following result is due to Hartshorne and Hirschowitz [13]. It is one of the 2 main tools we use.

Theorem 2. *Let C be a curve in $Z(d, g; n)$ with $H^1(C, N_C)=0$ and let D be a line intersecting C in k ($k=1$ or $k=2$) distinct points, the intersection being quasi-transversal. Then $C\cup D$ is in $Z(d+1, g+k-1; n)$.*

Note that by a Mayer—Vietoris type exact sequence, $H^1(C\cup D, N_{C\cup D})=0$. Thus we may apply several times Theorem 2. We are interested only in the case $k=2$, which we assume during this discussion. By a Mayer—Vietoris type exact sequence, $h^1(C, \mathcal{O}_C(1))=h^1(C\cup D, \mathcal{O}_{C\cup D}(1))$. Assume $n=d-g+1$. A general element C of $Z'(d, g; d-g+1)$ has $H^1(C, N_C)=0$; this is a well-known consequence of [4]. Since every smooth, non degenerate curve of genus $g+1$ and degree $d+1=g+n$ in $\mathbf{P}^n$ is special, for a general secant D to C, $C\cup D\in Z'(d+1, g+1; d-g+1)$. Now assume $n<d-g+1$ and C a smooth curve in $Z'(d, g; n)$. Then C is the projection of $C'\in Z'(d, g; d-g+1)$ and there exists a unique secant D' to C' such that $C\cup D$ is a projection of $C'\cup D'$. Thus $C'\cup D'\in Z'(d+1, g+1; d-g+1)$ and, by projecting

the deformation to $\mathbf{P}^n$, we obtain $C \cup D \in Z'(d+1, g+1; n)$. We will use always Theorem 2 in this form.

The other degeneration theorem we need was proved, but unfortunately not stated un full generality, in [1] Prop. II. 1.

Let C be a smooth curve of degree d in $\mathbf{P}^n$ and consider $\mathbf{P}^k$, $k<n$, as a linear subspace of $\mathbf{P}^n$. Denote by $Pr_d(C, \mathbf{P}^k)$ the closure in Hilb $\mathbf{P}^k$ of the set of general projections of C in $\mathbf{P}^k$. We need many reducible elements in $Pr_d(C, \mathbf{P}^k)$. The proofs of [1] do not use in an essential way the assumption "$\mathcal{O}_C(1)$ not special".

Let X be a smooth curve and L a very ample line bundle on X. We write φ_L for the embedding of X given by the sections of L. Note that for a point P of X, $L(P):= L\otimes_{\mathcal{O}_X}\mathcal{O}_X(P)$ is very ample if and only if $h^0(X, L(P))=h^0(X, L)+1$.

Theorem 3. ([1] Prop. II. 1) *Let X be a nonsingular, non degenerate curve embedded in $\mathbf{P}^n$, $\deg X=d$. Let $P_1, \dots, P_k$ be distinct points of X. Put $L:=\mathcal{O}_X(1)$ and $M:= L(P_1+\dots+P_k)$. Assume $h^0(X, M)=k+h^0(X, L)$. Let D_i, $i=1, \dots, k$, be a line intersecting X only at P_i and quasi-transversally; assume that the D_i's are disjoint. Then $X\cup D_1\cup\dots\cup D_k\in Pr_{d+k}(\varphi_M(X), \mathbf{P}^n)$.*

We will use Theorem 3 also to control the postulation of the intersection of a general element of $Z'(d, g; n)$ with a hyperplane not in general position. We recall that the proof of Theorem 3 in [1] used a family of projections from $\mathbf{P}^s$, $s=h^0(X, M)-1$, into $\mathbf{P}^n$. In the proof of [1] Prop. II. 1, it was constructed an affine smooth curve U, $0\in U$, and a closed subset V of $\mathbf{P}^n\times U$ such the restriction to V of the projection on the second factor has the following property: p: $V\to U$ is a flat family of curves with, for $t\neq 0$, $p^{-1}(t)$ a general projection of $\varphi_M(X)$, $p^{-1}(0)=X\cup D_1\cup\dots\cup D$.

Thus the proof gave a family of isomorphism $h_t: p^{-1}(t)\cong\varphi_M(X)$ for $t\neq 0$ and h_0: $X\cong\varphi_M(X)$. Here is a typical example of the use of Theorem 3 to control an intersection. Consider C smooth, $C\in Z'(2g-5, g; 3)$ with $\mathcal{O}_C(1)\cong K_C(-P_1-P_2-P_3)$ with the P_i distinct and not collinear. These condition are satisfied for general C. Take the plane H spanned by the P_i's. Suppose we want to control the intersection of C with H, if C is general. We consider a degeneration of C to the union of $X\in Z'(g+2, g; 3)$, X smooth, and $g-7$ lines D_i intersecting X (Theorem 3) at the point R_i. We use the notations U, V, p, h_t for this degeneration. There exists uniquely determined points P_1^t, P_2^t, P_3^t on $p^{-1}(t)$ such that $\mathcal{O}_{p^{-1}(t)}(1)\cong K_{p^{-1}(t)}(-P_1^t-P_2^t-P_3^t)$. Restricting possibly U, we may assume that for every t, the points p_i^t span a plane H_t; here we choose a good X, with general choice of the R_j's. By semicontinuity the postulation of $p^{-1}(t)\cap H_t$ for general t is bounded by the postulation of $(X\cup\bigcup D_j)\cap H_0$. In $(X\cup\bigcup D_j)\cap H_0$ the points $(\bigcup D_j)\cap H_0$ can be choosen in general position. Thus we have only to control $X\cap H_0$.

Notations: If x is a point of $\mathbf{P}^3$, $\chi(x)$ means its first infinitesimal neighborhood in $\mathbf{P}^3$. Let Z be a subscheme of $\mathbf{P}^3$ and H a plane. $\mathrm{Res}_H(Z)$ is its residual scheme with respect to H. For example if Z is reduced, $\mathrm{Res}_H(Z)$ is the union of the irreducible components of Z not contained in H; if $x \in H$, $\mathrm{Res}_H(\chi(x))=x$. Similar notation are used in any projective space, with H a hyperplane. For a subscheme Z of $\mathbf{P}^3$, $\mathcal{I}_Z$ is its ideal sheaf; for any integer t we have the restriction map $\varrho_Z(t)$: $H^0(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(t)) \to H^0(Z, \mathcal{O}_Z(t))$. We put $h^i(Z, L):= \dim H^i(Z, L)$. N_Z is the normal bundle or the normal sheaf.

§ 2.

In this section we give some preliminary results about curves of genus g and degree $g+2$ in $\mathbf{P}^3$. Every such curve is special. We will need to know that for $4 \leqq g \leqq 11$, a general element of $Z'(g+2, g; 3)$ has maximal rank.

Lemma 1. *For* $4 \leqq g \leqq 11$ *a general element of* $Z'(g+2, g; 3)$ *has maximal rank*

Proof: For $g=4$ and $g=5$ every smooth element in $Z'(g+2, g; 3)$ is projectively normal and in particular has maximal rank. In fact, the canonical curves of genus 4 are complete intersection of a quadric and a cubic. By the generalized Halphen's bound of the genus proved in [7], every smooth element of $Z'(7,5;3)$ is linked by 2 cubic surfaces to a plane conic and thus it is projectively normal. By [4] a general element C in $Z'(7, 5; 3)$ has $H^1(C, N_C)=0$. Consider the following assertions:

U_3: there exists in $\mathbf{P}^3$ a curve C of genus 6 and degree 8 with $h^0(\mathbf{P}^3, \mathcal{I}_C(3))=1$.

U_4: there exists a triple (Y, D, S) where Y is the union of a smooth C in $Z'(8, 6; 3)$ and 2 lines D_i, $i=1, 2$, with D_1 secant to C, D_2 intersects D_1 and C, D is a secant line to Y and $S \subset D$, $\mathrm{card}(S)=2$, with $H^0(\mathbf{P}^3, \mathcal{I}_{Y \cup S}(4))=0$.

U_5: there exists a curve $Y \in Z'(13, 11; 3)$ with Y union of a smooth curve C of genus 8 and degree 10 and 3 lines D_i, $i=1, 2, 3$, with D_1 secant to C, D_2 and D_3 intersecting C and D_1, with $h^2(\mathbf{P}^3, \mathcal{I}_Y(5))=1$.

It is easy to show that U_3, U_4 and U_5 imply Lemma 1. In fact U_4 implies that $Y \cup D$ is not contained in any surface of degree 4 and by semicontinuity the same happens for a general curve of genus 10 and degree 12 (or genus 11 and degree 13). For the proof of the assertions U_i we use the method of [12] and [14].

a) *Proof of* U_3: We take a smooth curve Z in $\mathbf{P}^3$ of genus 5 and degree 7, hence not contained in any quadric. We consider a general plane H intersecting Z at 7 distinct points in uniform position [9], hence in particular no 3 of them collinear. Let D be a line in H containing 2 points of Z; let P be a general point of H. By Theorem 2 it is sufficient to prove that no cubic surface contains Z, D and P. Take $f \in H^0(\mathbf{P}^3, \mathscr{I}_{Z\cup D\cup\{p\}}(3))$ and call g its restriction to H. We want to prove that g vanishes. Indeed g vanishes on a line, on 5 points no 4 of them collinear and at P; the 5 points determine a unique conic A and we may assume P not contained in A. Thus f is divided by the equation of H and Z is contained in a quadric, unless $f=0$.

b) The implications $U_3 \Rightarrow U_4$ and $U_4 \Rightarrow U_5$ are similar. We sketch only the second one. By Theorem 2, semicontinuity and U_4 there is a smooth curve Z in $Z'(10, 8; 3)$ and 2 points A, B such that $H^0(\mathbf{P}^3, \mathscr{I}_{Z\cup\{A,B\}}(4))=0$. We take a general plane H; we may assume that B is a general point of H. Call $P_i, i=1, \ldots, 10$, the points in $Z\cap H$. Let L, D, R be lines in H with L spanned by P_1 and P_2, D by B and P_3, R by B and P_4. For general B we may assume that $B \notin L$ and that L, D, R, contain no P_i with $i \geqq 5$. Put $Y=Z\cup L\cup D\cup R\cup\chi(B)$. Y can be deformed to the union of Z, a secant line L' to Z, and lines D', R' intersecting Z and L' (see [10] fig. 11 p. 260 or the pictures in [14]). Thus $Y \in Z'(13, 11; 3)$ and by semicontinuity it is sufficient to show that $Y\cup A$ is not contained in any surface of degree 5. Since $\mathrm{Res}_H(Y)=Z\cup B$, this follows from U_4 and the fact that the P_i, $i=5, \ldots, 10$, are not contained in a conic because Y is not contained in a quadric and the P_i, for general H, are in uniform position.

§ 3.

In this section we prove Theorem 1 for $g=7$. We have simply to prove the existence of a canonical curve of genus 7 in $\mathbf{P}^3$ with maximal rank. The strategy of the proof is the same as in Lemma 1, [12], [14], [2], [3]. The same strategy will be used for $g=8, 9, 11$.

A_4: there exists a nonsingular curve C in $\mathbf{P}^3$ of genus 7, degree 10 with $h^1(C, \mathcal{O}_C(1))=1$ and $h^0(\mathbf{P}^3, \mathscr{I}_C(4))=1$.

A_5: there exists in $\mathbf{P}^3$ a canonical curve of genus 7 with $h^0(\mathbf{P}^3, \mathscr{I}_C(5))=2$.

It is easy to show that a canonical curve C satisfying A_5 has maximal rank. In fact by A_5 $H^1(\mathbf{P}^3, \mathscr{I}_C(5))=0$ and by [2] Lemma 5.1, $H^1(\mathbf{P}^3, \mathscr{I}_C(t))=0$ for $t \geqq 5$. Furthermore C cannot be contained in a quartic surface, otherwise it would be contained in too many reducible quintic surfaces.

First we will prove A_4 and then that A_4 implies A_5.

a) *Proof of* A_4: Let D be a canonical curve of genus 5 and degree 8 in $\mathbf{P}^3$. By [7] D has not maximal rank but it is always [7] p. 55, with $h^0(\mathbf{P}^3, \mathscr{I}_D(4))=7$ i.e. $h^1(\mathbf{P}^3, \mathscr{I}_D(4))=0$.

By the remarks after Theorem 2 for every 2 general secant A, B to D, $D\cup A\cup$

$B \in Z'(10, 7; 3)$. Thus it is sufficient to prove that we may find 2 secants A, B with $h^0(\mathbf{P}^3, \mathscr{I}_{D \cup A}(4)) = 4$, $h^0(\mathbf{P}^3, \mathscr{I}_{D \cup A \cup B}(4)) = 1$.

The existence of A is easy. Assume we cannot choose such an A. Then for every secant line A to D, $h^0(\mathbf{P}^3, \mathscr{I}_{D \cup A}(4)) \geqq 5$. Let H be a general plane; in H we have 8 points P_i, $i=1, \ldots, 8$, of D in uniform position [9]. Take $A = P_1 P_2$. Since $\mathscr{I}_D(4)$ is generated by global sections, for every line $P_i P_j$, $i, j > 2$, $P_i P_j \cap A$ gives a condition for $V := H^0(\mathbf{P}^3, \mathscr{I}_D(4))$, while by assumption $P_i P_j$ gives at most 2 conditions for V. Thus we may find f in V such that f vanishes on a point P contained neither in H nor in the unique cubic surface [7] containing D and on the lines $A, P_3 P_4, P_3 P_5, P_3 P_6$. This implies that f vanishes on H, contradicting the choice of P.

The proof of the existence of B is similar but slightly more complicated. Assume that for a general secant A and for every secant B, $h^0(\mathbf{P}^3, \mathscr{I}_{D \cup A \cup B}(4)) \geqq 2$, i.e. for every point P in $\mathbf{P}^3$, there exists $F \in V$, F vanishing on A, B, P and $F \neq 0$. We take a general plane H and put $H \cap D = \{P_i\}$, $i=1, \ldots, 8$, with the P_i in uniform position. We take as P a general point on the line $L := P_2 P_3$. Put $L' = P_1 P_4$, $x = L \cap L'$. Since x can be choosen as a general points of $\mathbf{P}^3$ (for a general point there are at least 2 secants to D by the genus formula on a plane), we have $h^0(\mathbf{P}^3, \mathscr{I}_{D \cup A \cup x}(4)) = 3$ and by assumptions we can find such a F vanishing on D, A, P, L' and thus on L. If $P_5 P_6$ contains x, for example, then certainly $P_5 P_7$, $P_5 P_8$, $P_6 P_7$, $P_6 P_8$ do not contain x and thus at least 4 of the $P_i P_j$, $i, j \geqq 5$, intersect $L \cup L'$ in 2 points. Note that by monodromy [9] we may assume that no 3 lines $P_i P_j$ have a common point. Let I be one of the $P_i P_j$ not containing x. If the 2 points in $I \cap (L' \cup L)$ give independent conditions for $W := H^0(\mathbf{P}^3, \mathscr{I}_{D \cup A}(4))$, then F would vanish on I and thus on H since if, say, $I = P_6 P_7$, we may assume, for general A, that P_5, P_8 and $A \cap H$ are not collinear. This contradicts the generality of A, since we may assume that $D \cup A$ is not contained in any cubic surface.

Thus for every such I, $I \cap (L \cup L')$ give at most one condition for W. We may assume that none of the points of $P_i P_j \setminus D$ is a base point of W. In fact W has, outside $D \cup A$, a base locus S contained strictly in the unique cubic surface T containing D, since W contains irreducible quartics. Thus from a general point of T passes a secant to D not intersecting S, i.e. the general secant to D intersect the base locus of W only at D. Thus every quartic surface G containing D, A, L contains also every point of $L' \cap I$ for at least 2 such lines I; thus it contains 5 points of L' and hence L'. But in this statement P does not appear. It means that every quartic surface containing D, A, L contains the other secant lines $P_i P_j$ and thus H. This is a contradiction, since we may assume A not contained in the unique cubic surface containing D.

b) A_4 implies A_5. *Proof:* Let C be given by A_4. Thus $\mathcal{O}_C(1) \cong K_C(-B_1 - B_2)$

for some B_1, B_2 in C. By semicontinuity and irreducibility we may assume that C has general moduli and $B_1 \neq B_2$.

Claim: We may assume that $H \cap C$ is reduced (except eventually a double point at B_i) and that $H \cap C \setminus \{B_1, B_2\}$ give 8 conditions for the cubic in H.

First we show that the claim implies A_5 by the general method of [12], [14]. Let L_i, $i=1, 2$, be a general line in H containing B_i and put $y = L_1 \cap L_2$. Consider $Y = C \cup L_1 \cup L_2 \cup \chi(y)$, where $\chi(y)$ means the first infinitesimal neighborhood of y in $\mathbf{P}^3$. Let P, P' be general points in H. Y can be deformed to a curve Y' union of C and of 2 disjoint lines through B_1, B_2 (see the pictures in [14] and [10] fig. 11 p. 260). By Theorem 2 $Y' \in Z'$ (12, 7; 3). By semicontinuity it is sufficient to prove that $V := H^0(\mathbf{P}^3, \mathscr{I}_{Y \cup P \cup P'}(5))$ vanishes. Take f in V. Then the restriction g of f to H vanishes on L_1, L_2, 8 points of $H \cap C$ and 2 general points P, P', By the claim g vanishes on H. Since the residual scheme of Y to H is $C \cup y$ and y can be any general point of $\mathbf{P}^3$, by A_4 $f=0$. Now we prove the claim. First we want to use that C has general moduli to apply some very special case of the Brill—Noether theory proved by Griffiths and Harris [6] and show (in a silly way) that we may assume that the residual scheme T in H of $C \cap H$ with respect to L_1, L_2 is reduced. Equivalently we want to show that for H general, $H \cap C$ has no triple point and no double point, except possibly B_1 or B_2. Consider the embedding h of C in $\mathbf{P}^4$ given by the complete linear system $H^0(C, \mathcal{O}_C(1))$. For a general hyperplane R through $h(B_1)$ and $h(B_2), h(C) \cap R$ has at most double points at $h(B_i)$. Otherwise by Bertini's theorem the linear series $|K_C(-B_1 - B_2)|$, which is a g_8^2, would have at least 2 base points, while by Brill—Noether theory [6] C has no g_6^2, since it has general moduli. Then we want to use Theorem 3 and the discussion after it to control, outside the B_i's, the postulation of $C \cap H$. By Theorem 3 we can degenerate C in $\mathrm{Pr}_{10}(h(W), \mathbf{P}^3)$ to $C' \cup L$, where there is an isomorphism s from C to C', $\deg C' = 9$ and L is a line. Thus it is sufficient to control $C' \cap U$ where U is a general plane through $s(B_1)$ and $s(B_2)$. Again we may assume $C' \cap U$ reduced. We have to prove that the 7 points in $C' \cap U \setminus \{s(B_1), s(B_2)\}$ give independent conditions for cubic. By [5], p. 714, it is sufficient to prove that C' has no 5-secant. But a 5-secant to C' gives a g_4^1 on C', contradicting the fact that C' has general moduli.

§ 4.

The case $g=8$ is easy. We consider the following assertions:

B_4: the general element C of $Z'(10, 8; 3)$ has $h^0(\mathbf{P}^3, \mathscr{I}_C(4)) = 2$;
B_5: the general element C in $Z'(12, 8; 3)$ has $h^0(\mathbf{P}^3, \mathscr{I}_C(5)) = 3$;
B_6: there exists a canonical curve C in $\mathbf{P}^3$ with $h^0(\mathbf{P}^3, \mathscr{I}_C(6)) = 7$.

B_4 is a particular case of Lemma 1. By [2] Lemma 5.1, B_6 implies $\varrho_C(t)$ surjective for $t\geqq 6$. Thus such a canonical curve C has maximal rank if and only if it is contained in no quintic surface. This follows very easily from the proof that B_4 implies B_5 below (or see part c) in the next section).

a) B_4 implies B_5. *Proof:* Let C be given by B_4 with $K_C(-P_1-P_2-P_3-P_4)$. By irreducibility and semicontinuity we may assume C of general moduli and the P_i's different. Consider a general plane H through P_1 and P_2. Let L_i, $i=1,2$, general lines in H with $P_i\in L_i$ and put $x=L_1\cap L_2$. Put $Y=C\cup L_1\cup L_2\cup\chi(x)$; it is sufficient to show that $h^0(\mathbf{P}^3, \mathcal{I}_Y(5))=3$. First x, moving H, can be arbitrary and thus we may choose a point A not in H such that $H^0(\mathbf{P}^3, \mathcal{I}_{C\cup\{A,x\}}(4))=0$. Since $\mathrm{Res}_H(Y)=C\cup x$, to prove B_5 it is sufficient to show that if $f\in H^0(H, \mathcal{O}_H(3))$ vanishes on $C\cap H, L_1, L_2$ and 2 fixed points P, P' (choosen in general position) then $f=0$. Equivalently we show that the residual scheme Z of $C\cap H$ to $L_1\cup L_2$ in H gives 8 conditions for the cubic passing through it. First we may assume Z reduced (8 distinct points, possibly one of the P_i). By Bertini's theorem it is sufficient to show that $K_C(-2P_1-2P_2-P_3-P_4)$ has at most one base point. If not, there exists a 3-dimensional family of g_6^1 on C, contradicting Brill—Noether theory [6]. Now we have to show that the 8 points in Z give independent conditions for the cubic passing through them. This fails if and only if 5 of the points in Z are collinear or all the points are in a conic. The first case implies easily the existence of infinite g_5^1 on C, contradicting [6]. If for every such plane H, Z is contained in a conic S, this happens in particular, if H is a plane containing P_1, P_2 and P_3. Interchanging the role of P_2 and P_3 we obtain that S contains $C\cap H$. By the exact sequence

(1) $$0\to\mathcal{I}_C(1)\to\mathcal{I}_C(2)\to\mathcal{I}^{(2)}_{C\cap H,H}\to 0,$$

and the linear normality of C, C is contained in a quadric surface, contradicting B_4.

b) The proof of $B_5\Rightarrow B_6$ is similar, easier and omitted.

§ 5.

Here we prove Theorem 1 for $g=9$. Consider the following assertions:

E_6: there exists $C\in Z'(15, 9; 3)$ with $h^0(\mathbf{P}^3, \mathcal{I}_C(6))=2$.
E_7: there exists $Y\in Z'(16, 9; 3)$ with $h^0(\mathbf{P}^3, \mathcal{I}_Y(7))=16$.

a) A_5 implies E_6. *Proof:* Let Z be given by A_5; take a general secant line L to Z. Consider a flat map $p\colon V\to U$, where V is closed in $\mathbf{P}^3\times U$, p is the restriction of the projection on the second factor, U is an affine smooth curve with $0\in U$, $p^{-1}(0)=Z\cup L$ and $C_t:=p^{-1}(t)$ a smooth element of $Z'(13, 8; 3)$

for $t \neq 0$. For every $t \neq 0$ there is a unique point P_t on C_t such that $\mathcal{O}_{C_t}(1) \cong K_{C_t}(-P_t)$. Since p is proper and U a smooth curve, the morphism u from $U \setminus \{0\}$ into V with $u(t) := P_t$ can be extended to a section s of p.

Claim: $s(0) \in L$.

Assuming the claim, we will prove E_6. We consider a plane H containing the general secant L. We may assume $H \cap Z$ reduced and in uniform position [9]. We consider in H a line L' through $s(0)$ and not intersecting, outside possibly $s(0)$, Z. We want to prove that $Z \cup L \cup L' \in Z'(14, 8; 3)$. Indeed consider a general point $x \in L'$ and let L'_t be the line spanned by x and $s(t)$; in $\mathbf{P}^3 \times U$ consider the union J of V and the L'_t (with $L'_0 := L'$); the restriction to J of the projection on U is flat [10].

Now consider a line L'' in H not intersecting Z. Put $Y = Z \cup L \cup L' \cup L''$. We have $Y \in Z'(15, 9; 3)$ by theorem 2. The usual game will show $h^0(\mathbf{P}^3, \mathcal{I}_Y(6)) = 2$. Indeed in H for $H^0(H, \mathcal{O}_H(6))$ we have 3 lines and 10 points of $Z \cap H$. Since the points in $Z \cap H$, are in uniform position, the only problem arise if $Z \cap H$ is contained in a cubic for general H. By [8], since $\deg Z = 12$, this implies Z contained in a cubic surface, contradiction.

Now we prove the claim. The relative dualizing sheaf $\omega_{V/U}$ is locally free and for every $t \in U$ $h^0(C_t, \omega_{C_t}(-1)) = 1$ (with $C_0 := Z \cup L$). By changing basis, restricting possibly U, $p_*(\omega_{V/U}(-1))$ is free of rank 1 and a non-zero section of it induces a non-zero section u_t on every fiber. Since u_t, $t \neq 0$, vanishes exactly at $s(t)$ while u_0 vanishes on L, we have proved the claim.

b) E_6 implies E_7. *Proof:* Let C be given by E_6, with $\mathcal{O}_C(1) \cong K_C(-P)$. Take a general plane H through P and a line L in H intersecting C only at P. Then it is easy to show $h^0(\mathbf{P}^3, \mathcal{I}_{C \cup L}(7)) = 16$, using the remark after Theorem 3 to control $C \cap H$. Indeed degenerating C to the union of 4 lines and C' of degree 11, we have only to control 10 points for $\mathcal{O}_H(6)$; no 8 of the points can be collinear since C' has no g^1_2.

c) Now we can prove Theorem 1 for $g=9$. E_7 implies that for a general canonical curve of genus 9 Y, $\varrho_Y(n)$ is surjective for $n \geqq 7$. Thus it is sufficient to show that a general canonical curve of genus 9 in not contained in a sestic surface. This will follows easily from E_6. Let C be given by E_6 with $\mathcal{O}_C(1) \cong K_C(-P)$. We have to prove that for a general line D through P, $C \cup D$ is not contained in any sestic surface. Let x be a general point of $\mathbf{P}^3$, not in the base locus of $F := H^0(\mathbf{P}^3, \mathcal{I}_C(6))$. Let S be the only sestic surface containing C and x. In a neighborhood G of x, every $y \in G \cap S$ is not a base point of F and for general $y \in G \cap S$, the line Px is not contained in S, unless S is a cone with vertex P. Thus it is sufficient to show that that a general $S \in F$ is not a cone with vertex P. Let C' be the projection of C from P into a plane. If $\deg C' \geqq 6$, then C cannot be contained in 2 sestic cones with vertex P. If $\deg C'=2$, C is contained in too many sestic surfaces.

§ 6.

For $g=11$ we consider the following assertions:

D_6: There exists in $\mathbf{P}^3$ a nonsingular curve $C \in Z'(15, 11; 3)$ with $h^0(\mathbf{P}^3, \mathcal{I}_C(6))=4$.

D_7: There exists a nonsingular $C \in Z'(17, 11; 3)$ with $\mathcal{O}_C(1) \cong K_C(-P_1-P_2-P_3)$, with the P_i distinct, a plane H containing P_1, P_2, P_3, 3 lines $L_i, i=1, 2, 3$, $P_i \in L_i$, with $A:=L_1 \cap L_2, B:=L_1 \cap L_3, I:=L_2 \cap L_3$ and

$$h^0(\mathbf{P}^3, \mathcal{I}_{C \cup \{A, B, I\}}(7)) = 8.$$

D_8: There exists in $\mathbf{P}^3$ a canonical curve of genus 11 with $h^0(\mathbf{P}^3, \mathcal{I}_C(8))=15$.

a) B_5 implies D_6. *Proof:* Let C be a smooth curve of genus 8 and degree 12 satisfying B_5. Take a general plane H intersecting C at 12 distinct points in uniform position [9]. Consider the union Y of C and of 3 lines D, R, L in H with D secant to C, R intersecting C at a point and L disjoint from C. By Theorem 2 it is sufficient to show that the 9 points of $C \cap H$ not in $D \cup R$ give independent conditions for $\mathcal{O}_H(3)$. This happen if and only if either 5 are collinear or 8 in a conic or all are contained in 2 irreducible cubics. By definition of uniform position the last case contradicts Bezout's theorem while the first 2 cases imply that C is contained in a plane or a quadric, by the exact sequence (1) in Section 4.

c) D_6 implies D_7. *Proof:* Let X be given by D_6 with $\mathcal{O}_X(1) \cong K_X(-P_1-P_2-P_3-P_4-P_5)$. As usual we may assume X of general moduli and the P_i distinct. First we want to show that we may assume no 4 of the P_i coplanar. Let g be the embedding of C in $\mathbf{P}^5$ by the complete linear system $K_X(-P_1-P_2-P_3-P_4-P_5)$. It is sufficient to show, for instance, that $g(P_1), g(P_2), g(P_3), g(P_4)$ are not coplanar. If this happen, $h^0(X, K_X(-2P_1-2P_2-2P_3-2P_4-P_5))=0$ for every P_5 i.e. $h^0(X, \mathcal{O}_X(2P_1+2P_2+2P_3+2P_4))=2$. Since the general element of a base point-free g^1_8 is reduced, we obtain a 4-dimensional family of g^1_8 on C, contradicting Brill—Noether theory. Furthermore we may assume that for every plane R spanned by $3P_i$, say $P_1, P_2, P_3, R \cap X$ is reduced. By Bertini's theorem, it is sufficient to show that $|K(-P_4-P_5-2P_1-2P_2-2P_3)|$ has at most a base point, not at P_i, $i=1, 2$ or 3. The first part follows from Brill—Noether theory for g^2_{10}; if the linear system considered has P_1 as base point for general P_i, interchanging P_1 and P_2 (or P_3) along a path, we obtain P_2 and P_3 as base points, contradiction.

Now consider the plane U spanned by P_3, P_4, P_5 and take in U 2 general lines L_i, $i=4, 5$, with $P_i \in L_i$; put $x := L_4 \cap L_5$. Consider $Z = X \cup L_4 \cup L_5 \cup \chi(x)$. We have $Z \in Z'(17, 11; 3)$. First we want to prove $h^0(\mathbf{P}^3, \mathcal{I}_Z(7)) = 11$. By the discussion after Theorem 3 we can degenerate X to the union of 2 lines and of $Y \cong X$, $\deg Y = 13$; thus we have only to prove that 11 distinct points in $Y \cap U$ give independent conditions for $H^0(U, \mathcal{O}_U(5))$. No 7 of them are collinear since Y has no g_6^1. Let S be an irreducible conic containing $n \geqq 5$ of them and such that among the remaining $11-n$ of them, no 5 are collinear. By adding $11-n$ points on S, we win easily since the remaining $11-n$ points give independent conditions for cubics [5], p. 714.

Consider the plane R spanned by P_1, P_2, P_3 and take in R 2 general lines $L_i, i=1, 2, P_i \in L_i$. Put $L_3 = U \cap R$, $A = L_1 \cap L_2$, $B = L_1 \cap L_3$, $I = L_2 \cap L_3$. We have to prove that A, B, I give 3 independent conditions for $H^0(\mathbf{P}^3, \mathcal{I}_Z(7))$. We want to repeat the omitted proof with the residual scheme to a plane. A gives a condition since it is a general point of R and Z is not contained in a quintic. B and I are general points on L_3; as in the first part of the proof, we have to control 10 points of $Y \cap R$, A and 2 general points on L_3. The only problem now arises if on L_3, apart from P_3, there are 5 more points of Y. If this happens, we change in the last construction L_3 with L_4 and take as plane R' spanned by P_1, P_2, P_4 or L_3 with L_5 and a plane R'' containing P_1, P_2, P_5. It is impossible that $Y \cap U$ contains 5 points on R, 5 points on R' and 5 points on R'' outside P_1, P_2, P_3.

d) D_7 implies D_8. *Proof:* Let $(C, P_1, P_2, P_3, H, L_1, L_2, L_3, A, B, I)$ be given by D_7. Put $Z = C \cup L_1 \cup L_2 \cup L_3 \cup \chi(A) \cup \chi(B) \cup \chi(I)$. We have $\mathrm{Res}_H(Z) = C \cup \{A, B, I\}$ and we apply D_7 and the usual game. We have only to control as in c) 10 of the points of Y, $Y \cong C$, $\deg Y = 13$, with respect to $\mathcal{O}_H(5)$.

e) D_8 implies that for a general canonical curve T in $\mathbf{P}^3$, $\varrho_T(t)$ is surjective for $t \geqq 8$. By irreducibility it is sufficient to show that a general T is not contained in a surface of degree 7. We use the notations of c). Let D_3 be a general line in U with $P_3 \in D_3$ and put $M = D_3 \cap L_3$, $N = D_3 \cap L_5$ and $T = X \cup D_3 \cup L_4 \cup L_5 \cup \chi(M) \cup \chi(N) \cup \chi(x)$. We want to show that for a general choice of the P_i $h^0(\mathbf{P}^3, \mathcal{I}_T(7)) \leqq 4$. For this it is sufficient to show

1) $h^0(\mathbf{P}^3, \mathcal{I}_{X \cup \{x, M\}}(6)) = 2$;
2) at least 9 of the 10 points in $U \cap (X \setminus \{P_3, P_4, P_5\})$ give independent conditions for $\mathcal{O}_U(4)$.

Assume 1) is false and take any point x in U such that $h^0(\mathbf{P}^3, \mathcal{I}_{X \cup \{x\}}(6)) = 3$. Then every $f \in H^0(\mathbf{P}^3, \mathcal{I}_{X \cup \{x\}}(6))$ vanishes on the line xP_4, thus f vanishes on a cone in U with vertex P_4; D_6 gives the contradiction.

Now we prove 2). If 10 of the points considered are on an irreducible conic or cubic, or 5 or 6 (but not 7) of them are collinear, 2) holds. The only problem arise if 7 of them are collinear, but this contradicts the generality of X. By smoothing T we obtain a nonsingular curve $V \in Z'(18, 11; 3)$ with $h^0(\mathbf{P}^3, \mathscr{I}_V(7)) \leqq 4$. Then we conclude as in part c) of Section 4.

§ 7.

The following example shows why the constructions used for the proof of Theorem 1 are so long and baroque. Let C be a smooth curve of genus 5 and degree 7 in $\mathbf{P}^3$. We have seen in the proof of Lemma 1 that it is projectively normal; here it is sufficient that it is not contained in a quadric. We have $\mathcal{O}_C(1) \cong K_C(-P)$ for a unique point P in C. Let x be a point of C. Consider $\mathbf{P}^4$ as a hyperplane H of $\mathbf{P}^3$. Let D be a line in $\mathbf{P}^4$ through x, D not contained in H. Then $C \cup D$ is in an irreducible component of $\mathrm{Hilb}\,\mathbf{P}^4$ containing a smooth curve if and only if $x = P$.

Indeed if $x = P$, $C \cup D$ can be deformed to a smooth canonical curve $X = C$, by a straightforward extension of the proof of Theorem 3 ([1], Prop. II. 1). Think of $\mathbf{P}^4$ as a hyperplane R in $\mathbf{P}^5$ and let U be another hyperplane. Let $X \subset U$ be a linearly normal canonical curve isomorphic to C, $h: C \to X$ be a hysomorphism. We may assume that C is the projection of X from $h(P)$. With a family of projections of X into R which tends to the projection from $h(P)$ (see [1]) we obtain $C \cup D \in Z'(8, 5; 4)$.

Now assume $x \neq P$. Assume that $C \cup D$ can be smoothed, necessarly to a family of canonical curves of genus 5. By projecting the deformation into H, we see that for a general line L through x, $C \cup L \in Z'(8, 5; 3)$: any $C \cup L$ is a projection of $C \cup D$.

Claim: $C \cup L$ is not contained in any cubic surface.

By semicontinuity and [1] the claim will give a contradiction. By the usual game the claim will follow if there exists a plane F through x such that for a general line E in F containing x, $\mathrm{Res}_{E,F}(C \cap F)$ is not contained in a conic. In particular, since, for $x = P$, $C \cup D$ can be smoothed, this never occurs for a plane containing P. Let T be a general plane containing x and P. In T there exists a conic Q containing $\mathrm{Res}_{E,T}(C \cap T)$ and a conic Q' containing $\mathrm{Res}_{E',T}(C \cap T)$ for general lines E, E' through x, P respectively. Since C is not hyperelliptic and cannot have infinite g_3^1, $Q = Q'$; indeed C has no 5-secant and the only 4-secant contained possibly in a general plane F is Px. Thus $C \cap F$ is contained in a conic. The exact sequence (1) in Section 4 gives the contradiction.

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